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w-JORDAN NEAR-RINGS II

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Abstract: In previous papers [3, 4] we have studied near-rings with invariant series of ideals. In the present paper we study near-rings with invariant series whose factors are without proper subnear-rings and we characterize those whose zero-symmetric part is an ideal. Moreover we continue the study of near-rings with invariant series whose factors are of prime order and we provide a complete characterisation for those of length 2.

Introduction

In previous papers we have studied near-rings with invariant series of ideals. In [3] we observed that numerous results, particularly concerning closure problems, do not depend on the near-ring structure and therefore are valid in a more general ambit, that is for universal algebras or, at least, for Ω-groups. In [4] we considered several classes of near-rings: simple (S_0), simple and strongly monogenic (S_1), N_0-simple (S_2), without proper subnear-rings (S_3), of prime order (S_4), and we studied near-rings with an invariant series whose factors belong to S_w ($w \in \{0,1,2,3,4\}$), called *w-Jordan near-rings.* While in [4] we turned our attention to the zero-symmetric case, here we present the results of the study of finite 3-Jordan near-rings, completely characterizing those whose zero-symmetric part is an ideal. Moreover, we continue the study of the 4-Jordan near-rings, begun in [4], and we provide a complete characterization for those of length 2.

Hereafter N will indicate a left near-ring and we refer to [10] without mentioning this explicitely. In particular, we shall use the term *"mixed"* to describe a near-ring N, with $N_0 \neq \{0\}$ and $N_c \neq \{0\}$. In general N_0 is a right ideal of N and N_c is an invariant subnear-ring. Furthermore, a zero-symmetric near-ring without proper N-subgroups H such that $HN = \{0\}$ is called *A-simple.* A near-ring is *N-simple* if it is without proper N-subgroups, that is if the additive group N^+ does not contain proper subgroups which are proper right ideals of the multiplicative semigroup $N^{\cdot}$. N is *N_0-simple* if it is without proper N_0-subgroups. In [7] a near-ring N is called *p-singular* if the order of N is divisible by a prime number p but the order of every proper subnear-ring of N is not divisible by p. $A_s(N) = \{x \in N/xA = \{0\}\}$, $(A_d(N) = \{x \in N/Ax = \{0\})$ denote the left (right) annihilator of N and $A(N) = A_s(N) \cap A_d(N)$, the annihilator of N; $r(n) = \{x \in N/nx = 0\}$ denotes the annihilator of $n \in N$; $r(n)$ is always a right ideal. If $N = N_1 \supset N_2 \supset \ldots \supset N_n = \{0\}$ is an invariant series of N, we will indicate $N_i/N_{i+1}, N_i/N_{i+2}, \ldots, N_i/N_{i+k}$ respectively with $N_i', N_i'', \ldots, N_k^k$ and with $f_i', f_i'', \ldots f_k^k$ the corresponding canonical epimorphisms.

1. 3J-near-rings

Definition 1. Let N be a finite near-ring. An *a-series* is an invariant series $N = N_1 \supset N_2 \supset \ldots \supset N_n = \{0\}$ whose factors belong to S_3 and such that, for every zero-symmetric N_i' and constant N_j' with $i < j$, $|N_j'|$ does not divide $|N_i'|$. The term *a-near-ring* describes a finite near-ring with an a-series.

Lemma 1. *A mixed a-near-ring N with an a-series*

$$(\alpha) \qquad N = N_1 \supset N_2 \supset \ldots \supset N_n = \{0\}$$

such that N_1' is zero-symmetric and N_i' is constant for every $i \in \{2, \ldots, n-1\}$ is isomorphic to $N_0 \oplus N_c$. Moreover there exists in N another series $N = M_1 \supset M_2 \supset \ldots \supset M_n = \{0\}$ such that $M_{n-1} \simeq N_1'$, hence zero-symmetric, and $M_{i-1}' \simeq N_i'$, therefore constant, for every $i \in \{2, 3, \ldots, n-1\}$.

