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TORSION-FREE MODULES AND SYZYGIES

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Abstract: We show how a number of well-known results follow from a characterization of torsion-free modules.

In this note we bring to light a result which seems to be lying beneath the surface of a number of well known theorems and, once stated, from which these theorems may be readily derived. To wit, let R be a Noetherian ring and M a finitely generated R-module admitting a finite free resolution

$$\mathbf{F}\colon 0 \longrightarrow F_n \xrightarrow{\phi_n} \cdots \xrightarrow{\phi_2} F_1 \xrightarrow{\phi_1} F_0 \longrightarrow M \longrightarrow 0.$$

We observe that M is torsion-free if and only if the ideal of minors associated to the ith map in the resolution has depth greater than or equal to $i+1$. The similarity with this statement and the one appearing in the celebrated exactness theorem of Buchsbaum and Eisenbud [3] is not coincidental. The result is essentially equivalent to their theorem. However, it seems that bringing it to the fore allows one to see precisely how the conditions of their theorem yield exactness. The result also serves as the inductive step in an analogous characterization for M to be a kth syzygy. Using this result, we can derive a theorem of Auslander–Bridger appearing in [1] and extend a result of Bruns concerning the structure of kth syzygies to non-Cohen–Macaulay local rings. Finally, though the proposition below doesn't seem to explicitly appear in any of the standard references on the subject, undoubtedly it is not new. Our primary purpose here is to demonstrate the central place it occupies.

Let $\mathbf{F}$ be given as above i.e., each F_i is a free R-module of finite rank and ϕ_i is a $\text{rank}(F_{i-1}) \times \text{rank}(F_i)$ matrix with entries in R. The rank of ϕ_i is the size of the largest non-vanishing minor of ϕ_i and we will write $I(\phi_i)$ for the ideal generated by minors of size $\text{rank}(\phi_i)$. If $\text{rank}(\phi_i) = 0$, take $I(\phi_i) = R$. With this we may state the result as follows.

Proposition. *Let M and $\mathbf{F}$ be as above. Then M is a torsion-free R-module if and only if* $\text{depth}(I(\phi_i)) \geq i+1$ *for* $i = 1, \ldots, n$.

Proof. We begin the proof with a couple of remarks. First, recall that for M as above, M is torsion-free if and only if every prime ideal associated to M is an associated prime of R. Furthermore, as the hypotheses and conclusions of the proposition are preserved under localization, we are free to localize at a prime ideal at any point in the argument. Finally, recall that if R is local and the projective dimension of M (denoted $\text{p.d.}_R(M)$) is finite, then the Auslander–Buchsbaum formula states that $\text{depth}(M) + \text{p.d.}_R(M) = \text{depth}(R)$ (see [2]).

Now, suppose that M is torsion-free. Let i be the largest integer for which $\text{depth}(I(\phi_i)) \leq i$. We seek a contradiction. Select a prime ideal P containing $I(\phi_i)$ such that $\text{depth}(R_P) \leq i$. It follows that $I(\phi_j) \nsubseteq P$, for $j > i$. Therefore, upon localizing at P, the sequence $\mathbf{F}$ splits at F_i. Localizing at P and changing notation, we have $I(\phi_{i+1}) = R$, $I(\phi_i) \neq R$ and $\text{depth}(R) \leq i$. Thus $F_i = \text{image}(\phi_{i+1}) \oplus F_i'$ and we may truncate $\mathbf{F}$ to obtain an exact sequence

$$\mathbf{F}': 0 \longrightarrow F_i' \xrightarrow{\phi_i} \cdots \xrightarrow{\phi_2} F_1 \xrightarrow{\phi_1} F_0 \longrightarrow M \longrightarrow 0.$$

Thus $\text{p.d.}_R(M) \leq i$. If $\text{p.d.}_R(M) < i$, $\text{image}(\phi_j)$ is projective (free) for some $j \leq i-1$, so the truncated sequence splits to the left of F_j and it follows that $I(\phi_i) = R$, which isn't so. Thus $\text{p.d.}_R(M) = i$. Since $\text{depth}(R) \leq i$, the Auslander–Buchsbaum implies $i = \text{depth}(R) = \text{p.d.}_R(M)$. Consequently $\text{depth}(M) = 0$. Therefore P, the maximal ideal of R, is an associated prime of M, and hence R, since M is torsion-free. Thus $i = \text{depth}(R) = 0$ and this is the contradiction we sought.

Conversely, suppose the depth condition holds. Let P be an associated prime of M. We must show that P is an associated prime of R. We may assume that R is a local ring and P is its maximal ideal.

