Mathematica Pannonica
5/1 (1994), 67 – 77

RELATIONSHIPS BETWEEN DISTANCE DOMINATION PARAMETERS

Michael A. **Henning**

Department of Mathematics, University of Natal, P.O.Box 375 Pietermaritzburg 3200, South Africa

Ortrud R. **Oellermann**

Department of Mathematics and Computer Science, Brandon University, Brandon, Manitoba, Canada

Henda C. **Swart**

Department of Mathematics, University of Natal, Durban 4001, South Africa

Received October 1992

AMS Subject Classification: 05 C 99

Keywords: Distance domination, distance independent domination, total distance domination.

Abstract: For any integer $n \geq 2$ a set D of vertices of a graph G of order p is defined to be a $P_{\leq n}$-dominating set (total $P_{\leq n}$-dominating set) of G if every vertex in $V(G) - D$ (respectively $V(G)$) is at distance at most $n-1$ from some vertex in D other than itself. The $P_{\leq n}$-domination number, $\gamma_n(G)$ (total $P_{\leq n}$-domination number $\gamma_n^t(G)$) is the minimum cardinality among all $P_{\leq n}$-dominating sets (total $P_{\leq n}$-dominating sets) of G. It is shown that if G is a connected graph on $p \geq 2n$ vertices, then $\gamma_n(G) + \gamma_n^t(G) \leq 2p/n$. A set I of vertices in a graph G is $P_{\leq n}$-independent if the distance between every two vertices of I is at least n. A $P_{\leq n}$-dominating set that is also $P_{\leq n}$-independent is called a $P_{\leq n}$-independent dominating set. The minimum cardinality among all $P_{\leq n}$-independent dominating sets in a graph G is the $P_{\leq n}$-independent domination number of G and is denoted by $i_n(G)$. It is shown that if G is a connected graph of order $p \geq n$, then $i_n(G) + (n-1)\gamma_n(G) \leq p$.

The terminology and notation of [2] will be used throughout. Recall that a *dominating set* (*total dominating set*) D of a graph G is a set of vertices of G such that every vertex of $V(G)-D$ (respectively, $V(G)$) is adjacent to some vertex of D. The *domination number* (*total domination number*) of G is the minimum cardinality of a dominating set (total dominating set) of G. Further, the *distance* $d(u,v)$ between two vertices u and v of G is the length of a shortest $u-v$ path if one exists, otherwise $d(u,v)=\infty$. In [5] generalizations of the above-mentioned domination parameters are defined and studied. For an integer $n \geq 2$, a set D of vertices of a graph G is defined to be a $P_{\leq n}$*-dominating set* (*total* $P_{\leq n}$*-dominating set*) of G if every vertex in $V(G)-D$ (respectively $V(G)$) is at distance at most $n-1$ from some vertex in D other than itself. The $P_{\leq n}$*-domination number* $\gamma_n(G)$ (*total* $P_{\leq n}$*-domination number* $\gamma_n^t(G)$) is the minimum cardinality of a $P_{\leq n}$-dominating set (total $P_{\leq n}$-dominating set) of G. Hence $\gamma_2(G)=\gamma(G)$ and $\gamma_2^t(G)=\gamma_t(G)$.

In [5] sharp bounds for the $P_{\leq n}$-domination number and total $P_{\leq n}$-domination number of a graph are established. In particular the following two results were obtained.

Theorem A. *If G is a connected graph of order $p \geq n$, then $\gamma_n(G) \leq$ $\leq p/n$.*

Theorem B. *If G is a connected graph of order $p \geq 2$, then*

$$\gamma_n^t(G) = 2 \qquad \text{for } 2 \leq p \leq 2n-1$$

and

$$\gamma_n^t(G) \leq \frac{2p}{2n-1} \qquad \text{for } p \geq 2n-1.$$

We now investigate relationships between these two generalized domination parameters. Observe that if G is a connected graph on p vertices with $2 \leq p \leq 2n-1$, then $\mathrm{rad}(G) \leq n-1$ and so $\gamma_n(G)+$ $+\gamma_n^t(G)=3$. We thus consider graphs of order $p \geq 2n$. Allan, Laskar and Hedetniemi [1] showed that, if G is a connected graph of order $p \geq 3$, then $\gamma(G)+\gamma_t(G) \leq p$. The following theorem generalizes this result.

Theorem 1. *For an integer $n \geq 2$, if G is a connected graph of order $p \geq 2n$, then*

$$\gamma_n(G)+\gamma_n^t(G) \leq 2p/n.$$

Proof. Let $n \geq 2$ be an integer. If T is a spanning tree of a connected graph G of order at least $2n$ and $\gamma_n(T)+\gamma_n^t(T) \leq 2p(G)/n$, then

$\gamma_n(G) + \gamma_n^t(G) \le \gamma_n(T) + \gamma_n^t(T) \le 2p(G)/n$. Hence we shall prove the theorem by establishing its validity for a tree G. We proceed by induction on the order of a tree of order at least $2n$.

