Mathematica Pannonica
5/1 (1994), 119 – 130

ON TOTALLY UMBILICAL SUBMANIFOLDS OF MANIFOLDS WITH CERTAIN RECURRENT CONDITION IMPOSED ON THE CURVATURE TENSOR

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Received September 1991

AMS Subject Classification: 53 B 25, 53 B 30

Keywords: Pseudo-Riemannian geometry, recurrent manifolds, totally umbilical submanifolds.

Abstract: Totally umbilical submanifolds in manifolds which are generalisation of recurrent manifolds are investigated. At the end of the paper two examples are given.

1. Introduction

Investigating *conformally flat Riemannian manifolds of class one*, i.e. manifolds characterized by the property that at least $n-1$ principal normal curvatures are equal to one another, R. N. Sen and M. C. Chaki ([10]) found that if the remaining one is zero, then the curvature tensor satisfies

$$(1)\qquad R_{hijk,l} = 2a_l R_{hijk} + a_h R_{lijk} + a_i R_{hljk} + a_j R_{hilk} + a_k R_{hijl}\,,$$

where the "comma" denotes covariant derivative with respect to the metric. Hereafter, Riemannian manifolds with condition (1) imposed on the curvature tensor were examined ([1], [2], [3]). Some further generalisations of the condition (1) for various tensor fields were considered by L. Tamássy and T. Q. Binh ([11]). In [3] the present author proved

Proposition ([3]). *If the curvature tensor satisfies*

$$R_{hijk\,,\,l} = \sum_p \overset{p}{v}_{i_1} R_{i_2 i_3 i_4 i_5}\,,$$

where the sum includes all permutation p of the indices (h,i,j,k,l) and $\left\{\overset{p}{v} = \left(\overset{p}{v}_1, \ldots, \overset{p}{v}_n\right)\right\}$ is a set of some vectors, then there exists a vector a_l such that relation (1) *holds.*

Hence it follows that on a *recurrent manifold*, i.e. on a manifold satisfying the condition

$$R_{hijk\,,\,l} R_{pqrs} - R_{hijk} R_{pqrs\,,\,l} = 0\,,$$

at each point where R_{hijk} does not vanish relation (1) is satisfied. Moreover, it was proved that on a neighbourhood of a generic point the vector a_l is a gradient ([3]).

In the paper we begin investigation of totally umbilical submanifolds of manifold satisfying the condition (1) for some vector field a_l. Throughout the paper all manifolds under consideration are assumed to be smooth connected Hausdorff manifolds and their metrics need not be definite.

2. Preliminaries

Let N be an n-dimensional Riemannian manifold with not necessarily definite metric g_{rs}, covered by a system of coordinate neighbourhoods $\{U; x^r\}$. We denote by Γ^k_{ij}, R_{hijk}, R_{hk}, R the Christoffel symbols, the curvature tensor, the Ricci tensor and the scalar curvature of N respectively. Here and in the sequel the indices h, i, j, k, l, r, s, t, u run over the range $1, 2, \ldots, n$. Let M be an m-dimensional manifold covered by a system of coordinate neighbourhoods $\{V; y^a\}$ immersed in manifold N and let $x^r = x^r(y^a)$ be its local expression in N. Then the local components g_{ab} of the induced metric tensor of M are related to g_{rs} by $g_{ab} = g_{rs} B^r_a B^s_b$, where $B^r_a = \frac{\partial x^r}{\partial y^a}$. In what follows we shall adopt the convention

$$B^{rs}_{ab} = B^r_a B^s_b\,, \quad B^{rst}_{abc} = B^r_a B^s_b B^t_c\,, \quad B^{rstu}_{abcd} = B^r_a B^s_b B^t_c B^u_d\,.$$

We denote by Γ^c_{ab}, K_{abcd}, K_{ad}, K the Christoffel symbols, the curvature tensor, the Ricci tensor and the scalar curvature of M with respect to g_{ab} respectively. Here and in the sequel the indices a, b, c, d, e, f run over the range $1, 2, \ldots, m$ $(m < n)$. The *van der Waerden-Bertolotti covariant derivative* ([12], [13]) of B^r_a is given by

(2) $$B^r_{a\,/\,b} = B^r_{a\cdot b} + \Gamma^{\,r}_{st} B^{st}_{ab} - B^r_c \Gamma^{\,c}_{ab}\,,$$

where the "comma" and the dot denote covariant derivative with respect to g_{ab} and partial derivative.