Since $\operatorname{depth}(M) = 0$, $\operatorname{depth}(R) \leq n$ (by the Auslander–Buchsbaum formula). Moreover, as $\operatorname{depth}(I(\phi_n)) \geq n+1$, we must have $I(\phi_n) = R$. Thus the sequence $\mathbf{F}$ splits at F_{n-1}, so we may truncate as before to obtain

$$\mathbf{F}': 0 \longrightarrow F'_{n-1} \xrightarrow{\phi_{n-1}} \cdots \xrightarrow{\phi_2} F_1 \xrightarrow{\phi_1} F_0 \longrightarrow M \longrightarrow 0.$$

By induction on n (the case $n = 0$ is trivial), M is torsion-free, so P is associated to R, as desired. ◊

Corollary A (Buchsbaum–Eisenbud). *Let R be a Noetherian domain and*

$$\mathbf{F}: 0 \longrightarrow F_n \xrightarrow{\phi_n} \cdots \xrightarrow{\phi_2} F_1 \xrightarrow{\phi_1} F_0 \longrightarrow 0$$

a complex of finitely generated free R-modules. Then $\mathbf{F}$ is acyclic (i.e., $\ker(\phi_i) = \operatorname{image}(\phi_{i+1})$ for $i > 0$) if and only if: (i) $\operatorname{rank}(\phi_i) + \operatorname{rank}(\phi_{i+1}) = \operatorname{rank}(F_i)$ *and* (ii) $\operatorname{depth}(I(\phi_i)) \geq i$, *for $i = 1, \ldots, n$.*

Proof. Let K denote the quotient field of R and suppose the conditions hold. We proceed by induction on n. If $n = 1$, the complex is exact by McCoy's theorem. Assume $n > 1$. Condition (i) implies that $\mathbf{F} \otimes K$ is an acyclic complex of vector spaces. Hence the ith homology module is a torsion module, for $i > 0$. In particular, $H_1(\mathbf{F})$ is torsion. On the other hand, by induction $H_i(\mathbf{F}) = 0$ for $i = 2, \ldots, n$. Thus $\mathbf{F}$ resolves the cokernel of ϕ_2. By the Prop., condition (ii) implies that the cokernel of ϕ_2 is torsion-free. Hence its submodule $H_1(\mathbf{F})$ is torsion-free. Thus $H_1(\mathbf{F})$ is both torsion and torsion-free, and therefore zero. That is, $\mathbf{F}$ is acyclic. Conversely, if $\mathbf{F}$ is acyclic, then $\mathbf{F} \otimes K$ is an acyclic complex of vectors spaces, so (i) holds. Clearly $\operatorname{depth}(I(\phi_1)) \geq 1$. Moreover, the cokernel of ϕ_2 is torsion-free, so (ii) holds by the proposition. ◊

Remark. Of course the Buchsbaum–Eisenbud theorem holds for any Noetherian ring, but we have presented the domain case to exhibit more clearly how conditions (i) and (ii) determine exactness. However, essentially the same proof works in general (with the aid of some additional linear algebra). For example, one can show that the conditions (i) and $\operatorname{depth}(I(\phi_i)) \geq 1$ hold if and only if the complex $\mathbf{F} \otimes K$ is split exact, where K now denotes the total quotient ring of R. Hence if (i) and (ii) hold, $H_i(\mathbf{F})$ is torsion on the one hand and torsion-free on the other (by the Prop., as in the proof above) and therefore zero.

Corollary B. *Let R be a Noetherian ring and M a finitely generated R-module. Then M is the kth syzygy in a finite free resolution of an R-module N if and only if M admits a finite free resolution $\mathbf{F}$ (as before) satisfying* $\operatorname{depth}(I(\phi_i)) \geq i + k$ *for $i = 1, \ldots, n$.*

Proof. Suppose that M is the kth syzygy in a finite free resolution of the R-module N. Then $M = \text{image}(\psi_k)$, where ψ_k is the kth map in the resolution of N. We may take for $\mathbf{F}$, the resolution of M, that portion of the resolution for N whose first map is ψ_{k+1}. That $\text{depth}(I(\phi_i)) \geq$ $\geq i + k$ for $i = 1, \dots, n$ now follows from the Buchsbaum–Eisenbud theorem.