Let T be a tree of order $2n$. Then $\operatorname{diam} T \le 2n-1$, and so $\operatorname{rad} T \le n-1$ or T is bicentral with $\operatorname{rad} T \le n$. If $\operatorname{rad} T \le n-1$, then a central vertex of T is within distance $n-1$ from every vertex of T, while a central vertex, together with any other vertex of T, forms a total $P_{\le n}$-dominating set of T. Hence in this case, $\gamma_n(T)+\gamma_n^t(T) = 3 < < 2p(T)/n$. If, however, $\operatorname{rad} T = n$, then the central vertices of T form a total $P_{\le n}$-dominating set (and hence certainly a $P_{\le n}$-dominating set) of T and so $\gamma_n(T) + \gamma_n^t(T) = 4 = 2p(T)/n$. Hence the theorem is true for a tree of order $2n$.

Assume that $\gamma_n(T') + \gamma_n^t(T') \le 2p(T')/n$ for all trees T' with $2n \le p(T') < k$, and let T be a tree of order k. If $\operatorname{diam} T \le 2n-1$, then $\gamma_n(T) + \gamma_n^t(T) \le 4 < 2p(T)/n$. So we may assume that $\operatorname{diam} T \ge 2n$.

Suppose that there exists an edge e of T such that both components of $T-e$ are of order at least $2n$. Let T_1 and T_2 be the components of $T-e$. Then $2n \le p(T_i) < k$ and so, by the induction hypothesis, for $i \in \{1,2\}$, T_i has a $P_{\le n}$-dominating set D_i and a total $P_{\le n}$-dominating set D_i' with $|D_i| + |D_i'| = \gamma_n(T_i) + \gamma_n^t(T_i) \le 2p(T_i)/n$. Then $D_1 \cup D_2$ is a $P_{\le n}$-dominating set of T and $D_1' \cup D_2'$ is a total $P_{\le n}$-dominating set of T with $\gamma_n(T) + \gamma_n^t(T) \le |D_1 \cup D_2| + |D_1' \cup D_2'| \le 2p(T)/n$. For the remainder of the proof we shall therefore assume that, for each edge e of T, at least one of the (two) components of $T-e$ is of order less than $2n$. In particular, we note that $2n \le \operatorname{diam} T \le 4n-2$. Let $\operatorname{diam} T = d$ and let u, v be two vertices of T such that $d(u,v) = d \ge 2n$. Let the $u-v$ path in T be denoted by $P: u = u_0, u_1, \dots, u_d = v$. To complete the proof we consider four lemmas.

Lemma 1. *If* $2n < p(T) \le 3n-2$, *then* $\gamma_n(T) + \gamma_n^t(T) < 2p(T)/n$.

Proof. Let T_1, T_2 and T_3 denote the components of $T - u_{n-1}u_n$, $T - - u_{d-n}u_{d-n+1}$ and $T - \{u_{n-1}u_n, u_{d-n}u_{d-n+1}\}$, respectively, containing u, v and u_n respectively. Since $p(T) \le 3n-2$, it follows that $d \le 3n - - 3$; so $d(u_{n-1}, u_{d-n+1}) = d + 2 - 2n \le n-1$. Moreover, since P is a longest path in T, the vertex u_{n-1} (u_{d-n+1}) is at distance at most $n-1$ from every vertex in T_1 (T_2, respectively). As $p(T_3) = p(T) - - (p(T_1) + p(T_2)) \le 3n-2-2n = n-2$, every vertex of T_3 is within distance $n-2$ from both u_{n-1} and u_{d-n+1} in T. It follows that $\gamma_n(T) = = \gamma_n^t(T) = \big|\{u_{n-1}, u_{d-n+1}\}\big| = 2$; so $\gamma_n(T) + \gamma_n^t(T) = 4 < 2p(T)/n$. This completes the proof of Lemma 1. $\Diamond$

Lemma 2. *If* $p(T) \geq 3n-1$ *and* $2n \leq d \leq 3n-3$, *then* $\gamma_n(T)+\gamma_n^t(T) \leq \leq 2p(T)/n$.

Proof. Let T_1, T_2 and T_3 be defined as in the proof of Lemma 1. Since $d \leq 3n-3$, $d(u_{n-1}, u_{d-n+1}) \leq n-1$. Moreover, as P is a longest path in T, $u_{n-1}(u_{d-n+1})$ is at distance at most $n-1$ from every vertex in T_1 (T_2, respectively).

If $p(T_3) \leq n-1$, then every vertex of T_3 is within distance $n-1$ from both u_{n-1} and u_{d-n+1}; consequently, $\gamma_n(T)+\gamma_n^t(T) = 4 < < 2p(T)/n$.