The vector field H^r defined by $H^r = \frac{1}{m} g^{ab} B^r_{a\,/\,b}$ is called the *mean curvature vector* of M. Using (2) and the equation

$$\Gamma^{\,a}_{bc} = \left(B^r_{b\cdot c} + \Gamma^{\,r}_{st} B^{st}_{bc}\right) B^u_d g^{da} g_{ru}$$

we obtain on M

$$g_{rs} H^r B^s_a = 0\,.$$

The *Schouten curvature tensor* H^r_{ab} of M is defined by

$$H^r_{ab} = B^r_{a\,/\,b}\,.$$

If the tensor H^r_{ab} satisfies the condition

$$H^r_{ab} = g_{ab} H^r\,,$$

then M is said to be a *totally umbilical submanifold* of N.

Let N^r_x $(x, y, z = m+1, \ldots, n)$ be pairwise orthogonal unit vectors normal to M. Then

(3) $$g_{rs} N^r_x N^s_x = e_x\,, \quad g_{rs} N^r_x N^s_y = 0 \quad (x \neq y)\,, \quad g_{rs} N^r_x B^s_a = 0$$

and

(4) $$g^{rs} = B^{rs}_{ab} g^{ab} + \sum_x e_x N^r_x N^s_x\,,$$

where e_x is the indicator of the vector N^r_x. On a totally umbilical submanifold M of a manifold N the Gauss and Codazzi equations take the form ([7])

(5) $$K_{abcd} = R_{rstu} B^{rstu}_{abcd} + H(g_{bc} g_{ad} - g_{bd} g_{ac})$$

and

(6) $$R_{rstu} B^{rst}_{abc} N^u_x = A_{ax} g_{ac} - A_{bx} g_{ac}$$

respectively, where

$$H = g_{rs} H^r H^s\,, \quad A_{ax} = H_{x\cdot a} + \sum_y e_y L_{ayx} H_y\,, \quad H_y = H^r N^s_y g_{rs}$$

and

$$L_{azy} = g_{rs} N^r_y N^s_{z\,/\,a}\,.$$

Moreover, we have ([6], [7])

(7) $$R_{rstu} H^r B^{stu}_{bcd} = \frac{1}{2}\left(g_{bc} H_d - g_{bd} H_c\right)\,, \quad H_c = H_{/\,c}\,,$$

$$
\begin{aligned}
(8)\qquad K_{abcd\,/\,e} = {} & R_{hijk;l}B^{hijkl}_{abcde} + H_e\left(g_{bc}g_{ad} - g_{bd}g_{ac}\right) + \\
& + \frac{1}{2}\Big[H_a\left(g_{bc}g_{ed} - g_{bd}g_{ec}\right) + H_b\left(g_{ec}g_{ad} - g_{ed}g_{ac}\right) + \\
& + H_c\left(g_{be}g_{ad} - g_{bd}g_{ae}\right) + H_d\left(g_{bc}g_{ae} - g_{be}g_{ac}\right)\Big],
\end{aligned}
$$

where the semicolon denotes covariant derivative with respect to the metric of the ambient space,

$$
(9)\qquad H^r_{\ /a} = -HB^r_a + \sum_z e_z A_{az} N^r_z .
$$

We shall also use

Lemma 1 ([8]). (I) *Let (A_i), (B_i) be two sequences of numbers which are linearly independent as elements of the space $\mathbb{R}^n$. If T_{ij}, S_{ij} are numbers satisfying conditions*