Proof. Since N_1' is zero-symmetric, $N_2 \supseteq N_c$. Hence $N_2 = (N_2)_0 + N_c$. Because the following factors are constant, for every $i \in \{3, 4, \ldots, n\}$, $N_i \supseteq (N_2)_0$. Because $N_n = \{0\}$, we even get $(N_2)_0 = \{0\}$. Thus N_c is an ideal and N_0 is without proper subnear-rings, in fact it is isomorphic to N_i'. In [7] it was shown that a finite near-ring is without proper subnear-rings iff it is a simple p-singular near-ring and, in this case, its order is divisible by at most a prime number. Hence $|N_0| = p^b$ (p prime) and by Def. 1, $|N_i'| \neq p$ for every $i \in \{2, \ldots, n-1\}$.

Now we prove that $|N^+/N_0^+|$ is prime with p and consequently N_0 is an ideal of N. In fact, if $|N^+/N_0^+| = kp$, then $|N_c^+| = kp$ and, because $|N_2/N_3| \neq p$, we have $|N_3| = k_1 p$. In this way we get $|N_n| = k_{n-2}\, p = 0$ and this is absurd. Thus $|N^+/N_0^+|$ is relatively prime to p, $|N_0| = p^b$ and N_0^+, which is normal, is the unique Sylow p-subgroup of N^+. Since the homomorphic image of a Sylow p-subgroup is contained in a Sylow p-subgroup, N_0^+ is fully invariant in N^+. Therefore, since the left translations are endomorphisms of N^+, $nN_0 \subseteq N_0$ for every $n \in N$ and N_0 is an ideal of N. From this we can conclude that $N \simeq N_0 \oplus N_c$.

Finally, if we denote $M_i = N_{i+1} + N_0$, we have $M_{n-1} = N_0$ and the series $N = M_1 \supset M_2 \supset \ldots \supset M_{n-1} \supset M_n = \{0\}$ is the series required. ◊

The following theorem gives a complete characterization of all a-near-rings.

Theorem 1. *A finite near-ring with an invariant series whose factors belong to S_3 is an a-near-ring iff its zero-symmetric part is an ideal.*

Proof. Suppose N a finite near-ring with an invariant series whose factors belong to S_3, and $N_0 \triangleleft N$. This series is a refinement of $N \supset \supset N_0 \supset \{0\}$. Therefore in such a series there are no zero-symmetric factors that precede the constant ones. Thus this series is an a-series.

Conversely, let

$$(\alpha) \qquad N = N_1 \supset N_2 \supset \ldots \supset N_n = \{0\}$$

be an a-series of a finite near-ring. First we show that in N there is a series $N = M_1 \supset M_2 \supset \ldots \supset M_n = \{0\}$ such that the constant factors precede the zero-symmetric ones. Let j be the highest index of the series (α) such that N_j' is zero-symmetric, $N_{j+1}', \ldots, N_{j+h}'$ are constant and N_{j+k}' is zero-symmetric for every $k > h$. Consider now the subseries of (α) $N_j \supset N_{j+1} \supset \ldots \supset N_{j+h}$ and its image

$$(\beta) \qquad G = G_j \supset G_{j+1} \supset \ldots \supset G_{j+h} \supset G_{j+h+1} = \{0\}$$

under the homomorphism f_j^{h+1}, where $G_t = f_j^{h+1}(N_t)$ for $t \in \{j, j+1, \ldots, j+H\}$. Since $G'_j = G_j/G_{j+1}$ is isomorphic to N'_j, G'_j is zero-symmetric and G'_t is constant for every $t \in \{j+1, \ldots, j+h\}$. Therefore the series (β) is an a-series of G satisfying the hypotheses of Lemma 1. So applying Lemma 1, there is in G a series

$$(\gamma) \qquad G = F_j \supset F_{j+1} \supset \ldots \supset F_{j+h} = \{0\}$$

such that F_{j+h} is isomorphic to G'_j and is the zero-symmetric part of G, that is G_0, whereas F_t is isomorphic to G'_{t+1} for every $t \in \{j, j+1, \ldots, j+h-1\}$ and therefore constant. Using Lemma 1 again, we are able to say that F_t is fully invariant in G for every $t \in \{j, j+1, \ldots, j+h\}$.