Suppose that $n \leq p(T_3) \leq 2n-1$. Then $p(T) \geq 3n$ and $\operatorname{diam} T_3 \leq \leq 2n-2$; so $\operatorname{rad} T_3 \leq n-1$. We show that there exists a central vertex of T_3 that is distance at most $n-1$ from u_{n-1} or u_{d-n+1}. If this is not the case, then, for w a central vertex of T_3, w is at distance $n-1$ from both u_n and u_{d-n}. Since $d(u_n, u_{d-n}) = d-2n \leq n-3$, w is not a vertex of the $u_n - u_{d-n}$ path. Let $Q : v = w_0, w_1, \ldots, w_s$ be the shortest path from w to a vertex of the $u_n - u_{d-n}$ path. Then, necessarily, $w_s = u_j$ for some $j \in \{n+1, \ldots, d-n-1\}$ and $V(Q) \cap V(P) = \{u_j\}$. Let T' and T'' denote the components of $T_3 - ww_1$ containing w_1 and w respectively. Since the $w_1 - u_n$ path (of order $n-1$) does not contain the vertex u_{d-n}, we observe that $p(T') \geq n$. Further, if $p(T'') \leq n-1$, then it follows that w_1 is a central vertex of T_3 at distance $n-1$ from both u_{n-1} and u_{d-n+1}, which contradicts our assumption. Hence $p(T'') \geq \geq n$, and so $p(T_3) \geq 2n$, which again produces a contradiction. Hence there exists a central vertex w (say) of T_3 that is at distance at most $n-1$ from u_{n-1} or u_{d-n+1}, and from each vertex of T_3. Thus $D = = \{u_{n-1}, u_{d-n+1}, w\}$ is a total $P_{\leq n}$-dominating set (and so certainly a $P_{\leq n}$-dominating set) of T; so $\gamma_n(T) + \gamma_n^t(T) \leq 6 \leq 2p(T)/n$.

If $p(T_3) \geq 2n$, then it follows from the induction hypothesis that T_3 has a $P_{\leq n}$-dominating set D' and a total $P_{\leq n}$-dominating set D'' with $|D'| + |D''| = \gamma_n(T_3) + \gamma_n^t(T_3) \leq 2p(T_3)/n$. So $D_1 = = D' \cup \{u_{n-1}, u_{d-n+1}\}$ is a $P_{\leq n}$-dominating set of T and $D_2 = D'' \cup \cup \{u_{n-1}, u_{d-n+1}\}$ is a total $P_{\leq n}$-dominating set of T with $\gamma_n(T) + +\gamma_n^t(T) \leq |D_1| + |D_2| + 4 \leq 2p(T_3)/n + 2(p(T_1) + p(T_2))/n = 2p(T)/n$. This completes the proof of Lemma 2. ◊

Lemma 3. *If* $3n-2 \leq d \leq 4n-3$, *then* $\gamma_n(T) + \gamma_n^t(T) \leq 2p(T)/n$.

Proof. Necessarily there exists an integer i, $1 \leq i \leq d-1$, such that the components of $T - u_{i-1}u_i$ and $T - u_iu_{i+1}$ containing u are, respectively, of order less than $2n$ and of order at least $2n$. From the assumption that, for every edge e of T, $T-e$ contains a component of order at most $2n-1$, it follows that $d-2n+1 \leq i \leq 2n-1$.

Let T_1' and T_2' be the components of $T-u_i$ containing u and v, respectively. We note that T_1' and T_2' are both of order less than $2n$. Further, let $\deg u_i = r$ and denote by $T_1', T_2', \ldots, T_r'$ the components of $T-u_i$ and by w_i the vertex in T_i' adjacent to u_i in $T (i = 1, 2 \ldots, r)$. We note that $w_1 = u_{i-1}$ and $w_2 = u_{i+1}$. If $r \geq 3$, then for $j \in \{3, \ldots, r\}$ we observe that, since one component of $T - u_i w_j$ contains P and is therefore of order at least $2n$, the component T_j' is of order at most $2n-1$.

We consider two possibilities.

Case 1: Suppose that $i = 2n-1$ or $i = d-2n+1$. Without loss of generality, we may assume (relabelling the path P by $v = u_0, u_1, \ldots, u_d = u$ if necessary) that $i = 2n-1$. Since $p(T_1') \leq 2n-1$, $T_1' \cong P_{2n-1}$ and $\{u_{n-1}\}$ is a $P_{\leq n}$-dominating set of T_1'. We consider two possibilities.

Case 1.1: Suppose that $d = 3n-2$. Then $u_{2n-1} = u_{d-n+1}$ and every vertex of T_2' is within distance $n-1$ from u_{2n-1}. Consequently, if $r = 2$, then $\gamma_n(T)+\gamma_n^t(T) \leq |\{u_{n-1}, u_{2n-1}\}|+|\{u_{n-1}, u_{2n-2}, u_{2n-1}\}| = 5 \leq 2(3n-1)/n \leq 2p(T)/n$. We now consider the case where $r \geq 3$. Let $\{3, \ldots, r\} = I = I_1 \cup I_2 \cup I_3$ where

$$I_1 = \{j \in I \mid p(T_j') \leq n-1\},$$
$$I_2 = \{j \in I \mid n \leq p(T_j') \leq 2n-2\},$$
$$I_3 = \{j \in I \mid p(T_j') = 2n-1\}.$$