$$
T_{ij}A_k + T_{jk}A_i + T_{ki}A_j + S_{ij}B_k + S_{jk}B_i + S_{ki}B_j = 0 ,
$$

$$
T_{ij} = T_{ji} , \quad S_{ij} = S_{ji}
$$

then there exist numbers D_i such that

$$
T_{ij} = -B_iD_j - B_jD_i , \quad S_{ij} = A_iD_j + A_jD_i .
$$

(II) *Let T_{ij}, A_k be numbers satisfying conditions*

$$
T_{ij}A_k + T_{jk}A_i + T_{ki}A_j = 0 , \quad T_{ij} = T_{ji} .
$$

Then either each T_{ij} is zero or each A_i is zero.

Lemma 2 ([4], Lemma 1). *Let M be a Riemannian manifold of dimension $n \geq 3$. If B_{hijk} is a tensor field on M such that*

$$
(10)\qquad B_{hijk} = -B_{ihjk} = B_{jkhi} , \quad B_{hijk} + B_{hjki} + B_{hkij} = 0 ,
$$

$$
B_{hijk\,/\,[lm]} = 0 ,
$$

and a_l, A_l are vectors fields on M satisfying

$$
a_r R^r{}_{ijk} = g_{ij}A_k - g_{ik}A_j ,
$$

then

$$
A_l\left[B_{hijk} - \frac{S}{n(n-1)}\left(g_{ij}g_{hk} - g_{ik}g_{hj}\right)\right] = 0 ,
$$

where $S = B_{pqrs}g^{ps}g^{qr}$.

Lemma 3 ([9], Lemma 3). *If c_j, p_j, B_{hijk} are numbers satisfying* (10) *and*

$$
c_lB_{hijk} + p_hB_{lijk} + p_iB_{hljk} + p_jB_{hilk} + p_kB_{hijl} = 0 ,
$$

then either each $b_j = c_j + 2p_j$ is zero or each B_{hijk} is zero.

3. Main results

Theorem 1. *Let M ($\dim M > 2$) be a totally umbilical submanifold of the manifold N satisfying the condition (1) for some vector field a_i. Then the relation*

$$(g_{rs}H^rH^s - a_rH^r)C_{abcd} = 0 \tag{11}$$

holds on M, where C_{abcd} are components of the Weyl conformal curvature tensor of the submanifold M.

Proof. Transvecting (1) with $H^hB^{ijkl}_{bcde}$ and applying (5) and (7) we obtain

$$\begin{aligned} &R_{hijk\,,\,l}H^hB^{ijkl}_{bcde} = \\ &= a_e(g_{bc}H_d - g_{bd}H_c) + VK_{ebcd} - VH(g_{bc}g_{ed} - g_{bd}g_{ec}) + \\ &+\frac{1}{2}a_b(g_{ec}H_d - g_{ed}H_c) + \frac{1}{2}a_c(g_{be}H_d - g_{bd}H_e) + \frac{1}{2}a_d(g_{bc}H_e - g_{be}H_c), \end{aligned} \tag{12}$$

where $a_e = a_rB^r_e$, $V = a_rH^r$. On the other hand, differentiating covariantly the left hand side of (7), in virtue of (9), (5) and (6), we get

$$\begin{aligned} &\left[R_{hijk}H^hB^{ijk}_{bcd}\right]_{,\,e} = R_{hijk\,,\,l}H^hB^{ijkl}_{bcde} - HK_{ebcd} + \\ &+H^2(g_{bc}g_{ed} - g_{bd}g_{ec}) + g_{bc}E_{de} - g_{bd}E_{ce} - g_{ce}S_{bd} + g_{de}S_{bc}, \end{aligned} \tag{13}$$

