

# NON-REGULARITY OF SOME TOPOLOGIES ON $\mathbb{R}^n$ STRONGER THAN THE STANDARD ONE<sup>1</sup>

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**Abstract:** The finest topology on  $\mathbb{R}^n$  ( $n \geq 2$ ) which induces the Euclidean topology on each line is not regular and have big character and extent. The same holds for the finest topology which induces the Euclidean topology on each line parallel to a coordinate axis; this latter topology is symmetrizable.

## 0. Introduction

Investigating Minkowski space  $M$  (the “real” 4-dimensional space-time continuum) Zeeman suggested some alternative topologies for  $M$ , [12, 13]. The *fine topology* on  $M$  induces the 3-dimensional Euclidean topology on every space-axis and the 1-dimensional Euclidean topology on every time-axis, and it is the finest topology satisfying this property, [13]. The reader could consult also [1], [4], [5], [7, 8, 9, 10] for a more detailed view of this topic. In this paper we investigate some of the properties of the finest topology on  $\mathbb{R}^n$  which induces the Euclidean topology on each line in  $\mathbb{R}^n$  (resp. each line parallel to a coordinate axis).

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## 1. Preliminaries

Let  $\mathcal{A}_*^n = \{\ell: \ell \text{ is a line in } \mathbb{R}^n\}$  and  $\mathcal{A}_+^n = \{\ell \in \mathcal{A}_*^n: \ell \text{ is parallel to a coordinate axis}\}$ . Let  $\mathbb{R}_*^n$  (resp.  $\mathbb{R}_+^n$ ) be the set  $\mathbb{R}^n$  with the following topology: a set  $U \subseteq \mathbb{R}^n$  is open in  $\mathbb{R}_*^n$  (resp.  $\mathbb{R}_+^n$ ) if and only if  $U \cap \ell$  is open with respect to the standard topology on  $\ell$ , for each line  $\ell \in \mathcal{A}_*^n$  (resp.  $\ell \in \mathcal{A}_+^n$ ). The topology of  $\mathbb{R}_*^n$  is stronger than the standard topology of  $\mathbb{R}^n$ , and the topology of  $\mathbb{R}_+^n$  is stronger than the topology of  $\mathbb{R}_*^n$ . For instance, any circle without a point is closed in  $\mathbb{R}_*^n$  although it is not closed in the standard topology of  $\mathbb{R}^n$ ; any non-parallel to a coordinate axis line without a point is closed in  $\mathbb{R}_+^n$  although it is not closed in  $\mathbb{R}_*^n$ . The space  $\mathbb{R}_+^n$  is symmetrizable by the symmetric  $s(x, y)$  defined as follows:

$$s(x, y) = \begin{cases} \|x - y\| & \text{if } x \text{ and } y \text{ lie on a line that is} \\ & \text{parallel to a coordinate axis;} \\ 1 & \text{otherwise} \end{cases}$$

( $\|\cdot\|$  stands for the usual norm in  $\mathbb{R}^n$ ). We recall that a *symmetric*  $s$  on a topological space  $X$  is a function from  $X \times X$  into  $\mathbb{R}$  such that: a)  $s(x, y) = s(y, x) \geq 0$  for each  $x, y \in X$ ; b)  $s(x, y) = 0$  iff  $x = y$ ; c) a set  $U \subseteq X$  is open iff for each  $x \in U$  there exists  $r > 0$  such that the "ball"  $K(x, r) = \{y: s(x, y) < r\}$  is contained in  $U$  ([2], [3], see also [11]).

In Section 2 we prove that, for  $n \geq 2$ , both  $\mathbb{R}_*^n$  and  $\mathbb{R}_+^n$  are not regular. The first, for  $n = 3$ , answers a question of Prof. Otto Laback (of Technical University Graz, Austria). The second, for  $n = 2$ , answers a question of Prof. Stoyan Nedev (Institute of Mathematics, BAN, Sofia, Bulgaria). In Section 3 we investigate some cardinal functions of  $\mathbb{R}_*^n$  and  $\mathbb{R}_+^n$  (density, weight, character, extent and spread) and the results once more show that  $\mathbb{R}_*^n$  and  $\mathbb{R}_+^n$  are not regular. Throughout the paper  $\mathbb{Q}$  denotes the set of all rational numbers. The  $i$ -th coordinate of an  $x \in \mathbb{R}^n$  is denoted by  $x_i$ ,  $x = (x_1, \dots, x_n)$ .