Conversely, suppose M admits a finite free resolution $\mathbf{F}$ satisfying the required depth condition. We proceed by induction on k. When $k =$ $= 1$, M is torsion-free (by the Prop.) and it is well known that M can be embedded in a free module (when R is not a domain, this requires that M have finite projective dimension, which we are assuming). Therefore M is the first syzygy in a resolution of the cokernel of this embedding. Now suppose that $k > 1$. Let $f_1, \dots, f_n$ generate $\text{Hom}(M, R)$ and take $u\colon M \to R^n$ to be the so-called *universal pushforward* (see [6]). In other words, for each $m \in M$, $u(m)$ is the column vector whose jth entry is $f_j(m)$. Let $C = \text{cokernel}(u)$. Using * to denote R duals, we have exact sequences

$$0 \longrightarrow M \longrightarrow R^n \longrightarrow C \longrightarrow 0$$

$$0 \longrightarrow C^* \longrightarrow (R^n)^* \longrightarrow M^* \longrightarrow 0$$

where exactness in the first sequence follows because M is torsion-free, and exactness in the second sequence follows from the definition of universal pushforward. Let $Q \subseteq R$ be a prime ideal with $\text{depth}\,(R_Q) \leq$ $\leq k$. Then $I(\phi_1) \nsubseteq Q$, so M_Q is a free R_Q module. Therefore M_Q^* is free, so the second sequence splits over R_Q. Therefore the dual of the second sequence (i.e., the "double dual") splits over R_Q. Since M_Q is free this shows that $C_Q = C_Q^{**}$ and that the first sequence splits over R_Q. It follows that if we let ϕ_0 denote the composition $F_0 \to M \to R^n$, then $\text{depth}\,(I(\phi_0)) \geq k + 1$ and C admits a resolution satisfying the given depth condition for $k - 1$. By induction C is a $(k-1)$st syzygy of the required form, so M is a kth syzygy, as desired. $\Diamond$

Corollary C (Auslander–Bridger). *Let R be a local ring satisfying Serre's condition S_k and M a finitely generated R-module having finite projective dimension. Then M is a kth syzygy if and only if M satisfies S_k $(k \geq 1)$.*

Proof. Recall that a finitely generated R-module N satisfies S_k if for all prime ideals P in the support of N, $\text{depth}_{R_P}(N_P) \geq \min(k, \dim(R_P))$. Now, let M satisfy S_k and $\mathbf{F}$ be a projective resolution. We want to see that $\mathbf{F}$ satisfies the depth condition of Cor. B. As in the proof of the

Prop., we let i be the largest integer for which depth$(I(\phi_i)) \leq i+k-1$ and we select a prime ideal P containing $I(\phi_i)$ with depth$(R_P) \leq i+$ $+k-1$. If we localize at P, then p.d.$(M_P) = i > 0$. Thus depth$(M_P) \leq$ $\leq k-1$, by the Auslander–Buchsbaum formula. Since M satisfies S_k, this implies depth$(M_P) =$ depth(R_P). Thus M_P is free, so $i = 0$, contradiction.

Conversely, suppose that M is a kth syzygy and $\mathbf{F}$ is a resolution of M. We may assume that $\mathbf{F}$ satisfies the depth condition of Cor. B. Let $P \subseteq R$ be prime ideal. If $\dim(R_P) \leq k$, then $I(\phi_i) \not\subseteq P$, so M_P is R_P-free. Thus depth$(M_P) =$ depth$(R_P) = \dim(R_P)$, since R satisfies S_k. If $\dim(R_P) \geq k+1$, depth$(R_P) = k+i$, for some $i \geq 0$, as R satisfies S_k. Thus $I(\phi_{i+1}) \not\subseteq P$. Hence, p.d.$(M_P) \leq i$, so depth$(M_P) \geq k$. Thus M satisfies S_k. $\Diamond$

Remark. In [4] Bruns proves the following result which shows how to construct kth syzygies of rank k from kth syzygies having rank greater than k. Let (R, m) be a Cohen–Macaulay local ring and M a finitely generated R-module having finite projective dimension. Suppose that M is a kth syzygy having rank $k+s$, for $s \geq 1$. Then there exists a free submodule $F \subseteq M$ such that $F \cap mM = mF$, rank$(F) = s$, and M/F is a kth syzygy. In the corollary below, we use Cor. B to extend Bruns' theorem to non-Cohen–Macaulay rings. In order to do this, we need to observe that choosing basic elements on subsets of Spec(R) determined by depth conditions can be done analogously to the more standard method of choosing basic elements on subsets of Spec(R) determined by height conditions. We follow the treatment given in [6] (which is based upon [4]).