If $j \in I_1$, then u_{2n-1} is within distance $n-1$ from every vertex of T_j'. If $j \in I_2$, then since $p(\langle V(T_j') \cup \{u_{2n-1}\}\rangle) \leq 2n-1$, T_j' contains a vertex z_j such that $\{z_j\}$ is a $P_{\leq n}$-dominating set of T_j' and $d(u_{2n-1}, z_j) \leq n-1$. If $j \in I_3$, then $\operatorname{rad} T_j' \leq n-1$. Let x_j be a central vertex of T_j'. It follows, therefore, that $\gamma_n(T) \leq |\{u_{n-1}, u_{2n-1}\}| + |\bigcup_{j\in I_2} \{z_j\}| + |\bigcup_{j\in I_3} \{x_j\}| = 2 + |I_2| + |I_3|$ and $\gamma_n^t(T) \leq |\{u_{n-1}, u_{2n-2}, u_{2n-1}\}| + |\bigcup_{j\in I_2} \{z_j\}| + |\bigcup_{j\in I_3} \{x_j, w_j\}| = 3 + |I_2| + 2|I_3|$; so $\gamma_n(T) + \gamma_n^t(T) \leq 5 + 2|I_2| + 3|I_3|$. However, $p(T) \geq d+1+n|I_2|+(2n-1)|I_3| = 3n-1+n|I_2|+(2n-1)|I_3|$. Hence $2p(T)/n \geq 6 - 2/n + 2|I_2| + (4-2/n)|I_3| \geq 5 + 2|I_2| + 3|I_3| \geq \gamma_n(T) + \gamma_n^t(T)$.

Case 1.2: Suppose that $3n-1 \leq d \leq 4n-3$. Then $d-n+1 > 2n-1$ and so $u_{d-n+1} \in V(T_2')$. Further, since $p(T_2') \leq 2n-1$,

$\{u_{d-n+1}\}$ is a $P_{\leq n}$-dominating set of T_2'. Since $d \leq 4n-3$, we observe that $d(u_{d-n+1}, u_{2n-1}) = d - 3n + 2 \leq n - 1$.

If $r = 2$, then

$$\gamma_n(T) + \gamma_n^t(T) \leq |\{u_{n-1}, u_{d-n+1}\}| + |\{u_{n-1}, u_{2n-2}, u_{2n-1}, u_{d-n+1}\}| = = 6 \leq 2(3n)/n \leq 2p(T)/n.$$

If $r \geq 3$, then let $I = \{3, \dots, r\} = I_1 \cup I_2 \cup I_3 \cup I_4$ where

$$I_1 = \{j \in I \mid p(T_j') \leq 4n - d - 3\},$$
$$I_2 = \{j \in I \mid 4n - d - 2 \leq p(T_j') \leq n - 1\},$$
$$I_3 = \{j \in I \mid n \leq p(T_j') \leq 2n - 2\},$$
$$I_4 = \{j \in I \mid p(T_j') = 2n - 1\}.$$

If $j \in I_1$, then, since $d(u_{d-n+1}, u_{2n-1}) = d - 3n + 2$, it follows that u_{d-n+1} is within distance $n-1$ from every vertex of T_j'. If $j \in I_2$, then u_{2n-1} is within distance $n-1$ from every vertex of T_j'. If $j \in I_3$, then T_j' contains a vertex z_j such that $\{z_j\}$ is a $P_{\leq n}$-dominating set of T_j' and $d(u_{2n-1}, z_j) \leq n - 1$. If $j \in I_4$, then $\operatorname{rad} T_j' \leq n - 1$. Let x_j be a central vertex of T_j'. We now consider two possibilities.

Case 1.2.1: Suppose that $|I_2| \geq 1$. Then it follows that $\gamma_n(T) \leq \leq |\{u_{n-1}, u_{2n-1}, u_{d-n+1}\}| + |\bigcup_{j \in I_3} \{z_j\}| + |\bigcup_{j \in I_4} \{x_j\}| = 3 + |I_3| + |I_4|$ and $\gamma_n^t(T) \leq |\{u_{n-1}, u_{2n-2}, u_{2n-1}, u_{d-n+1}\}| + |\bigcup_{j \in I_3} \{z_j\}| + |\bigcup_{j \in I_4} \{x_j, w_j\}| = = 4 + |I_3| + 2|I_4|$; so $\gamma_n(T) + \gamma_n^t(T) \leq 7 + 2|I_3| + 3|I_4|$. However, $p(T) \geq (d+1) + (4n-d-2)|I_2| + n|I_3| + (2n-1)|I_4| \geq 4n - 1 + n|I_3| + + (2n-1)|I_4|$. Hence $2p(T)/n \geq 8 - 2/n + 2|I_3| + (4 - 2/n)|I_4| \geq 7 + + 2|I_3| + 3|I_4| \geq \gamma_n(T) + \gamma_n^t(T)$.

