

## DEPTH OF DENDROIDS

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**Abstract:** It is shown that for every countable ordinal  $\alpha$  there exists a fan whose depth is  $\alpha$  and that depth of uniformly arcwise connected dendroids (in particular smooth fans) is less than or equal to two.

### 1. Introduction

S. D. Iliadis has constructed in [4] an uncountable family of hereditarily decomposable and hereditarily unicoherent continua  $X(\alpha)$  numbered with ordinals  $\alpha < \omega_1$  having the property that

(1.1) for each  $\alpha < \omega_1$  the depth of the  $\alpha$ -th member of  
the family is just  $\alpha$  (see also [7], Th. 24, p. 24).

The continua  $X(\alpha)$  are arclike (they are called snake-like in [1] and [4]), and thus  $X(1)$  only, being an arc, is arcwise connected. A question can be asked if it is possible to construct such a family (having property (1.1)) composed exclusively of arcwise connected continua. In this paper we construct a family of continua satisfying (1.1) and moreover such that each member of the family is hereditarily arcwise connected, hereditarily unicoherent, and has exactly one ramification point. In

other words the family consists of fans. All the continua  $X(\alpha)$  were planable (as arclike ones, since each arclike continuum is planable, see [1], Th. 4, p. 654). Our family also keeps this property.

It is shown in the final part of the paper that if we replace the assumption of arcwise connectedness by a stronger one, uniform arcwise connectedness, then no such a family does exist. Namely depth of uniformly arcwise connected continua is either 1 if they are locally connected, or 2 otherwise. Since smoothness of dendroids implies uniform arcwise connectedness, depth of a smooth dendroid is at most 2.

## 2. Preliminaries

A *continuum* means a compact connected metric space. A continuum is said to be *hereditarily unicoherent* provided that the intersection of any two its subcontinua is connected. A continuum that is hereditarily unicoherent and arcwise connected is called a *dendroid*. A locally connected dendroid is called a *dendrite*. A point  $p$  in a dendroid  $X$  is called an *end point* of  $X$  if  $p$  is an end point of every arc contained in  $X$ . The set of all end points of a dendroid  $X$  is denoted by  $E(X)$ . A point  $p$  in a dendroid  $X$  is called a *ramification point* of  $X$  if  $p$  is the vertex of a simple triod contained in  $X$  (i.e. if there are three points  $a$ ,  $b$  and  $c$  in  $X$  such that any two of the three arcs  $pa$ ,  $pb$  and  $pc$  have the point  $p$  in common only). The set of all ramification points of a dendroid  $X$  is denoted by  $R(X)$ . A dendroid having exactly one ramification point is called a *fan*, and the point is called the *vertex* of the fan. A fan is said to be *countable* provided that the set of all its end points is countable.

A continuum  $X$  is said to be *hereditarily decomposable* provided that every subcontinuum of  $X$  is the union of two its proper subcontinua. A hereditarily decomposable and hereditarily unicoherent continuum is called a  $\lambda$ -*dendroid*. Given a  $\lambda$ -dendroid  $X$  we denote by  $\mathcal{P}(X)$  the family of all subcontinua  $S$  of  $X$  such that for each finite cover of  $X$  the elements of which are subcontinua of  $X$ , the continuum  $S$  is contained in a member of the cover. A (transfinite) well-ordered sequence (numbered with ordinals  $\alpha$ ) of nondegenerate subcontinua  $X_\alpha$  of a  $\lambda$ -dendroid  $X$  is said to be *normal* provided that the following conditions are satisfied:

$$(2.1) \quad X_1 = X;$$

$$(2.2) \quad X_{\alpha+1} \in \mathcal{P}(X_\alpha);$$

$$(2.3) \quad X_\beta = \bigcap \{X_\alpha : \alpha < \beta\} \text{ for each limit ordinal } \beta.$$

The *depth*  $k(X)$  of a  $\lambda$ -dendroid  $X$  is defined as the minimum ordinal number  $\gamma$  such that the order type of each normal sequence of subcontinua of  $X$  is not greater than  $\gamma$ . The reader is referred to [4], [6] and [8] for an additional information about this concept. The following three important facts concerning the depth will be needed in the present paper (see [4], Ths. 1, 2 and 3, p. 94 and 95).