## 2. Non-regularity of $\mathbb{R}_*^n$ and $\mathbb{R}_+^n$

**Lemma 2.1.** *Let  $n \geq 2$  and  $U$  is a subset of  $\mathbb{R}^n$  which is open with respect to the standard topology of  $\mathbb{R}^n$ . Then there exist a set  $F \subseteq U$  such that:*

- (a)  $\text{cl } F = \text{cl } U$ , where "cl" denotes the standard closure in  $\mathbb{R}^n$ .

(b) Each line in  $\mathbb{R}^n$  passes through at most two points of  $F$  (and hence,  $F$  is closed in both  $\mathbb{R}_*^n$  and  $\mathbb{R}_+^n$ ).

**Proof.** The lemma is trivial if  $U = \emptyset$ , so let  $U \neq \emptyset$ . Let  $B(x, r)$  denotes the ball  $\{y \in \mathbb{R}^n: \|x - y\| < r\}$ . Then the family  $\mathcal{B}(U) = \{B(x, \frac{1}{k}): x \in \mathbb{Q}^n, k \in \mathbb{N}, B(x, \frac{1}{k}) \subseteq U\}$  is countable and we can write it as  $\mathcal{B}(U) = \{B_i: i \in \mathbb{N}\}$ . By induction we will pick points  $x^i \in B_i$  and define sets  $F_i = \{x^j: j \leq i\}$  such that:

(\*) each line in  $\mathbb{R}^n$  passes through at most two points of  $F_i$ .

Finally we will set  $F = \cup\{F_i: i \in \mathbb{N}\}$ .

In order to do this let us pick an  $x^1 \in B_1$  and let  $F_1 = \{x^1\}$ . Let us suppose that, for some  $i \in \mathbb{N}$  and for each  $j \leq i$  we have picked points  $x^j \in B_j$  such that the condition (\*) holds. Let  $\mathcal{A}(F_i)$  be the family of these lines in  $\mathbb{R}^n$  that passes through exactly two points of  $F_i$ . Since  $\mathcal{A}(F_i)$  is finite, the set  $B_{i+1} \setminus (\cup \mathcal{A}(F_i))$  is not empty. We can pick a point  $x^{i+1}$  from it and define  $F_{i+1} = \{x^j: j \leq i + 1\}$ . The condition (\*) holds for  $F_{i+1}$ , because  $x^{i+1} \notin \cup \mathcal{A}(F_i)$ . The induction step is completed.

Now, let  $F = \cup\{F_i: i \in \mathbb{N}\}$ . It is clear that (b) holds. In order to prove (a) let  $y \in \text{cl } U$ . For any standard neighbourhood  $V$  of  $y$ ,  $U \cap V \neq \emptyset$ . Because  $\mathbb{Q}^n$  is dense in  $\mathbb{R}^n$ , we can pick an  $x \in \mathbb{Q}^n \cap U \cap V$ . There is some  $k \in \mathbb{N}$  such that  $B(x, \frac{1}{k}) \subseteq U \cap V$ . There is some  $i \in \mathbb{N}$  for which  $B(x, \frac{1}{k}) = B_i$  and hence  $x^i \in U \cap V$ . Hence  $F \cap V \neq \emptyset$  and  $y \in \text{cl } F$ ;  $\text{cl } U \subseteq \text{cl } F$ .  $\diamond$

**Proposition 2.2.** For  $n \geq 2$ , the spaces  $\mathbb{R}_*^n$  and  $\mathbb{R}_+^n$  are not regular.

**Proof.** Because  $\mathbb{R}_*^2$  (resp.  $\mathbb{R}_+^2$ ) is a closed subspace of  $\mathbb{R}_*^n$  (resp.  $\mathbb{R}_+^n$ ) it suffices to show that  $\mathbb{R}_*^2$  and  $\mathbb{R}_+^2$  are not regular.