Case 1.2.2: Suppose that $|I_2| = 0$. Then it follows that $\gamma_n(T) \leq \leq |\{u_{n-1}, u_{d-n+1}\}| + |I_3| + |I_4| = 2 + |I_3| + |I_4|$ and $\gamma_n^t(T) \leq 4 + |I_3| + 2|I_4|$; so $\gamma_n(T) + \gamma_n^t(T) \leq 6 + 2|I_3| + 3|I_4|$. However, $p(T) \geq d + 1 + n|I_3| + + (2n-1)|I_4| \geq 3n + n|I_3| + (2n-1)|I_4|$. Hence $2p(T)/n \geq 6 + 2|I_3| + + 3|I_4| \geq \gamma_n(T) + \gamma_n^t(T)$.

Case 2: Suppose that $d - 2n + 2 \leq i \leq 2n - 2$. Then, since $d \geq \geq 3n - 2$, $n \leq d - 2n + 2 \leq i \leq 2n - 2 \leq d - n$. Hence $u_{n-1}(u_{d-n+1})$ is a vertex of T_1' (T_2', respectively). In fact, as P is a longest path in T and as $p(T_i') \leq 2n - 1$ ($1 \leq i \leq 2$), $\{u_{n-1}\}$ ($\{u_{d-n+1}\}$) is a $P_{\leq n}$-dominating set of T_1' (T_2', respectively). Furthermore, since $i \leq \leq 2n - 2$, $d(u_{n-1}, u_i) = i - n + 1 \leq n - 1$ and since $i \geq d - 2n + + 2$, $d(u_{d-n+1}, u_i) = d - n + 1 - i \leq n - 1$. Consequently, if $r = 2$, then $\gamma_n(T) + \gamma_n^t(T) \leq |\{u_{n-1}, u_{d-n+1}\}| + |\{u_{n-1}, u_i, u_{d-n+1}\}| = 5 \leq$ $\leq 2(3n-1)/n \leq 2p(T)/n$.

If $r \geq 3$, then let $I = \{3, \ldots, r\} = I_1 \cup I_2 \cup I_3 \cup I_4$, where

$$I_1 = \{j \in I \mid p(T_j') < \max(2n-i-1, 2n+i-d-1)\},$$
$$I_2 = \{j \in I \mid \max(2n-i-1, 2n+i-d-1) \leq p(T_j') \leq n-1\},$$
$$I_3 = \{j \in I \mid n \leq p(T_j') \leq 2n-2\},$$
$$I_4 = \{j \in I \mid p(T_j') = 2n-1\}.$$

If $j \in I_1$, then $p(T_j') \leq 2n-i-2$ or $p(T_j') \leq 2n+i-d-2$. If $p(T_j') \leq 2n-i-2$, then since $d(u_{n-1}, u_i) = i-n+1$, it follows that u_{n-1} is within distance $n-1$ from every vertex of T_j'. If $p(T_j') \leq 2n+$ $+i-d-2$, then, since $d(u_{d-n+1}, u_i) = d-n+1-i$, it follows that u_{d-n+1} is within distance $n-1$ from every vertex of T_j'. If $j \in I_2$, then u_i is within distance $n-1$ from every vertex of T_j'. If $j \in I_3$, then T_j' contains a vertex z_j such that $\{z_j\}$ is a $P_{\leq n}$-dominating set of T_j' and $d(u_i, z_j) \leq n-1$. If $j \in I_4$, then $\operatorname{rad} T_j' \leq n-1$. Let x_j be a central vertex of T_j'. We now consider two possibilities.

Case 2.1: Suppose that $|I_2| \geq 1$. Then it follows that $\gamma_n(T) \leq$ $\leq |\{u_{n-1}, u_i, u_{d-n+1}\}| + |\bigcup_{j\in I_3} \{z_j\}| + |\bigcup_{j\in I_4} \{x_j\}| = 3 + |I_3| + |I_4|$ and $\gamma_n^t(T) \leq |\{u_{n-1}, u_i, u_{d-n+1}\}| + |\bigcup_{j\in I_3} \{z_j\}| + |\bigcup_{j\in I_4} \{x_j, w_j\}| = 3 + |I_3| +$ $+ 2|I_4|$; so $\gamma_n(T) + \gamma_n^t(T) \leq 6 + 2|I_3| + 3|I_4|$.

If $\max(2n-i-1, 2n+i-d-1) = 2n-i-1$, then $p(T) \geq$ $\geq (d+1) + (2n-i-1)|I_2| + n|I_3| + (2n-1)|I_4| \geq 2n+d-i+n|I_3| +$ $+(2n-1)|I_4| \geq 3n + n|I_3| + (2n-1)|I_4|$, since $d-i \geq n$. Hence $2p(T)/n \geq 6 + 2|I_3| + (4-2/n)|I_4| \geq 6 + 2|I_3| + 3|I_4| \geq \gamma_n(T) + \gamma_n^t(T)$.