Let now $M_t = (f_j^{h+1})^{\circ}(F_t)$, that is $F_t = M_t/N_{j+h+1}$. We prove that M_t is an ideal of N for every $t \in \{j, j+1, \ldots, j+h+1\}$. In fact M_{j+h} is a left ideal of N because every endomorphism of N_j which fixes N_{j+h+1}, fixes also M_{j+h}. Let $\varepsilon\colon N_j \to N_j$ be an endomorphism such that $\varepsilon(N_{j+h+1}) \subseteq N_{j+h+1}$. This endomorphism induces an endomorphism ε' in N_j/N_{j+h+1}, put $\varepsilon'(n_j + N_{j+h+1}) = \varepsilon(n_j) + N_{j+h+1}$. Let m now be an element of M_{j+h}. Obviously $m + N_{j+h+1} \in F_{j+h}$, therefore $\varepsilon'(m + N_{j+h+1}) = \varepsilon(m) + N_{j+h+1} \in F_{j+h}$ because, by Lemma 1 F_{j+h} is fully invariant in G. Hence there is an element $m' \in M_{j+h}$ and an element $n \in N_{j+h+1}$ such that $\varepsilon(m) = m' + n$, thus $\varepsilon(m) \in M_{j+h}$. Since every left translation γ_n, restricted to N_j is an endomorphism of N_j which fixes N_{j+h+1}, we have $nM_{j+h} \subseteq M_{j+h}$ for every $n \in N$. Thus M_{j+h} is a left ideal of N. Now we prove that $M_{j+h} = N_0 \cap N_j$ and, from this, that M_{j+h} is a right ideal. If $m \in N_0 \cap N_j$, then $f_j^{h+1}(m) \in G_0 = F_{j+1}$, therefore $m \in M_{j+h}$. On the other hand, M_{j+h} is obviously contained in N_j and it is zero-symmetric because one of its ideals and the respective factors are zero-symmetric. Thus we can conclude that M_{j+h} is an ideal of N.

Finally we show that $M_t = M_{j+h} + N_{t+1}$ for every $t \in \{j, j+1, \ldots, j+h\}$ and thus, M_t, as a sum of two ideals, is an ideal.

The series (γ) is obtained using Lemma 1, therefore $F_t = F_{t+h} + G_{t+1}$, where $F_{j+h} = G_0$. Hence $M_t/N_{j+h+1} = M_{j+h}/N_{j+h+1} + N_{t+1}/N_{j+h+1}$ and thus $M_t = M_{j+h} + N_{t+1}$. Finally we observe that M_{j+h} is exactly the zero-symmetric part of N_j and consequently $(N_j)_0$ is an ideal. $\Diamond$

It should be noted that not all the finite $3J$-near-rings are a-near-rings. For example $N = M_a(\mathbb{Z}p)$,[(1)] p prime is a $3J$-near-ring ($N \supset N_c \supset \{0\}$ is a series whose factors are in S_3) but N_0 is not an ideal.

2. 4J-near-rings of length 2

In this paragraph we study near-rings with an invariant series $N \supset \supset I \supset \{0\}$ whose factors are near-rings of prime order. Consequently they are near-rings of the order pq, where p, q are prime numbers. It is well known that the additive group of such a near-ring is a direct or semidirect sum of cyclic groups of prime order or is itself cyclic of the order p^2. We can also establish the following:

Proposition 1. *A $4J$-near-ring of length 2 has only one proper ideal or it is isomorphic to the direct sum of two of its ideals.*

Proof. Let $N \supset I \supset \{0\}$ be the invariant series of a $4J$-near-ring N with N/I and I of prime order: that means I is a maximal ideal. If J is another ideal of N, $I+J$ is also an ideal, thus $I = J$ or $I \oplus J = N$. $\Diamond$

From Th. 1 of [4] we know that a near-ring of prime order is constant or zero-symmetric. In the latter case, it is either an A-simple and strongly monogenic near-ring or a zero-ring. Thus, we will denote by

η'_c	the class of constant near-rings;
η'_0	the class of zero-symmetric near-rings;
$\mathcal{A}$	the class of A-simple and strongly monogenic near-rings;
$\mathcal{O}$	the class of zero-rings.