where $E_{de} = \sum_x e_xA_{dx}A_{ex} = E_{ed}$ and $S_{bc} = R_{hijk}H^hB^{ij}_{bc}H^k = S_{cb}$. Then, substituting (12) into (13) and taking into account relation (7), we find

$$\begin{aligned} &(H - V)K_{ebcd} = \\ &= H(H - V)(g_{bc}g_{ed} - g_{bd}g_{ec}) + g_{bc}E_{de} - g_{bd}E_{ce} - g_{ce}S_{bd} + g_{de}S_{bc} + \\ &+a_e(g_{bc}H_d - g_{bd}H_c) - \frac{1}{2}(g_{bc}H_{,\,de} - g_{bd}H_{,\,ce}) + \\ &+\frac{1}{2}(g_{ec}a_bH_d - g_{ed}a_bH_c + g_{be}a_cH_d - g_{bd}a_cH_e + g_{bc}a_dH_e - g_{be}a_dH_c). \end{aligned} \tag{14}$$

Hereafter, contracting (14) with g^{ed} and alternating the resulting equation in (b, c), we obtain

$$a_bH_c = a_cH_b\,. \tag{15}$$

Therefore, alternating (14) in (e, b) and using (15), we get

$$(16)\qquad 2(H-V)K_{ebcd}=$$
$$=2H(H-V)(g_{bc}g_{ed}-g_{bd}g_{ec})+$$
$$+g_{bc}(E_{de}+S_{de})-g_{bd}(E_{ce}+S_{ce})+g_{de}(E_{bc}+S_{bc})-g_{ce}(E_{bd}+S_{bd})+$$
$$+g_{bc}a_eH_d-g_{bd}a_cH_e+g_{ed}a_bH_c-g_{ec}a_bH_d-$$
$$-\frac{1}{2}(g_{bc}H_{,ed}-g_{bd}H_{,ce}+g_{ed}H_{,bc}-g_{ce}H_{,bd}),$$

whence we obtain

$$(17)\quad 2(H-V)K_{bc}=2(m-1)H(H-V)g_{bc}+(m-2)(E_{bc}+S_{bc})+$$
$$+g_{bc}(E+S+P-\frac{1}{2}Q)+(m-2)a_bH_c-\frac{m-2}{2}H_{,bc}$$

and

$$(18)\quad 2(H-V)K=(m-1)[2mH(H-V)+2(E+S)+2P-Q],$$

where

$$E=E_{bc}g^{bc},\quad S=S_{bc}g^{bc},\quad P=a_bH_cg^{bc},\quad Q=H_{,bc}g^{bc}.$$

Finally, using equations (16)–(18), by an immediate calculations, we check that (11) holds good. ◊

Transvecting (1) with B^{hijkl}_{abcde} and making use of (5) and (8) we find

$$(19)\quad K_{abcd,e}=2a_eK_{abcd}+a_aK_{ebcd}+a_bK_{aecd}+a_cK_{abed}+a_dK_{abce}+$$
$$+2Z_e(g_{bc}g_{ad}-g_{bd}g_{ac})+Z_a(g_{bc}g_{ed}-g_{bd}g_{ec})+$$
$$+Z_b(g_{ec}g_{ad}-g_{ed}g_{ac})+Z_c(g_{be}g_{ad}-g_{bd}g_{ae})+Z_d(g_{bc}g_{ae}-g_{be}g_{ac}),$$

where $Z_e=\frac{1}{2}H_{,e}-a_eH$, whence we obtain

$$(20)\qquad K_{bc,e}=2a_eK_{bc}+a_bK_{ec}+a_cK_{be}+a_fK^f{}_{bce}+a_fK^f{}_{cbe}+$$
$$+2mg_{bc}Z_e+(m-2)(g_{ec}Z_b+g_{be}Z_c),$$

$$(21)\qquad K_{,e}=2a_eK+4a_fK^f{}_e+2(m-1)(m+2)Z_e.$$

Suppose, that at a point $x\in M$ the relation

$$(22)\quad K_{abcd,e}=2b_eK_{abcd}+b_aK_{ebcd}+b_bK_{aecd}+b_cK_{abed}+b_dK_{abce}$$

is satisfied for a certain vector b_e. Then we have

$$(23)\qquad K_{bc,e}=2b_eK_{bc}+b_bK_{ec}+b_cK_{be}+b_fK^f{}_{bce}+b_fK^f{}_{cbe}.$$