If $\max(2n-i-1, 2n+i-d-1) = 2n+i-d-1$, then $p(T) \geq$ $\geq (d+1) + (2n+i-d-1)|I_2| + n|I_3| + (2n-1)|I_4| \geq 2n+i+$ $+n|I_3| + (2n-1)|I_4| \geq 3n + n|I_3| + (2n-1)|I_4|$, since $i \geq n$. Hence $2p(T)/n \geq 6 + 2|I_3| + 3|I_4| \geq \gamma_n(T) + \gamma_n^t(T)$.

Case 2.2: Suppose that $|I_2| = 0$. Then it follows that $\gamma_n(T) \leq$ $\leq |\{u_{n-1}, u_{d-n+1}\}| + |I_3| + |I_4| = 2 + |I_3| + |I_4|$ and $\gamma_n^t(T) \leq 3 + |I_3| +$ $+2|I_4|$; so $\gamma_n(T) + \gamma_n^t(T) \leq 5 + 2|I_3| + 3|I_4|$. However, $p(T) \geq d+1+$ $+n|I_3| + (2n-1)|I_4| \geq 3n-1+n|I_3| + (2n-1)|I_4|$. Hence $2p(T)/n \geq$ $\geq 6-2/n + 2|I_3| + (4-2/n)|I_4| \geq 5 + 2|I_3| + 3|I_4| \geq \gamma_n(T) + \gamma_n^t(T)$.

This completes the proof of Lemma 3. $\Diamond$

Lemma 4. *If $d = 4n-2$, then $\gamma_n(T) + \gamma_n^t(T) \leq 2p(T)/n$.*

Proof. Suppose that $d = 4n-2$. Then, using the notation introduced in the first two paragraphs of the proof of Lemma 3, it follows that $i = 2n-1$. Furthermore, since $p(T_i') \le 2n-1$, we therefore have $T_i' \cong P_{2n-1}$ $(1 \le i \le 2)$ and so $\{u_{n-1}\}$ ($\{u_{3n-1}\}$) is a $P_{\le n}$-dominating set of T_1' (T_2', respectively). We observe, however, that u_{2n-1} is at distance n from both u_{n-1} and u_{3n-1}. Consequently, if $r = 2$, then $\gamma_n(T)+\gamma_n^t(T) = \left|\{u_{n-1}, u_{2n-1}, u_{3n-1}\}\right| + \left|\{u_{n-1}, u_{2n-2}, u_{2n}, u_{3n-1}\}\right| = = 7 \le 2(4n-1)/n = 2p(T)/n$.

If $r \ge 3$, then let $I = \{3, \dots, r\} = I_1 \cup I_2 \cup I_3 \cup I_4$, where

$$I_1 = \{j \in I \mid p(T_j') \le n-2\},$$
$$I_2 = \{j \in I \mid p(T_j') = n-1\},$$
$$I_3 = \{j \in I \mid n \le p(T_j') \le 2n-2\},$$
$$I_4 = \{j \in I \mid p(T_j') = 2n-1\}.$$

If $j \in I_1$, then every vertex of T_j' is within distance $n-1$ from the vertices u_{2n-2}, u_{2n-1} and u_{2n}. If $j \in I_2$, then u_{2n-1} is within distance $n-1$ from every vertex of T_j'. If $j \in I_3$, then T_j' contains a vertex z_j such that $\{z_j\}$ is a $P_{\le n}$-dominating set of T_j' and $d(z_j, u_{2n-1}) \le n-1$. If $j \in I_4$, then $\operatorname{rad} T_j' \le n-1$. Let x_j be a central vertex of T_j'. We now consider two possibilities.

Case 1: Suppose that $|I_2| \ge 1$. Then it follows that $\gamma_n(T) \le \le \left|\{u_{n-1}, u_{2n-1}, u_{3n-1}\}\right| + \left|\bigcup_{j\in I_3}\{z_j\}\right| + \left|\bigcup_{j\in I_4}\{x_j\}\right| = 3 + |I_3| + |I_4|$ and $\gamma_n^t(T) \le \left|\{u_{n-1}, u_{2n-2}, u_{2n-1}, u_{2n}, u_{3n-1}\}\right| + \left|\bigcup_{j\in I_3}\{z_j\}\right| + \left|\bigcup_{j\in I_4}\{w_j, x_j\}\right|$; so $\gamma_n(T) + \gamma_n^t(T) \le 8 + 2|I_3| + 3|I_4|$. However, $p(T) \ge 4n-1+(n- -1)|I_2| + n|I_3| + (2n-1)|I_4| \ge 5n-2+n|I_3| + (2n-1)|I_4|$. Hence $2p(T)/n \ge 10-2/n+2|I_3|+(4-2/n) > 8+2|I_3|+3|I_4| \ge \gamma_n(T)+\gamma_n^t(T)$.