Moreover we will denote by $[\mathcal{S}, \mathcal{T}]$ the class of near-rings with an invariant series $N \supset I \supset \{0\}$ such that $N/I \in \mathcal{S}$ and $I \in \mathcal{T}$, where $\mathcal{S}, \mathcal{T} \in \{\eta'_c, \eta'_0, \mathcal{A}, \mathcal{O}\}$. In this way, $4J$-near-rings of length 2 are the union of $[\eta'_c, \eta'_0]$, $[\eta'_0, \eta'_c]$, $[\eta'_0, \eta'_0]$, $[\eta'_c, \eta'_c]$, where $\eta'_0 = \mathcal{A} \cup \mathcal{O}$. Moreover we observe that the structure of the near-rings of $[\eta'_c, \eta'_c]$ is that of groups of order pq and therefore well-known.

Let $N \equiv A +_{\varphi} B$ be a semidirect sum of additive groups A and B. By Clay method, every near-ring on N can be constructed but, generally, from a multiplicative view-point, isomorphic images of semidirect

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(1) $M_a(\mathbb{Z}p) = \mathrm{Hom}(\mathbb{Z}p, \mathbb{Z}p) + M_c(\mathbb{Z}p)$, where $M_c(\mathbb{Z}p) = \{f\colon \mathbb{Z}p \to \mathbb{Z}p / f$ is constant$\}$.

summands are not even substructures. In [1] we study those functions introduced by Clay that provide the semidirect summands with a well defined multiplicative structure. We call *Φ-sum* of near-rings A and B, a near-ring N obtained by semidirect sum of the additive groups of A and B with a suitable Clay function preserving the multiplicative structure on semidirect summands.

CLASS $[\eta'_c, \eta'_0]$

The following theorem characterizes the near-rings belonging to $[\eta'_c, \eta'_0]$.

Theorem 2. *A near-ring N belongs to $[\eta''_c, \eta'_0]$ iff $N = A +_\Phi B$ with $f(\langle 0,0\rangle) = O_A$ and $\bar{f}(N) = \mathrm{id}$, where $A \in \eta'_0$ and $B \in \eta'_c$.*

Proof. If $N \in [\eta'_c, \eta'_0]$, then N_0 is an ideal. Thus N is isomorphic to $N_0 +_\Phi N_c$, where $f(\langle 0,0\rangle) = O_{N_0}$ and $\bar{f}(N) = \mathrm{id}$ (see [1], Th. 1, Cor. 1). Moreover, both N_0 and $N/N_0 \simeq N_c$ are of prime order. Conversely, suppose $N = A +_\Phi B$ with $f(\langle 0,0\rangle) = O_A$ and $\bar{f}(N) = \mathrm{id}$, where $A \in \eta'_0$ and $B \in \eta'_c$. Then A° is an ideal of N; $A^\circ = N_0$; ${}^\circ B = N_c$ and $N/A^\circ \simeq {}^\circ B$ (see [1] Prop. 2). Whence $N \in [\eta'_c, \eta'_0]$. $\Diamond$

By construction described in Th. 2 we obtain each element of $[\eta'_c, \eta'_0]$. We observe also that, because $[\eta'_c, \eta'_0] = [\eta'_c \mathcal{A}] \cup [\eta'_c, \mathcal{O}]$, an element of $[\eta'_c, \eta'_0]$ belongs to $\eta'_c, \mathcal{A}]$ or to $[\eta'_c, \mathcal{O}]$ according to the choice of A in $\mathcal{A}$ or in $\mathcal{O}$ respectively.

CLASS $[\eta'_0, \eta'_c]$

The following theorem characterizes the near-rings belonging to $[\eta'_0, \eta'_c]$.

Theorem 3. *A near-ring N belongs to $[\eta'_0, \eta'_c]$ iff either $N = A \oplus B$, where $A \in \eta'_0$ and $B \in \eta'_c$, or N is an abstract affine near-ring of order p^2.*

Proof. Now the order of N is pq, where p, q are prime numbers.