Subtracting (23) from (20), permuting cyclically the such obtained equality in (b,c,e) and adding the resulting equations, we get

$$K_{bc}(a_e - b_e) + K_{ce}(a_b - b_b) + K_{eb}(a_c - b_c) + \quad (24)$$
$$+(m-1)(g_{bc}Z_e + g_{ce}Z_b + g_{eb}Z_c) = 0 .$$

If $a_e - b_e$ and Z_e are linearly independent, then by Lemma 1 (I) we have $\operatorname{rank} g_{ab} \leq 2$. Thus, for $m > 2$, either $Z_b = 0$ or $Z_b \neq 0$ and $Z_e = f(a_e - b_e)$, for $f \in \mathbb{R} - \{0\}$. Subtracting (22) from (19), then substituting $Z_e = f(a_e - b_e)$ and applying Lemma 3 we get $(a_e - - b_e)[K_{abcd} + f(g_{bc}g_{ad} - g_{bd}g_{ac})] = 0$ at x. Thus, if $Z_e(x)$ does not vanish and $\dim M > 2$, then on a neighbourhood of x we have $f_{,e} = 0$. Moreover, we have

$$(a_f - b_f)K^f{}_{bce} + g_{bc}Z_e - g_{be}Z_c = 0 \quad (25)$$

at a point $x \in M$ where Z_e does not vanish.

From the above made considerations we are in a position to obtain

Theorem 2 (cf [5], Th. 3.3). *Let M be a totally umbilical submanifold of a manifold N satisfying the condition (1) for some vector field a_l and suppose that a_l is not orthogonal to M. If condition (22) is satisfied on M for some vector field b_b which does not vanish on a dense subset of M and $\dim M > 2$, then $Z_b = 0$ on M. Conversely, if $Z_b = 0$, then condition (22) holds on M with $b_b = a_b$.*

Theorem 3 (cf. [5], Ths. 3.6 and 3.7). *Let M ($\dim M > 2$) be a totally umbilical submanifold of a manifold N satisfying the condition (1) for some vector field a_l and suppose that a_l is not orthogonal to M. If Z_b does not vanish on a dense subset of M, then M is a space of constant curvature.*

Theorem 4. *Let M ($\dim M > 2$) be a totally umbilical submanifold of a manifold N satisfying the condition (1) for some vector field a_l. If M is semi-symmetric (i.e. $K_{abcd,[ef]} = 0$) and Z_b does not vanish on a dense subset of M, then M is a space of constant curvature and $Z_b + \frac{K}{m(m-1)}a_b = 0$.*

Proof. Follows from (25) and Lemma 2. ◊

Theorem 5 (cf. [5], Th. 4.1). *Let M ($\dim M > 2$) be a totally umbilical submanifold of the manifold N satisfying the condition (1). If the vector a_l is orthogonal to M, then M is a conformally symmetric manifold.*

Proof. If a_l is orthogonal to M, then $a_e = a_r B^r_e$ vanishes. Using the formulas (19)–(21), by an immediate calculations, we check that $C_{abcd,e} = 0$ holds on M. ◊

Theorem 6. *Let M ($\dim M > 2$) be a totally umbilical submanifold of a manifold N satisfying the condition (1) for some vector field a_l and suppose that a_l is not orthogonal to M. Then the relation*

(26) $$K_{abcd/e} = c_e K_{abcd}$$
holds on M for some vector field c_e which does not vanish on a dense subset of M, if and only if

(27) $$Z_e = 0$$
and

(28) $$a_e K_{abcd} + a_c K_{abde} + a_d K_{abec} = 0$$
on M.

