Mathematica Pannonica
5/1 (1994), 3 – 6

ERDŐS–MORDELL INEQUALITY FOR SPACE N–GONS

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Received January 1992

AMS Subject Classification: 52 A 15, 53 A 05

Keywords: Erdős–Mordell inequality, special polygons.

Abstract: The Erdős–Mordell inequality is extended on the space closed n-gons in E^3. The inequality holds for any point O of the convex hull of the n-gon. The equality is attained only for regular n-gons with the center O.

H. Ch. Lenhard [3] proved the following statement: *Let $\mathcal{A}$ be a closed n-gon with vertices $A_0, A_1, \dots, A_{n-1}$ bounding a star shaped region in the plane and let O be a point in the interior of $\mathcal{A}$ such that all sides of $\mathcal{A}$ are visible from O. Denote by R_i the distance of the points O and A_i and let r_i denote the distance of O to the line A_iA_{i+1} (where $A_n = A_0$). Then*

$$(1) \qquad \cos\frac{\pi}{n}\sum_{i=0}^{n-1} R_i \geq \sum_{i=0}^{n-1} r_i$$

holds. Equality holds only if $\mathcal{A}$ is a regular n-gon with center O.

The inequality (1) for $n = 3$ is known as the Erdős–Mordell inequality. L. Fejes-Tóth [2] conjectured (1) for convex plane n-gons. Lenhard's result confirms and generalizes Fejes-Tóth's conjecture, because (1) holds even for non-convex n-gons. In the present work we will give a further generalization of (1), which is given in the following

Theorem 1. *Let $\mathcal{A}$ be a closed n-gon in E^3 with vertices $A_0, A_1, \dots, A_{n-1}$ and let O be a point in the convex hull $K(\mathcal{A})$ of $\mathcal{A}$. Denote by*

R_i the distance of the points O and A_i and denote by r_i the distance of O to the line A_iA_{i+1}. Then inequality (1) holds. Equality in (1) is attained iff $\mathcal{A}$ is a plane regular n-gon and O is its center.

To prove Th. 1 we shall need the following lemma:

Lemma. *Let $\mathcal{A}$ be a closed space n-gon in E^3 with vertices $A_0, A_1, \dots, A_{n-1}$ and let O be a point in the convex hull $K(\mathcal{A})$ of $\mathcal{A}$. Writing $\varphi_i = |\sphericalangle A_iOA_{i+1}|$ we have*

$$\sum_{i=0}^{n-1} \varphi_i \geq 2\pi\,. \tag{2}$$

Proof. Our proof is based on the following statement given by I. Fáry [1]: *Let $\vec{u}, \vec{v}$ be two vectors and φ their angle. Denote by $\varphi(\sigma)$ the angle between the orthogonal projections of $\vec{u}$ and $\vec{v}$ in the direction σ. Then*

$$\varphi = \frac{1}{4\pi} \int\limits_{\Omega} \varphi(\sigma)\, d\Omega \quad \sigma \in \Omega,$$

where Ω is a unit sphere and σ its point determined by the direction σ. By applying this statement to the n-gon $\mathcal{A}$ we get

$$\sum_{i=0}^{n-1} \varphi_i = \frac{1}{4\pi} \int\limits_{\Omega} \sum_{i=0}^{n-1} \varphi_i(\sigma) d\Omega\,.$$

This equality reduces the space case to a planar one. To prove (2) it suffices to show that

$$\sum_{i=0}^{n-1} \varphi_i(\sigma) \geq 2\pi \quad \text{for all} \quad \sigma \in \Omega\,.$$

Let $\mathcal{A}_\sigma$ denote the orthogonal projection of $\mathcal{A}$ in the direction σ. From the definition of the convex hull, it follows that the point O belongs to the convex hull $K(\mathcal{A}_\sigma)$ of $\mathcal{A}_\sigma$. The convex hull $K(\mathcal{A}_\sigma)$ of $\mathcal{A}_\sigma$ is a polygon, whose vertices form a subset of the set of vertices of $\mathcal{A}_\sigma$. Because of convexity of $K(\mathcal{A}_\sigma)$, the sum of the angles between O and the vertices of $K(\mathcal{A}_\sigma)$ equals 2π. Our assertion now readily follows. ◊

For the proof of Th. 1 we shall need another geometrical result, known as a discrete case of Wirtinger's inequality: *Let $\mathcal{A}$ be a closed space n-gon in E^3 with vertices $A_0, A_1, \dots, A_{n-1}$ and with the centroid at the origin of the coordinate system. Then*

$$\sum_{k=0}^{n-1} |A_kA_{k+1}|^2 \geq 4\sin^2\frac{\pi}{n} \sum_{k=0}^{n-1} |A_k|^2\,. \tag{3}$$

Equality holds iff $\mathcal{A}$ is a plane affine-regular n-gon. Inequality (3) was stated for plane n-gons by B. H. Neumann [4]. For a proof of the general case see [5], [6].

Proof of Th. 1. We will proceed similarly as H. Ch. Lenhard. It is more convenient to use now the notation $|\sphericalangle A_iOA_{i+1}| = 2\varphi_i$. We will show that even

$$\cos\frac{\pi}{n}\sum_{i=0}^{n-1} R_i \geq \sum_{i=0}^{n-1}\sqrt{R_iR_{i+1}}\cos\varphi_i \geq \sum_{i=0}^{n-1} r_i \tag{4}$$

holds. Namely,

$$|OP_i| = \frac{2R_iR_{i+1}}{R_i+R_{i+1}}\cos\varphi_i\,,$$

where P_i is the intersection of the bisector of the angle R_iOR_{i+1} with the side A_iA_{i+1}. The second inequality in (4) now follows from the inequality between the harmonic and geometric mean, with equality only for $R_i = R_{i+1}$.

To prove the first inequality in (4), construct the central symmetric $2n$-gon $\mathcal{B}$ with vertices $B_0, B_1, \ldots, B_{2n-1}$, with $B_{2n} = B_0$, with the centroid at the point O as the origin of the coordinate system, so that

$$|B_i| = \sqrt{R_i}\,, \quad |\sphericalangle B_iOB_{i+1}| = \varphi_i\,, \quad B_{n+i} = -B_i\,, \quad i = 0, 1, \ldots, n-1\,.$$

Inequality (2) ensures, that this construction always gives at least one $2n$-gon $\mathcal{B}$. By applying the inequality (3) to the $2n$-gon $\mathcal{B}$ we get

$$\sum_{k=0}^{2n-1}|B_kB_{k+1}|^2 \geq 4\sin^2\frac{\pi}{2n}\sum_{k=0}^{2n-1}|B_k|^2\,, \tag{5}$$

which is equivalent to

$$\cos\frac{\pi}{n}\sum_{k=0}^{2n-1}|B_k|^2 \geq \sum_{k=0}^{2n-1}|B_k|\cdot|B_{k+1}|\cos\varphi_k\,.$$

Dividing both sides by 2, we get the left inequality in (4). Equality in (5) is attained if and only if the $2n$-gon is a plane affine-regular one which, together with the condition $R_i = R_{i+1}$, $i = 0, 1, \ldots, n-1$, gives that equality in (1) is attained iff the n-gon $\mathcal{A}$ is regular and O is its center. ◊

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