By Lemma 2.1, there is a set  $F \subseteq \mathbb{R}^2 \setminus \{O\}$  (where  $O = (0, 0)$ ) such that  $\text{cl } F = \mathbb{R}^2$  and  $F$  is closed in both  $\mathbb{R}_*^2$  and  $\mathbb{R}_+^2$ . We will show that the  $O$  and  $F$  have no disjoint neighbourhoods in  $\mathbb{R}_+^2$  (and hence, also in  $\mathbb{R}_*^2$ ).

Let  $O \in U$  and  $F \subseteq V$  where  $U$  and  $V$  are open subsets of  $\mathbb{R}_+^2$ . We will show that  $U \cap V \neq \emptyset$ . There is an  $\varepsilon > 0$  such that the horizontal interval  $J = \{x: |x_1| < \varepsilon, x_2 = 0\}$  is included in  $U$  ( $x_i$  denotes the  $i$ -th coordinate of a given point  $x$ ). For each  $x \in J$  and  $n \in \mathbb{N}$  let  $U(x, n)$  is the vertical interval with base  $x$  and highness  $\frac{1}{n}$ , i.e.  $U(x, n) = \{(x_1, \delta): 0 \leq \delta \leq \frac{1}{n}\}$ . For each  $n \in \mathbb{N}$  let  $A_n = \{x \in J: U(x, n) \subseteq U\}$ . Since  $J \subseteq U$  and  $U$  is open in  $\mathbb{R}_+^2$  we have that

$\cup\{A_n : n \in \mathbb{N}\} = J$ . Since  $J$  is of second category there are an  $n \in \mathbb{N}$  and a nonempty open interval  $J' \subseteq J$  such that the standard closure of  $A_n$  contains  $J'$ . The set  $P = \{x : x_1 \in J', 0 < x_2 < \frac{1}{n}\}$  is nonempty and open (in  $\mathbb{R}^2$ ) and hence we can pick a point  $y \in P \cap F$ . Since  $y \in V$  there exists  $\mu > 0$  such that the horizontal interval  $Y = \{z \in \mathbb{R}_+^2 : z_2 = y_2, |z_1 - y_1| < \mu\}$  is contained in  $V$ . Let  $x$  be a point from  $A_n \cap J'$  such that  $|x_1 - y_1| < \mu$ . Then  $U(x, n) \cap Y \neq \emptyset$  and hence  $U \cap V \neq \emptyset$ .  $\diamond$

**Corollary 2.3.** *The fine topology of Minkowski space  $M$  is not regular.*

**Proof.** We have that  $\mathbb{R}_+^2$  is a closed subspace of  $M$ . In fact,  $\mathbb{R}_+^2 \simeq T \times \mathbb{R} \hookrightarrow T \times \mathbb{R}^3 = M$ , where  $T = \mathbb{R}$  is the time,  $\simeq$  denotes homeomorphism,  $\hookrightarrow$  denotes homeomorphic embedding, and  $\times$  denotes product of sets (not topological product).  $\diamond$

### 3. Some cardinal functions of $\mathbb{R}_*^n$ and $\mathbb{R}_+^n$

Let us recall the definitions of the cardinal functions *weight*, *character* and *density*, denoted (for a given topological space  $X$ ) by  $w(X)$ ,  $\chi(X)$  and  $d(X)$  respectively:

$$w(X) = \min\{|\mathcal{B}| : \mathcal{B} \text{ is a base for the topology of } X\};$$

$$\chi(X) = \sup\{\chi(x, X) : x \in X\}, \text{ where}$$

$$\chi(x, X) = \min\{|\mathcal{B}_x| : \mathcal{B}_x \text{ is a base for the topology of } X \text{ at } x\};$$

$$d(X) = \min\{|A| : A \text{ is dense in } X\}.$$

A space  $X$  is called *separable* if  $d(X) = \aleph_0$ . For a detailed survey on cardinal functions see [6]. Another approach for showing that  $\mathbb{R}_*^n$  and  $\mathbb{R}_+^n$  are not regular is to use that for a regular space  $X$ ,  $w(X) \leq 2^{d(X)}$ . In fact,  $\mathbb{R}_*^n$  and  $\mathbb{R}_+^n$  are separable but they have weight and character strongly greater than  $\mathfrak{c} = 2^{\aleph_0}$ .