Proof. Suppose that relation (26) holds on M, i.e. at each point there exists a vector c_e satisfying (26). Consequently, we have on M

(29) $$c_e K_{abcd} + c_c K_{abde} + c_d K_{abec} = 0$$

and relation of the form (22) is also satisfied ([3], Prop. 1). According to the Th. 2, the last condition is equivalent to $Z_e = 0$. Hence, we have (19) with $Z_e = 0$. Substituting (26) and (27) into (19) we obtain

$$(-c_e + 2a_e)K_{abcd} + a_a K_{ebcd} + a_b K_{aecd} + a_c K_{abed} + a_d K_{abce} = 0\,,$$

whence, in virtue of Lemma 3, $c_e = 4a_e$. Therefore, using (29), we get (28) on M. Conversely, if $Z_e = 0$ and (28) holds on M, then (19) yields $K_{abcd/e} = 4a_e K_{abcd}$. ◊

Suppose now that $K_{abcd/e}(x) = 0$, $x \in M$. If a_e and Z_e are not linearly dependent, then (20) and Lemma 1(I) yield $\operatorname{rank} g_{ab} \le 2$. Thus, for $m > 2$, we have either

$$Z_e = 0 \quad \text{and} \quad a_e = 0 \quad \text{or}$$

$$Z_e = 0 \quad \text{and} \quad a_e \neq 0 \quad \text{or}$$

$$Z_e \neq 0\,, \quad a_e \neq 0 \quad \text{and} \quad Z_e = f a_e\,, \quad f \in \mathbb{R} - \{0\}\,.$$

Therefore relation (19) and Lemma 3 result in

Theorem 7. *Let M ($\dim M > 2$) be a totally umbilical submanifold of a manifold N satisfying the condition (1) for some vector field a_l. If a_l is orthogonal to M, then $Z_e = 0$ if and only if $K_{abcd/e} = 0$.*

Theorem 8. *Let M ($\dim M > 2$) be a totally umbilical submanifold of a manifold N satisfying the condition (1) for some vector field a_l and suppose that M is locally symmetric. If $a_e(x) \neq 0$ and $Z_e = 0$, then M is flat. If a_l is not orthogonal to M and Z_e does not vanish at any point of M, then M is a non-flat space of constant curvature.*

4. Some examples

Let N be an open subset of $\mathbb{R}^n$, ($n > 2$), endowed with the metric

$$\tilde{g}_{ij}dx^i dx^j = (dx^1)^2 + p^2 f_{\alpha\beta}dx^\alpha dx^\beta \,,$$

$\alpha, \beta, \gamma, \cdots = 2, \ldots, n$, where $f_{\alpha\beta}dx^\alpha dx^\beta$ is a flat metric and p is a function in x^1 variable satisfying the equation

$$pp'p''' + 3(p')^2 p'' - 4p(p'')^2 = 0 \,.$$

For suitable choosen of N there exist solutions such that the condition (1) holds on N and N is not recurrent ([3], Th. 6, Props. 5 and 6).

Let V be a flat manifold of dimension m endowed with the metric $h_{PQ}dx^P dx^Q$, $P, Q = n+1, \ldots, n+m$. On the manifold $N \times V$ define the metric

$$g_{rs}dx^r dx^s = \tilde{g}_{ij}dx^i dx^j + h_{PQ}dx^P dx^Q \,.$$

Then on $(N \times V, g)$, for suitable function p, the condition (1) is fulfilled while $R_{hijk\,/\,l} = c_l R_{hijk}$ is not satisfied.