**Fact 2.4.** *For every two  $\lambda$ -dendroids  $X$  and  $Y$  if  $Y \subset X$  then  $k(Y) \leq k(X)$ .*

**Fact 2.5.** *A  $\lambda$ -dendroid  $X$  is locally connected (i.e., it is a dendrite) if and only if  $k(X) = 1$ .*

**Fact 2.6.** *If a  $\lambda$ -dendroid  $Y$  is a continuous image of a  $\lambda$ -dendroid  $X$ , then  $k(Y) \leq k(X)$ .*

Let a continuum  $X$  with a metric  $d$  be given, and let  $A$  and  $B$  be two its closed subsets. The *Hausdorff distance*  $\text{dist}$  from  $A$  to  $B$  is defined by

$$\text{dist}(A, B) = \max \{ \sup \{ d(a, B) : a \in A \}, \sup \{ d(b, A) : b \in B \} \}.$$

As usual, the symbol  $\mathbb{N}$  stands for the set of all positive integers.

### 3. Fans

The main result of the paper is the following theorem.

**Theorem 3.1.** *For every ordinal number  $\alpha < \omega_1$  there exists a countable plane fan  $F[\alpha]$  having its depth  $k(F[\alpha])$  equal to  $\alpha$ .*

**Proof.** We proceed by transfinite induction. Let  $\alpha = 1$ . In the rectangular Cartesian coordinate system in the plane let  $v = (0, 0)$  be the origin. For each  $n \in \mathbb{N}$  put  $e(1, n) = (1/n, 1/n^2)$  and denote by  $I(1, n)$  the straight line segment joining  $v$  and  $e(1, n)$ . Then the union

$$(3.2) \quad F[1] = \bigcup \{ I(1, n) : n \in \mathbb{N} \}$$

is a plane fan having the origin  $v$  as its vertex. The set  $E(F[1]) = \{ e(1, n) : n \in \mathbb{N} \}$  of end points of  $F[1]$  is countable, so the constructed fan is countable by its definition. Furthermore, since  $F[1]$  is locally connected, we infer from Fact 2.5 that

$$(3.3) \quad k(F[1]) = 1.$$

Let  $\beta \geq 1$  be an ordinal number. Assume that countable plane fans  $F[\alpha]$  are defined for all ordinals  $\alpha < \beta$  in such a way that

$$(3.4) \quad v \text{ is the vertex of } F[\alpha];$$

$$(3.5) \quad F[\alpha] \text{ contains the union of a sequence of arcs } \{I(1, n) : n \in \mathbb{N}\};$$

$$(3.6) \quad k(F[\alpha]) = \alpha.$$

If  $\beta = \alpha + 1$ , we perform the following construction. For each  $n \in \mathbb{N}$  we define an arc  $I(\beta, n)$  such that

$$(3.7) \quad v \text{ is an end point of } I(\beta, n);$$

$$(3.8) \quad I(\beta, n) \cap F[\alpha] = \{v\};$$

$$(3.9) \quad I(\beta, m) \cap (I(\beta, 1) \cup \dots \cup I(\beta, m-1)) = \{v\} \\ \text{for each } m \in \mathbb{N} \text{ and } m \geq 2;$$

$$(3.10) \quad \text{dist}(I(\beta, n), \bigcup\{I(\alpha, m) : m \in \{1, \dots, n\}\}) < 1/2^n.$$

Put

$$(3.11) \quad F[\beta] = F[\alpha] \cup \bigcup\{I(\beta, n) : n \in \mathbb{N}\}.$$

Condition (3.10) implies that the arcs  $I(\beta, n)$  better and better approximate the unions  $\bigcup\{I(\alpha, m) : m \in \{1, \dots, n\}\}$  and therefore it guarantees that the resulting space  $F[\beta]$  is compact. Further, it can be observed by (3.10) and (3.11) that

$$(3.12) \quad \text{the union } \bigcup\{I(\beta, n) : n \in \mathbb{N}\} \text{ is a dense (and thus } F[\alpha] \\ \text{is a nowhere dense) subset of } F[\beta].$$

Connectedness of  $F[\beta]$  follows from (3.8), and thus (3.11) assures that  $F[\beta]$  is arcwise connected. Further, conditions (3.7)–(3.9) imply hereditary unicoherence of  $F[\beta]$ , so that  $F[\beta]$  is a dendroid, and also they lead to the equality  $R(F[\beta]) = R(F[\alpha]) = \{v\}$ , so  $F[\beta]$  is a fan having  $v$  as its vertex.