**Proposition 3.1.** *For  $n \geq 2$ ,  $\chi(\mathbb{R}_*^n) > \mathfrak{c}$  and  $\chi(\mathbb{R}_+^n) > \mathfrak{c}$ .*

**Proof.** We consider the case  $n = 2$ . Suppose that  $x = (x_1, x_2) \in \mathbb{R}^2$  and  $\mathcal{U} = \{U_\alpha : \alpha < \mathfrak{c}\}$  is a family of neighbourhoods of  $x$  (either in  $\mathbb{R}_*^2$  or in  $\mathbb{R}_+^2$ ); we shall show that  $\mathcal{U}$  cannot be a base at  $x$ . By induction, for each  $\alpha < \mathfrak{c}$  we shall pick a point  $x^\alpha \in U_\alpha \setminus \{x\}$  such that the following condition holds:

(C $_\alpha$ ) each line in  $\mathbb{R}^2$  contains at most two points of the set  $C_\alpha = \{x^\beta : \beta \leq \alpha\}$ .

Then the set  $C = \{x^\alpha : \alpha < \mathfrak{c}\}$  will be closed in  $\mathbb{R}_*^2$  and  $\mathbb{R}_+^2$ . The set  $V = \mathbb{R}^n \setminus C$  will be a neighbourhood of  $x$  (in  $\mathbb{R}_*^n$  and  $\mathbb{R}_+^n$ ) that does not contain any element of  $\mathcal{U}$  (because  $V$  misses each  $x^\alpha$ ,  $\alpha < \mathfrak{c}$ ).

Let  $x^0 \in U_0 \setminus \{x\}$  and  $C_0 = \{x^0\}$ . Let  $1 \leq \alpha < \mathfrak{c}$  and suppose that for each  $\gamma < \alpha$  the points  $x^\gamma$  have already been picked so that the conditions  $(C_\gamma)$  hold. There is an  $\varepsilon > 0$  such that the vertical interval  $J = \{y : y_1 = x_1, |y_2 - x_2| < \varepsilon\}$  is contained in  $U_\alpha$ . Since  $\alpha < \mathfrak{c}$  there is an  $h$ ,  $x_2 - \varepsilon < h < x_2 + \varepsilon$ , such that the horizontal line  $\tilde{h} = \{y : y_2 = h\}$  misses  $x^\gamma$ , for each  $\gamma < \alpha$ . There is a  $\delta > 0$  such that the horizontal interval  $H = \{y : |y_1 - x_1| < \delta, y_2 = h\}$  is included in  $U_\alpha \cap \tilde{h}$ . Let  $\mathcal{A} = \{\ell : \ell \text{ is a line in } \mathbb{R}^2 \text{ passing through two points of } \{x^\gamma : \gamma < \alpha\}\}$ . Since  $|\mathcal{A}| \leq \alpha^2 < \mathfrak{c}$  and  $\tilde{h} \notin \mathcal{A}$ , we have that  $|H \setminus \cup \mathcal{A}| = \mathfrak{c}$ . We can pick an  $x^\alpha \in H \setminus \cup \mathcal{A}$ ,  $x^\alpha \neq x$ , and define  $C_\alpha = \{x^\gamma : \gamma \leq \alpha\}$ . Because  $x^\alpha \notin \cup \mathcal{A}$ ,  $(C_\alpha)$  holds.  $\diamond$

**Corollary 3.2.** For  $n \geq 2$ ,  $w(\mathbb{R}_*^n) = \chi(\mathbb{R}_*^n) > \mathfrak{c}$  and  $w(\mathbb{R}_+^n) = \chi(\mathbb{R}_+^n) > \mathfrak{c}$ .

The author conjectures that  $w(\mathbb{R}_*^n) = \chi(\mathbb{R}_*^n) = w(\mathbb{R}_+^n) = \chi(\mathbb{R}_+^n) = 2^{\mathfrak{c}}$  (the assumption  $\mathfrak{c}^+ = 2^{\mathfrak{c}}$  implies these equations).