Example 1. Let M be an n-dimensional manifold covered by a system of coordinate neighbourhoods $\{W; y^a\}$, $a, b, \cdots = 1, \ldots, n$, immersed in $N \times V$ and let $x^1 = Q(y^a)$, $x^\alpha = y^\alpha$, $x^P = C_P$, $C_P =$ const be its local expression in $N \times V$. Then $B^1_d = Q_d$, $B^\alpha_\beta = \delta^\alpha_\beta$, $B^P_d = 0$, where $Q_d = Q_{\,/\,d}$. The covariant and contravariant components of the induced metric tensor of M are respectively

$$g_{11} = (Q_1)^2 \,, \quad g_{1\alpha} = Q_1 Q_\alpha \,, \quad g_{\alpha\beta} = Q_\alpha Q_\beta + \tilde{g}_{\alpha\beta} \,,$$

$$g^{11} = (Q_\alpha Q_\beta \tilde{g}^{\alpha\beta} + 1)(Q_1)^{-2} \,, \quad g^{1\alpha} = -Q_\beta \tilde{g}^{\beta\alpha}(Q_1)^{-1} \,, g^{\alpha\beta} = \tilde{g}^{\alpha\beta} \,.$$

The only components of the Christoffel symbols which may not vanish are

$$\Gamma^1_{11} = \frac{Q_{\cdot 11}}{Q_1} \,, \quad \Gamma^1_{1\alpha} = \frac{Q_{\cdot 1\alpha}}{Q_1} - \frac{p'}{p}Q_\alpha \,,$$

$$\Gamma^1_{\alpha\beta} = (Q_{\cdot\alpha\beta} - \frac{p'}{p}\tilde{g}_{\alpha\beta})(Q_1)^{-1} - 2\frac{p'}{p}Q_\alpha Q_\beta (Q_1)^{-1} - \overline{\Gamma}^\nu_{\alpha\beta} Q_\nu (Q_1)^{-1} \,,$$

$$\Gamma^\alpha_{\beta\gamma} = \frac{p'}{p}(Q_\beta \delta^\alpha_\gamma + Q_\gamma \gamma^\alpha_\beta) + \overline{\Gamma}^\alpha_{\beta\gamma} \,, \quad \Gamma^\alpha_{1\beta} = \frac{p'}{p}Q_1 \delta^\alpha_\beta \,,$$

where $\overline{\Gamma}^\alpha_{\beta\gamma}$ are Christoffel symbols of $f_{\alpha\beta}dx^\alpha dx^\beta$. Then, using (2), we check that $B^r_{a\,/\,b} = 0$, so the submanifold M is a totally geodesic one. Consequently, M is a totally umbilical submanifold in $N \times V$ and the vector field $Z_e = \frac{1}{2}H_{\,/\,e} - a_e H$ vanishes. Moreover, the components of the projection of the vector $\tilde{a}_l$ of $N \times V$ onto the submanifold M are $a_d = \tilde{a}_1 Q_d$. If $\tilde{a}_1 = \frac{p''}{p'} - \frac{p'}{p} \neq 0$ ([3]) and $Q_d \neq 0$, then, according to the Th. 2, condition (22) holds on M with $b_d = a_d$.

The components of the curvature tensor and the Ricci tensor of M are

$$K_{\alpha\beta\gamma\delta} = \frac{p''}{p}\left[Q_\beta Q_\gamma \tilde{g}_{\alpha\delta} - Q_\beta Q_\delta \tilde{g}_{\alpha\gamma} + Q_\alpha Q_\delta \tilde{g}_{\beta\gamma} - Q_\alpha Q_\gamma \tilde{g}_{\beta\delta}\right] +$$

$$+(p')^2(\tilde{g}_{\beta\gamma}\tilde{g}_{\alpha\delta} - \tilde{g}_{\beta\delta}\tilde{g}_{\alpha\gamma}),$$

$$K_{1\beta\gamma\delta} = \frac{p''}{p}Q_1(\tilde{g}_{\beta\gamma}Q_\delta - \tilde{g}_{\beta\delta}Q_\gamma),$$

$$K_{1\beta\gamma 1} = \frac{p''}{p}(Q_1)^2\tilde{g}_{\beta\gamma},$$

$$K_{1d} = (n-1)\frac{p''}{p}Q_1 Q_d,$$

$$K_{\alpha\beta} = (n-1)\frac{p''}{p}Q_\alpha Q_\beta + \left[\frac{p''}{p} + (n-2)(p')^2\right]\tilde{g}_{\alpha\beta}.$$