Case 2: Suppose that $I_2| = 0$. Then, if $|I_3| \ge 1$, it follows that $\gamma_n(T) \le \left|\{u_{n-1}, u_{3n-1}\}\right| + \left|\bigcup_{j\in I_3}\{z_j\}\right| + \left|\bigcup_{j\in I_4}\{x_j\}\right| = 2 + |I_3| + |I_4|$ and $\gamma_n^t(T) \le \left|\{u_{n-1}, u_{2n-2}, u_{2n-1}, u_{2n}, u_{3n-1}\}\right| + |I_3| + 2|I_4| \le 5 + |I_3| + 2|I_4|$; so $\gamma_n(T) + \gamma_n^t(T) \le 7 + 2|I_3| + 3|I_4|$. However, $p(T) \ge 4n-1+n|I_3| + + (2n-1)|I_4|$. Hence $2p(T)/n \ge 8 - 2/n + 2|I_3| + (4-2/n)|I_4| \ge 7 + + 2|I_3| + 3|I_4| \ge \gamma_n(T) + \gamma_n^t(T)$.

If $|I_3| = 0$, then it follows that $\gamma_n(T) \le \left|\{u_{n-1}, u_{2n-1}, u_{3n-1}\}\right| + + \left|\bigcup_{j\in I_4}\{x_j\}\right| = 3 + |I_4|$ and $\gamma_n^t(T) \le \left|\{u_{n-1}, u_{2n-2}, u_{2n}, u_{3n-1}\}\right| + 2|I_4| = = 4 + 2|I_4|$; so $\gamma_n(T) + \gamma_n^t(T) \le 7 + 3|I_4|$. However, $p(T) \ge 4n-1+$

$+ (2n-1)|I_4|$. Hence $2p(T)/n \geq 8 - 2/n + (4 - 2/n)|I_4| \geq 7 + 3|I_4| \geq$ $\geq \gamma_n(T) + \gamma_n^t(T)$.

This completes the proof of Lemma 4 and thus of Th. 1. $\Diamond$

That the bound in Th. 1 is best possible may be seen as follows: Let G be obtained from a connected graph H by attaching a path of length $n-1$ to each vertex of H. (The graph G is shown in Fig. 1.) Then $\gamma_n(G) + \gamma_n^t(G) = 2p(H) = 2p(G)/n$.

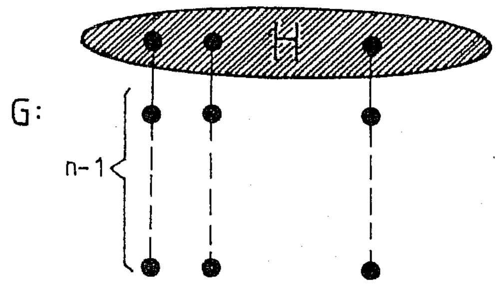

Fig. 1.

The fact that every maximal independent set of vertices in a graph is also a dominating set motivated Cockayne and Hedetmiemi [3] in 1974 to initiate the study of another domination parameter. A dominating set of vertices in a graph that is also an independent set is called an *independent dominating set.* The minimum cardinality among all independent dominating sets of a graph G is called the *independent domination number* of G and is denoted by $i(G)$.

The independent domination number of a graph and the distance domination parameters introduced earlier suggest yet another distance domination parameter. A set I of vertices in a graph G is $P_{\leq n}$*-independent* in G if every two vertices of I are at distance at least n apart in G. A $P_{\leq n}$-independent set of vertices in a graph that is also a $P_{\leq n}$-dominating set is called a $P_{\leq n}$*-independent dominating set.* The minimum cardinality among all $P_{\leq n}$-independent dominating sets of a graph G is called the $P_{\leq n}$*-independent domination number* of G and is denoted by $i_n(G)$. Hence $i_2(G) = i(G)$.

Before investigating relationships between the distance domination parameter i_n and the distance domination parameters γ_n and γ_n^t we need some additional concepts. A set of vertices $X \subset V(G)$ has *property* $\pi_n (n \geq 2)$ if and only if every nontrivial path of length $\ell \leq$ $\leq n-1$ in G contains at least ℓ vertices of X. A set of vertices with property π_n is called a *$P_{\leq n}$-cover* of G. So a $P_{\leq 2}$-cover of G is simply a cover of G. The minimum cardinality among all $P_{\leq n}$-covers of G is called the *$P_{\leq n}$-covering number* of G and is denoted by $\alpha_n(G)$. The maximum cardinality among all $P_{\leq n}$-independent sets is called the *$P_{\leq n}$-independence number* of G and is denoted by $\beta_n(G)$. Hence $\alpha_2(G)$ is simply the covering number $\alpha(G)$ and $\beta_2(G)$ is the independence number $\beta(G)$. The next Gallai-type result generalizes a well-known relationship between the covering number and independence number of a graph [4].