(i) Suppose $p \neq q$. In this case $N \supset N_c \supset \{0\}$ is an a-series, thus $N = N_0 \oplus N_c$, by Lemma 1.

(ii) Let $p = q$. In this case the order of N is p^2 and $N^+ = N_0^+ \oplus N_c^+$. If N_0 is an ideal of N, we still have $N = N_0 \oplus N_c$. If N_0 is not an ideal of N, we have $N_c N_0 = N_c$, because the order of N_c is a prime number. Moreover $r(N_c) = \{0\}$, due to $r(N_c) \subset N_0$. Thus N_c is a base of equality and N is an abstract affine near-ring (see [10], 9.85).

Conversely, if $N = A \oplus B$, where $A \in \eta'_0$ and $B \in \eta'_c$, it is clear that the theorem holds. Let N now be an abstract affine near-ring of order p^2. Obviously both N_0 and N_c are of the order p and, since N_c is an ideal, $N \in [\eta'_0, \eta'_c]$. ◊

Corollary 1. *A near-ring N belongs to $[\mathcal{O}, \eta'_c]$ iff $N \simeq A \oplus B$, where $A \in \mathcal{O}$ and $B \in \eta'_c$.*

Proof. Let $N \in [\mathcal{O}, \eta'_c]$. If N_0 is not an ideal of N, then $N_c N_0 = N_c$, thus $N_c = N_c N_0 = (N_c N_0) N_0 = N_c (N_0)^2 = \{0\}$, because N_0 is now a zero-ring. From this it follows that N_0 is an ideal of N, thus $N = N_0 \oplus N_c$ where $N_0 \in \mathcal{O}$ and $N_c \in \eta'_c$. The converse is trivial. ◊

Corollary 2. *A near-ring N belongs to $[\mathcal{A}, \eta'_c]$ iff either $N \simeq A \oplus B$, where $A \in \mathcal{A}$ and $B \in \eta'_c$, or N is an abstract affine near-ring of order p^2.*

In [2] we have shown a method for constructing abstract affine near-rings with a given zero-symmetric part and a given constant part (they are suitable Λ-sums). By using Th. 3 of [2], an abstract affine near-ring of order p^2 can be characterized as a Λ-sum of a field isomorphic to $\mathbb{Z}_p$ and the constant near-ring on $\mathbb{Z}_p$. We can also note that near-rings belonging to $[\eta'_0, \eta'_c] \cap [\eta'_c, \eta'_0]$ are direct sums of their zero-symmetric and constant parts. Those belonging to $[\eta'_0, \eta'_c] \setminus [\eta'_c, \eta'_0]$ are abstract affine near-rings of order p^2. Those belonging to $[\eta'_c, \eta'_0] \setminus [\eta'_0, \eta'_c]$ are Φ-sums (not direct sums) of two near-rings of prime order satisfying the conditions of Th. 2.

CLASS $[\eta'_0, \eta'_0]$

We now study the subclasses of $[\eta'_0, \eta'_0]$. Near-rings belonging to $[\mathcal{O}, \mathcal{O}]$ are characterized by the following

Theorem 4. *A near-ring N belongs to $[\mathcal{O}, \mathcal{O}]$ iff $|N| = pq$ and there is an ideal I of N such that $N^2 \subseteq I \subseteq A(N)$.*

Proof. If $N \in [\mathcal{O}, \mathcal{O}]$, obviously $|N| = pq$ and N has a proper ideal I, where I is a zero-ring of prime order including N^2. Moreover, suppose now that $NI \neq \{0\}$. We have $nI = I$ for some $n \in N$ and, because of $N^2 \subseteq I$, $I = nI = n^2 I = \{0\}$, a contradiction, due to I being a proper ideal. Thus $NI = \{0\}$ and $I \subseteq A_d(N)$. Now let $N^2 \neq \{0\}$, that means $nN = I$ for some $n \in N$. Thus $N^2 = I$ and $IN = N^2 N = NN^2 = NI = \{0\}$. From this it follows $I \subseteq A(N)$.