Moreover, the scalar curvature of M is given by

$$K = (n-1)\left[2\frac{p''}{p} + (n-2)(p')^2\right].$$

One can check, that M is a conformally flat submanifold in $N \times V$. Setting in (28) $a = \alpha$, $b = \beta$, $c = \gamma$, $d = \delta$, $e = \nu$ we easily obtain $(n-3)(n-1)Q_\nu = 0$. Thus we have

Proposition 1. *For each $n > 3$ and $t > n$ there exists t-dimensional manifold satisfying* (1) *admitting n-dimensional totally umbilical and conformally flat submanifold M such that the condition* (1) *holds on M whereas* (26) *is not satisfied.*

Example 2. Let M be an $(n-1)$-dimensional manifold covered by a system of coordinate neighbourhoods $\{W; z^\alpha\}$, $\alpha, \beta, \gamma = 2, \ldots n$, immersed in $N \times V$ and let $x^1 = C_1$, $x^P = C_P$, $C_P = \text{const}$, $x^\alpha = X^\alpha(z^2, \ldots, z^n)$ be its local expression in $N \times V$. Then we have $B^1_\alpha = B^P_\alpha = 0$, $B^\alpha_\beta = \frac{\partial X^\alpha}{\partial z^\beta}$, $g_{\alpha\beta} = \tilde{g}_{\mu\nu}B^{\mu\nu}_{\alpha\beta}$, whence $B^1_{\alpha\,\prime\,\beta} = -\frac{p'}{p}g_{\alpha\beta}$, $B^P_{\alpha\,\prime\,\beta} = 0$. Moreover, in virtue of (3) and (4), we get

$$B^\nu_{\alpha\,\prime\,\beta} = \left(B^\rho_{\alpha\cdot\beta} + \overline{\Gamma}^\rho_{\mu\eta}B^{\mu\eta}_{\alpha\beta}\right)\left(\sum_x e_x N^\tau_x N^\nu_x \tilde{g}_{\tau\rho}\right).$$

Setting $f_{\alpha\beta} = \delta_{\alpha\beta}$, $x^\alpha = z^2 + \cdots + z^n$ we get

$$a_\alpha = \tilde{a}_r B^r_\alpha = 0\,, \quad H^1 = -\frac{p'}{p}\,, \quad H^\alpha = H^P = 0\,,$$

$$g_{rs}H^r H^s - \tilde{a}_r H^r = \frac{p''}{p}\,, \quad Z_e = 0\,,$$

$$K_{\alpha\beta\gamma\delta} = 2(p')^2(p)^{-2}(g_{\beta\gamma}g_{\alpha\delta} - g_{\beta\delta}g_{\alpha\gamma})\,.$$

Hence we obtain

Proposition 2. *For each $n > 3$ and $t > n$ there exists t-dimensional manifold satisfying* (1) *admitting $(n-1)$-dimensional totally umbilical submanifold M such that the recurrence vector is orthogonal to M* (cf. Th. 5).

Proposition 3. *For each $n > 3$ and $t > n$ there exists t-dimensional manifold satisfying* (1) *admitting $(n-1)$-dimensional totally umbilical submanifold M such that $g_{rs}H^rH^s - a_rH^r$ does not vanish identically on M* (cf. Th. 1).

Proposition 4. *For each $n > 3$ and $t > n$ there exists t-dimensional manifold satisfying* (1) *admitting $(n-1)$-dimensional totally umbilical locally symmetric submanifold* (cf. Th. 7).

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