Basic Element Lemma. *Let (R, m) be a local ring and $M \subseteq R^n$ be a finitely generated R-module with well-defined rank $\geq k+1$. Suppose that M_P is a free summand of $(R^n)_P$ for all prime ideals P satisfying* depth$(R_P) \leq k$. *Then there exists a minimal generator $x \in M$ such that x is basic at P for all P satisfying* depth$(R_P) \leq k$.

Proof. We first recall that a submodule $M' \subseteq M$ is said to be *t-fold basic* at P if at least t minimal generators for M_P can be chosen from the image of M'. The proof now follows along the same lines as that of Cor. 2.6 in [6], once we verify the following statement. Let $\{x_1, \ldots, x_s\}$ be a subset of a set of generators for M such that M', the submodule they generate, is t-fold basic at all primes P satisfying depth$(R_P) \leq j-1$. Then M' is t-fold basic at all but finitely many primes P satisfying depth$(R_P) = j$. To see this, suppose that

$\{x_1, \dots, x_s, \dots, x_m\}$ is a set of generators for M and write A for the $n \times m$ matrix whose columns correspond to the x_i. Let A' denote the corresponding submatrix associated to M'. Then for any prime ideal P such that M_P is a summand of $(R^n)_P$, M' is t-fold basic at P if and only if $I_t(A')$, the ideal of $t \times t$ minors of A', is not contained in P. Now, since M' is t-fold basic at all P satisfying depth$(R_P) \leq$ $\leq j-1$, depth$(I_t(A')) \geq j$. If depth$(I_t(A')) \geq j+1$, the statement follows. Otherwise, letting $a_1, \dots, a_j$ be a maximal regular sequence in $I_t(A')$, it follows that $I_t(A') \subseteq P$ and depth$(R_P) = j$ if and only if $P \in \text{Ass}(R/(a_1, \dots, a_j)R)$. Since $\text{Ass}(R/(a_1, \dots a_j)R)$ is finite, the statement follows in this case as well. $\Diamond$

Corollary D. *Let (R, m) be a local ring and M a finitely generated R-module with finite projective dimension. Suppose that M is a kth syzygy having rank $k+s$, for $s \geq 1$. Then there exists a free submodule $F \subseteq M$ such that $F \cap mM = mF$,* rank$(F) = s$, *and M/F is a kth syzygy.*

Proof. We follow the path laid out in Bruns' original theorem. Let $\mathbf{F}$ as above be a projective resolution of M. We may assume that $\mathbf{F}$ satisfies the depth condition of Cor. B. Since depth$(I(\phi_i)) \geq k+1$, M_P is free for all prime ideals P satisfying depth$(R_P) \leq k$. As in the proof of Cor. B, we may use the universal pushforward of M to further assume that $M \subseteq R^n$ for some n, and M_P is a summand of $(R^n)_P$, whenever M_P is free. We now employ the Basic Element Lemma to find a minimal generator $x \in M$ which is basic at all P satisfying depth$(R_P) \leq k$. In particular, Rx is a free submodule of M and without loss of generality we may assume that x is the "first" generator of M. It follows that a minimal resolution for M/Rx has the form

$$\mathbf{F}': 0 \longrightarrow F_n \xrightarrow{\phi_n} \cdots \xrightarrow{\phi_2} F_1 \xrightarrow{\phi_1'} F_0' \longrightarrow M/Rx \longrightarrow 0$$

where ϕ_1' is the matrix obtained from ϕ_1 by deleting the first row and F_0' is the free R-module on one less generator than F_0. Furthermore, the choice of x implies that $(M/Rx)_P$ is free for all primes P satisfying depth$(R_P) \leq k$, so depth$(I(\phi_1')) \geq k+1$. Hence the resolution for M/Rx satisfies the depth condition of Cor. B. That is, M/Rx is a kth syzygy. The process may be repeated if M/Rx has rank greater than k. $\Diamond$

Remark. Unfortunately, Cor. B does not shed a lot of light on the Evans–Griffith Syzygy Theorem (see [6]), which states that kth syzygies with finite projective dimension have rank $\geq k$ (when the ring R

contains a field). Using Cor. B in a manner analogous to its use in Cor. D, one can easily see that it suffices to find a minimal generator x whose order ideal has depth $\geq k$. For then M/Rx would be a $(k-1)$st syzygy, and induction would yield the result. (For rings containing a field such x exists.) This is exactly the original line of thought followed by Evans and Griffith. The point of Cor. B is that one need not have any standing assumption on the ring (like the Cohen–Macaulay property) as long as one replaces height by depth in a characterization of kth syzygies. (See also [7] or [5], where for rings containing a field, the Evans–Griffith estimates on the ranks of syzygies are improved.)

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