Conversely, let N be a near-ring of order pq with a proper ideal I such that $N^2 \subseteq I \subseteq A(N)$. Obviously the orders of I and N/I are prime numbers. Moreover, by $N^2 \subseteq I \subseteq A(N)$, both I and N/I are zero-rings, thus $N \in [\mathcal{O}, \mathcal{O}]$. ◊

Corollary 3. *A near-ring N belonging to $[\mathcal{O}, \mathcal{O}]$ is a zero-near-ring iff $I \subset A(N)$, where I is as in* Th. 4.

Proof. Let N be in $[\mathcal{O}, \mathcal{O}]$ and, by Th. 4, suppose the ideal I to be strictly contained in $A(N)$. Then $A(N)/I = N/I$ implies $A(N) = N$. The converse is trivial. ◊

Between near-rings of $[\mathcal{O}, \mathcal{O}]$ we can characterize the non-zero near-rings by the following

Theorem 5. *A near-ring N with $N^2 \neq \{0\}$, belongs to $[\mathcal{O}, \mathcal{O}]$ iff $|N| = = p^2$ and $N^3 = \{0\}$.*

Proof. Let $N \in [\mathcal{O}, \mathcal{O}]$ and $N^2 \neq \{0\}$. From Th. 4 and Cor. 3 it turns out that $I = A(N)$. Thus $N^3 = NN^2 \subseteq NI = \{0\}$. Moreover, due to an element $n \in N$ so that $nN = I$. we have $\operatorname{im}\gamma_n = nN = I = \ker\gamma_n$. Thus $|N| = |I|^2 = p^2$. The converse is trivial. ◊

Corollary 4. *A near-ring N, where $N^2 \neq \{0\}$ and N^+ is a cyclic group, belongs to $[\mathcal{O}, \mathcal{O}]$ iff $|N| = p^2$ and $A(N) \neq \{0\}$.*

Proof. If $N \in [\mathcal{O}, \mathcal{O}]$, the corollary holds trivially. Conversely, let N be a near-ring with $N^2 \neq \{0\}$, $|N| = p^2$, $A(N) \neq \{0\}$ and suppose that N^+ is a cyclic group. It should be noted that N has a proper ideal $A(N)$, which is a zero-ring of order p. Moreover, due to $|N| = p^2$, $N/A(N)$ is also of order p. It remains to show that $N/A(N)$ is a zero-ring. Because of $N^2 \neq \{0\}$, we get $\gamma_n \neq O_n$ for some $n \in N$. Since $\gamma_n(N)$ is an additive subgroup of N^+, if $\gamma_n \neq O_N$ then $\gamma_n(N) = N$ or $\gamma_n(N) = A(N)$. Now, $\gamma_n(N) = N$ implies $\ker\gamma_n = \{0\}$ and the last condition is absurd, because of $A(N) \subseteq \ker\gamma_n$. Thus $nN = A(N)$ or $nN = \{0\}$. In each case $N^2 \subseteq A(N)$. ◊

Theorem 6. *A near-ring N belongs to $[\mathcal{O}, \mathcal{O}]$ iff it arises by defining a Clay function $F\colon N \to \operatorname{End}(N)$ on an additive group N of order pq, where $F(N) \subseteq \operatorname{End}_I(N)^{(2)}$ and $F(I) = \{O_N\}$, I being a normal subgroup of N.*

Proof. Let $N \in [\mathcal{O}, \mathcal{O}]$; obviously $|N| = pq$. Let I be an ideal of N. From Th. 3 we have $I \subseteq A(N)$, that is $IN = \{0\}$. Thus the Clay function coupled with the product of N, satisfies the required conditions.

(2) $\operatorname{End}_I(N) = \{f \in \operatorname{End}(N) / f(N) \subseteq I \subseteq \ker f\}$.

Conversely, let N be a group of order pq, let I be one of its normal subgroups and $F\colon N \to \mathrm{End}(N)$ a Clay function such that $F(N) \subseteq \subseteq \mathrm{End}_I(N)$ and $F(I) = \{O_N\}$. Obviously $I \subseteq \ker f$, $\forall f \in F(N)$, thus $NI = \{0\}$, that means I is a left ideal of N and a zero-ring. Analogously $f(N) \subseteq I$, $\forall f \in F(N)$, thus $N^2 \subseteq I$, that means I is a right ideal of N and N/I is a zero-ring. Whence $N \in [\mathcal{O}, \mathcal{O}]$. $\Diamond$

Near-rings belonging to $[\mathcal{A}, \mathcal{A}]$ or to $[\mathcal{O}, \mathcal{A}]$ are characterized by the following theorems.