**Proposition 3.3.** For  $n \geq 2$ , both  $\mathbb{R}_*^n$  and  $\mathbb{R}_+^n$  are separable.

**Proof.** We shall show that the set  $\mathbb{Q}^2$  is dense in  $\mathbb{R}_+^2$  (and hence in  $\mathbb{R}_*^2$ ). The cases  $n \geq 3$  are similar to this one. Let  $U$  be a nonempty open subset of  $\mathbb{R}_+^2$ . Let us pick an  $h \in \mathbb{R}$  such that the horizontal line  $\tilde{h} = \{x : x_2 = h\}$  intersects  $U$ . Since  $U \cap \tilde{h}$  is open in  $\tilde{h}$  there is a  $p \in \mathbb{Q}$  for which  $(p, h) \in U \cap \tilde{h}$ . Let  $\nu = \{x : x_1 = p\}$ . There is a  $q \in \mathbb{Q}$  for which the point  $(p, q) \in U \cap \nu$  and hence  $\mathbb{Q}^2 \cap U \neq \emptyset$ .  $\diamond$

Finally, let us recall that the *extent*  $e(X)$  of a space  $X$  is defined as

$$e(X) = \sup\{|C| : C \text{ is a closed discrete subspace of } X\},$$

and the *spread*  $s(X)$  is defined as

$$s(X) = \sup\{|C| : C \text{ is a discrete subspace of } X\}.$$

In the proof of Prop. 3.1, the set  $C$  (looked at as a subspace of either  $\mathbb{R}_*^n$  or  $\mathbb{R}_+^n$ ) is closed discrete, and  $|C| = \mathfrak{c}$ . So, we have proved:

**Proposition 3.4.** For  $n \geq 2$ ,  $e(\mathbb{R}_*^n) = e(\mathbb{R}_+^n) = s(\mathbb{R}_*^n) = s(\mathbb{R}_+^n) = \mathfrak{c}$ .

## References

- [1] ANCEL, F. D.: Collections of paths and rays in the plane which fix its topology, *Topology Appl.* **15** (1983), 99–110.
- [2] ARHANGEL'SKII, A. V.: Behaviour of metrizable in factor mappings, *Dokl. Akad. Nauk SSSR* **164/1** (1965), 9–12, in Russian (Translated as: *Soviet Math. Dokl.*)
- [3] ARHANGEL'SKII, A. V.: Mappings and spaces, *Uspekhi Mat. Nauk*, **21/4** (1966), 133–184, in Russian (Translated as: *Russian Math. Surveys*)
- [4] CURTIS, D. W. and RUBIN, L. R.: Sectionally continuous injections of Euclidean spaces, *Topology Appl.* **31** (1989), 159–168.
- [5] GÖBEL, R.: Zeeman topologies on space-times of General Relativity Theory, *Commun. Math. Phys.* **46** (1976), 289–307.
- [6] JUHÁSZ, I., ed.: Cardinal functions in topology, *Mathematical Centre Tracts* **34**, Mathematisch Centrum, Amsterdam, 1975.
- [7] LABACK, O.: On the continuity of bijections of  $\mathbb{R}^n$ , *Supplemento ai Rendiconti del Circolo Matematico di Palermo*, Ser. 2, **18** (1988), 337–341.
- [8] LABACK, O.: Topologies in general space-time, *Berichte der Mathematisch-Statistischen Sektion im Rechenzentrum Graz*, **138** (1980).
- [9] LABACK, O.: Fixing topologies by subspaces, *Berichte der Mathematisch-Statistischen Sektion im Rechenzentrum Graz*, **238** (1985).
- [10] LABACK, O.: On the timelike relation in space-time, *Acta Phys. Austriaca* **52** (1980), 293–299.
- [11] NEDEV, S.: O-metrizable spaces, *Trudy Moskov. Mat. Obshch.* **24** (1971), 201–236, in Russian (Translated as: *Trans. Moscow Math. Soc.*)
- [12] ZEEMAN, E. C.: Causality implies the Lorentz group, *Journal of Mathematical Physics* **5/4** (1964), 490–493.
- [13] ZEEMAN, E. C.: The topology of Minkowski space, *Topology* **6** (1967), 161–170.