Denote by  $e(\beta, n)$  this end point of  $I(\beta, n)$  which is distinct from  $v$ . It can easily be seen from the construction that

$$(3.13) \quad E(F[\beta]) = E(F[\alpha]) \cup \{e(\beta, n) : n \in \mathbb{N}\}.$$

Since  $E(F[\alpha])$  is countable by assumption, (3.13) implies that  $E(F[\beta])$  is countable, too, and thus the fan  $F[\beta]$  is countable.

If a covering of  $F[\beta]$  by finitely many subcontinua is considered, then condition (3.12) implies that  $F[\alpha]$  is contained in a member of the covering. Thus  $F[\alpha] \in \mathcal{P}(F[\beta])$ , whence we infer that if a normal sequence  $X_1, X_2, \dots$  of subcontinua of  $X = F[\beta]$  is considered, then the second term of it,  $X_2$ , is  $F[\alpha]$ , so we conclude from (3.6) that

$$(3.14) \quad k(F[\alpha + 1]) = \alpha + 1.$$

If  $\beta$  is a limit ordinal, we consider a sequence

$$(3.15) \quad \{\alpha_n : n \in \mathbb{N}\} \text{ with } \alpha_n < \beta \text{ and } \beta = \lim \alpha_n.$$

For each  $n \in \mathbb{N}$  we take a copy of the fan  $F[\alpha_n]$ . Roughly speaking, we locate these copies in the plane in such a manner that their union,  $F[\beta]$ , is obtained from  $F[\alpha_n]$ 's in the same way as  $F[1]$  is obtained from the segments  $I(1, n)$ . More rigorously, we assume that the considered copies of the fans  $F[\alpha_n]$  satisfy the following conditions:

$$(3.16) \quad \lim \text{diam } F[\alpha_n] = 0,$$

$$(3.17) \quad F[\alpha_n] \cap F[\alpha_m] = \{v\} \text{ if } n \neq m,$$

and we put

$$(3.18) \quad F[\beta] = \bigcup \{F[\alpha_n] : n \in \mathbb{N}\}.$$

Condition (3.16) guarantees compactness, and (3.17) implies connectedness (thus arcwise connectedness) and hereditary unicoherence of the resulting space  $F[\beta]$  which thereby is a countable plane fan. Finally Fact 2.4 implies by (3.6) and (3.18) that  $\alpha_n = k(F[\alpha_n]) \leq k(F[\beta])$ , whence by (3.15) we infer that

$$k(F[\beta]) = \beta.$$

Thus the fan  $F[\alpha]$  is defined for each ordinal number  $\alpha < \omega_1$ , and it satisfies the needed equality (3.6).  $\diamond$

**Remark 3.19.** After constructing, for each ordinal  $\alpha < \omega_1$ , an arclike continuum  $X(\alpha)$  with  $k(X(\alpha)) = \alpha$  a question is asked in [4], Remark 3, p. 98, whether there exists 1° a  $\lambda$ -dendroid, or 2° an arclike  $\lambda$ -dendroid  $X$  having the depth  $\omega_1$ . And it is shown in Section 3 of [6], p. 719, that an answer to 2° is negative, while an answer for the general case

(i.e. for  $1^\circ$ ) remains unknown. Thus, in connection with Th. 3.1, the following question seems to be natural.