Theorem 7. *A near-ring N belongs to $[\mathcal{A}, \mathcal{A}]$ iff $N \simeq A \oplus B$, where $A, B \in \mathcal{A}$.*

Proof. Let $N \in [\mathcal{A}, \mathcal{A}]$ and let $N \supset I \supset \{0\}$ be the invariant series such that N/I and I belong to $\mathcal{A}$. From Th. 6 of [4], we know that the radical $J_2(N)$ is nilpotent and $N/J_2(N)$ is a direct sum of A-simple and strongly monogenic near-rings. So $J_2(N) \notin \{N, I\}$. If it is also $J_2(N) \neq \{0\}$, then $J_2(N) + I$ is an ideal of N and, recalling that I is a maximal ideal, $J_2(N) \oplus I = N$. Thus $N/I \simeq J_2(N)$ is nilpotent, and this is absurd. So $J_2(N) = \{0\}$ and the theorem holds. $\Diamond$

Theorem 8. *A near-ring N belongs to $[\mathcal{O}, \mathcal{A}]$ iff $N \simeq A \oplus B$, where $A \in \mathcal{O}$ and $B \in \mathcal{A}$.*

Proof. It is trivial, by Th. 4 of [4]. $\Diamond$

It should be noted that $[\mathcal{O}, \mathcal{A}]$ is included in $[\mathcal{A}, \mathcal{O}]$. The following example shows that the inclusion is strict.

Example 1. As additive group we consider the symmetric group of degree 3 and we define the following product

$*$	0	a	b	c	x	y
0	0	0	0	0	0	0
a	0	a	a	a	0	0
b	0	a	c	b	y	x
c	0	a	b	c	x	y
x	0	0	0	0	0	0
y	0	0	0	0	0	0

In this way N is a near-ring, $I = \{0, x, y\}$ is the only ideal of N, $I \in \mathcal{O}$ and $N/I \in \mathcal{A}$. Thus $N \in [\mathcal{A}, \mathcal{O}]$, but $N \notin [\mathcal{O}, \mathcal{A}]$. We can note that the near-ring under definition is isomorphic to a Φ-sum $I +_\Phi A$, where $A = \{0, a\}$, and this Φ-sum is not a direct sum. In general, proper Φ-sums (not direct sums) of $[\mathcal{A}, \mathcal{O}]$ are characterized by the following theorems.

Theorem 9. *Let N be a proper Φ-sum, then N belongs to $[\mathcal{A}, \mathcal{O}]$ iff $N \simeq \simeq A +_\Phi B$, where $B \in \mathcal{A}$, $A \in \mathcal{O}$ and it is the only ideal of N, $f(\langle 0,0\rangle) = = O_A$, $\overline{f}(\langle 0,0\rangle) = O_B$, $\overline{f}_{a,b} = \overline{f}_{0,b}$ $\forall a \in A$, $\forall b \in B$.*

Proof. Let N be a proper Φ-sum of $[\mathcal{A}, \mathcal{O}]$. Then N has a left invariant subgroup B and only one proper ideal A belonging to $\mathcal{O}$. Thus we can represent N as $A +_\Phi B$ and the theorem holds by Prop. 4 of [2]. Conversely, let $N = A +_\Phi B$, where $A \in \mathcal{O}$ and $B \in \mathcal{A}$, and let $f(\langle 0,0\rangle) = O_A$, $\overline{f}(\langle 0,0\rangle) = O_B$, $\overline{f}_{a,b} = \overline{f}_{0,b}$ ($\forall a \in A$, $\forall b \in B$). From Prop. 4 of [2] we see that N is zero-symmetric, A° is an ideal of N isomorphic to A, that means $A^\circ \in \mathcal{O}$, and also $N/A^\circ \simeq {}^\circ B \simeq B \in \mathcal{A}$. Thus $N \in [\mathcal{A}, \mathcal{O}]$. ◊