Theorem 2. *If G is a connected graph of order $p \geq n$, then*

$$\alpha_n(G) + \beta_n(G) = p.$$

Proof. We note that X is a $P_{\leq n}$-cover if and only if $V(G) - X$ is a $P_{\leq n}$-independent set of vertices. So if X is a $P_{\leq n}$-cover of cardinality $\alpha_n(G)$, then $\alpha_n(G) = |X|$ and $|V(G) - X| = p - \alpha_n(G) \leq \beta_n(G)$. Similarly if Y is a $P_{\leq n}$-independent set of vertices of cardinality $\beta_n(G)$, $p - \beta_n(G) = |V(G) - Y| \geq \alpha_n(G)$. Thus $\alpha_n(G) + \beta_n(G) = p$. ◊

Allan, Laskar and Hedetniemi [1] showed that if G is a graph of order p that has no isolated vertices, then $\gamma(G) + i(G) \leq p$. We now present a generalization of this result.

Theorem 3. *If G is a connected graph of order $p \geq n \geq 2$, then*

$$i_n(G) + (n-1)\gamma_n(G) \leq p.$$

Proof. Let X be a $P_{\leq n}$-cover such that $\langle X \rangle$ contains as few components as possible of order less than $n-1$. We show that $\langle X \rangle$ has no components of order less than $n-1$. Suppose $\langle X \rangle$ has a component G_1 of order $p_1 \leq n-2$. Since G is connected, and $p \geq n$, there is a vertex $s \in S = V(G) - X$ that is adjacent with a vertex y in G_1 and a vertex z in $V(G) - V(G_1)$. Since S is $P_{\leq n}$-independent, z must belong to some component $G_2 \neq G_1$ of $\langle X \rangle$. Note that s is the only vertex of S which is adjacent to a vertex (or vertices) in G_1, for if t is any other vertex of S that is adjacent to a vertex of G $d(t, s) \leq n-1$, which is not possible since S is $P_{\leq n}$-independent.

Now if $p(G_1) = 1$, let $S' = (S - \{s\}) \cup \{y\}$. Otherwise if $p(G_1) \geq$ ≥ 2, let $x \neq y$ be a vertex of G_1 which is not a cut-vertex of G_1 and set $S' = (S - \{s\}) \cup \{x\}$. Then S' is a $P_{\leq n}$-independent set of cardinality $|V(G) - X|$. Since X is a $P_{\leq n}$-cover of cardinality $\alpha_n(G)$, it follows from Th. 2, that $|V(G) - X| = p - \alpha_n(G) = \beta_n(G)$, i.e., $|S'| = \beta_n(G)$. However, then $X' = V(G) - S'$ is a $P_{\leq n}$-cover of G of cardinality $\alpha_n(G)$

such that $\langle X' \rangle$ contains fewer components of order less than $n-1$ than $\langle X \rangle$. This contradicts our choice of X. Hence $\langle X \rangle$ has no components of order less than $n-1$.

Since G is connected, every vertex in $V(G)-X$ is adjacent with a vertex in X and, consequently

$$\gamma_n(G) \leq \gamma_{n-1}(\langle X \rangle).$$

Since $\langle X \rangle$ has no component of order smaller than $n-1$, it follows from Th. A that

$$\gamma_n(G) \leq \frac{p(\langle X \rangle)}{n-1} = \frac{|X|}{n-1} = \frac{\alpha_n(G)}{n-1}.$$

The fact that $\beta_n(G) = |V(G)-X| \geq i_n(G)$ and Th. 2 now imply that

$$i_n(G) + (n-1)\gamma_n(G) \leq \alpha_n(G) + \beta_n(G) = p. \ \Diamond$$

The bound given in Th. 3 is best possible as we now see. Let G be the graph shown in Fig. 1. Then $i_n(G) = \gamma_n(G) = p(H)$ and $i_n(G) + (n-1)\gamma_n(G) = np(H) = p(G)$. It is shown in [6] that if T is a tree of order $p \geq 2n-1$, then $i_n(T) + (n-1)\gamma_n^t(T) \leq p$.

References

[1] ALLAN, R. B., LASKAR, R. and HEDETNIEMI, S. T.: A note on total domination, *Discrete Math.* **49** (1984), 7–13.

[2] CHARTRAND, G. and LESNIAK, L.: *Graphs and Diagraphs*, Wadsworth and Brooks/Cole, Monterey CA, 1986.

[3] COCKAYNE, E. J. and HEDETNIEMI, S. T.: Indpendence graphs, *in: Proceedings of 5th Southeastern Conference on Combinatorics, Graph Theory and Computing. Utilitas Mathematicae*, Winnipeg (1974), 471–491.

[4] GALLAI, T.: Über extreme Punkt- und Kantenmengen, *Ann. Univ. Sci. Budapest. Eötvös Sect. Math.* **2** (1959), 133–138.

[5] HENNING, M. A., OELLERMANN, O. R. and SWART, H. C.: Bounds on distance domination parameters, *J. Combin. Inf. Syst. Sci.* **16** (1991), 11–18.

[6] HENNING, M. A., OELLERMANN, O. R. and SWART, H. C.: Relating pairs of distance domination parameters, *J. Comb. Math. Comb. Comp.* (to appear).