**Question 3.20.** Does there exist a fan  $X$  such that  $k(X) = \omega_1$ ?

**Remark 3.21.** If we consider, instead of arbitrary finite coverings of a  $\lambda$ -dendroid  $X$  by its subcontinua, finite coverings having a fixed number  $n \geq 2$  of elements, we get a (similarly defined) concept of the  $n$ -depth (see [8], p. 587). Since for this concept the results that correspond to Facts 2.4 and 2.5 are also true (see [9], Theorems 4 and 5), and since only Facts 2.4 and 2.5 were used in the proof of Th. 3.1 above, one can repeat all the arguments of that proof replacing the depth  $k(X)$  by the  $n$ -depth  $k_n(X)$  for any continuum  $X$  considered in that proof. In this way we have the following corollary to Th. 3.1.

**Corollary 3.22.** Let an ordinal number  $\alpha < \omega_1$  be given, and let  $F[\alpha]$  denote the countable plane fan of Th. 3.1. Then for each natural number  $n \geq 2$  the  $n$ -depth  $k_n(F[\alpha])$  of  $F[\alpha]$  equals  $\alpha$ .

#### 4. Uniformly arcwise connected dendroids

The cone  $F_c$  over the Cantor middle-thirds set is called the *Cantor fan*. A continuum  $X$  is said to be *uniformly pathwise connected* provided that it is a continuous image of the Cantor fan. The original definition of this concept, given in [5], p. 316, is more complicated, but it agrees with the above one by Th. 3.5 of [5], p. 322. A space  $X$  is said to be *uniformly arcwise connected* provided that it is arcwise connected and that for each  $\varepsilon > 0$  there is a  $j \in \mathbb{N}$  such that every arc in  $X$  contains  $j$  points that cut it into subarcs of diameters less than  $\varepsilon$ . By Th. 3.5 of [5], p. 322, each uniformly arcwise connected continuum is uniformly pathwise connected (but not conversely), and it is easy to see that for uniquely arcwise connected continua these two notions coincide (see [5], p. 316). In particular, the coincidence holds for dendroids.

**Proposition 4.1.** The depth  $k(F_c)$  of the Cantor fan  $F_c$  equals 2.

**Proof.** Denote by  $v$  the vertex of  $F_c$ . It can easily be observed that a subcontinuum  $S$  of  $F_c$  is in  $\mathcal{P}(F_c)$  if and only if it is an arc contained in the straight line segment  $vc$  for some end point  $c$  of  $F_c$ . Thus every normal sequence in  $F_c$  has the form  $\{F_c, S\}$  for some  $S \in \mathcal{P}(F_c)$ , and the conclusion follows.  $\diamond$

**Proposition 4.2.** If a dendroid  $X$  is uniformly arcwise connected, then  $k(X) \leq 2$ .

**Proof.** Since each such a dendroid is a continuous image of the Cantor fan  $F_c$  by the definition, the conclusion is an immediate consequence of Prop. 4.1 and Fact 2.6.  $\diamond$

**Remark 4.3.** Note that the converse implication to that of Prop. 4.2 is not true. Namely there is a non-uniformly arcwise connected fan  $F_{P_1}$  (see [2], (52), p. 201) for which we have  $k(F_{P_1}) = 2$ .

As particular examples of uniformly arcwise connected dendroids one can consider smooth ones. A dendroid  $X$  is said to be *smooth* provided that there is a point  $v \in X$  such that for each point  $x \in X$  and for each sequence of points  $\{x_n \in X : n \in \mathbb{N}\}$  tending to  $x$  we have  $vx = \text{Lim } vx_n$ . It is known that every smooth dendroid is uniformly arcwise connected (see [3], Cor. 16, p. 318). Thus Prop. 4.2 leads to the following corollary.

**Corollary 4.4.** *If a dendroid  $X$  is smooth, then  $k(X) \leq 2$ .*

**Remark 4.5.** It is evident from the construction of the fans  $F[\alpha]$  of Th. 3.1, especially by condition (3.12), that they are not smooth for  $\alpha > 1$ . And it follows from Cor. 4.3 that it is not possible to construct any family of smooth dendroids (therefore of smooth fans)  $X[\alpha]$  (for every  $\alpha < \omega_1$ ) having property (1.1), i.e. such that  $k(X[\alpha]) = \alpha$ .

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