Theorem 10. *A near-ring N belonging to $[\mathcal{A}, \mathcal{O}]$ such that $\overset{+}{N} = = \overset{+}{A} \oplus \overset{+}{B}$ and where $|A|$ and $|B|$ are prime numbers p and q, with $p \neq q$, can be represented as a Φ-sum.*

Proof. Let $N \in [\mathcal{A}, \mathcal{O}]$, with $\overset{+}{N} = \overset{+}{A} \oplus \overset{+}{B}$ and let $|A| = p$, $|B| = q$, $p \neq q$. Because N has a proper ideal, this one must be equal to A° or ${}^\circ B$. Suppose A° be the ideal of N. That means ${}^\circ B \in \mathcal{A}$. If $nb = 0$, $\forall n \in N$, $\forall b \in B$, then $n(a + b) = na \in {}^\circ A$, $\forall a \in A$, whence $N^2 \subseteq A^\circ$. This implies ${}^\circ B \simeq N/A^\circ \in \mathcal{O}$, which is a contradiction. Thus, there is an $n \in N$ for which $n{}^\circ B \neq \{0\}$. Because $n{}^\circ B$ is a proper subgroup of N of order q we have $n{}^\circ B = {}^\circ B$. Thus ${}^\circ B$ is a left invariant subgroup and, by Th. 1 of [2], $N = A +_\Phi B$. ◊

Finally, among Φ-sums of $[\mathcal{A}, \mathcal{O}]$ such that their additive group is a direct sum of two groups, we can characterize the non-monogenic case.

Theorem 11. *Let $N = A +_\Phi B$ belonging to $[\mathcal{A}, \mathcal{O}]$ with $\overset{+}{N} = \overset{+}{A} \oplus \oplus \overset{+}{B}$. Then N is non-monogenic iff $N \in [\mathcal{O}, \mathcal{A}]$.*

Proof. If $N \in [\mathcal{O}, \mathcal{A}]$, then $N \simeq A \oplus B$, by Th. 8, and, obviously, it is not monogenic.

Conversely, let N be a non-monogenic near-ring with $\overset{+}{N} = \overset{+}{A} \oplus \oplus \overset{+}{B}$, where $N = A +_\Phi B$ belongs to $[\mathcal{O}, \mathcal{A}]$. To show that $N = A \oplus \oplus B$, it is sufficient to prove that ${}^\circ B$ is a right ideal of N. Firstly, we prove that $f_{a,b} = O_A$, $\forall a \in A$, $\forall b \in B$. Suppose $b^2 = 0$. We have $(\langle a,b\rangle)^2\langle a,0\rangle = \langle f_{a,b}(a), b^2\rangle\langle a,0\rangle = \langle 0,0\rangle$, but also $\langle a,b\rangle(\langle a,b\rangle\langle a,0\rangle) = = \langle a,b\rangle\langle f_{a,b}(a),0\rangle = \langle f^2_{a,b}(a),0\rangle$. Thus $f^2_{a,b}(a) = 0$ and $f_{a,b}$ is not an automorphism. Whence $f_{a,b} = O_A$. Suppose now $b^2 \neq 0$. Then $b{}^\circ B = = {}^\circ B$ and $f_{a,b} \neq O_A$ imply $f_{a,b}(A) = A^\circ$. Thus $\langle a,b\rangle N = N$, but

N is not monogenic, whence $f_{a,b} = O_A$. For this reason $(\langle a,b\rangle + +\langle 0,\bar{b}\rangle)\langle a',b'\rangle - \langle a,b\rangle\langle a',b'\rangle = \langle a, b+\bar{b}\rangle\langle a',b'\rangle - \langle a,b\rangle\langle a',b'\rangle = \langle f_{a,b+\bar{b}}(a'),$ $(b+\bar{b})b'\rangle - \langle f_{a,b}(a'), bb'\rangle = \langle 0, (b+\bar{b})b' - bb'\rangle \in {}^{\circ}B$. Then ${}^{\circ}B$ is a right ideal. ◊

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