Mathematica Pannonica
4/2 (1993), 287 – 297

NEAR-RINGS GENERATED BY R-MODULES

James R. **CLAY**

Department of Mathematics, University of Arizona, Building #89, USA – Tucson, Arizona 85721

Hermann **KAUTSCHITSCH**

Institut für Mathematik, Universität für Bildungswissenschaften, A–9022 Klagenfurt, Universitätsstraße 65–67, Austria

Received October 1990

AMS Subject Classification: 16 A 20, 16 A 30, 16 A 96; 13 A 93.

Keywords: Nearring, R-module, composition-ring, radical, formal power series.

Abstract: In this paper a new way of constructing near-rings $(M \times R, +, \circ)$ from a ring R and an R-module M is described. The idea comes from the composition-ring $(R[[x]], +, \cdot, \circ)$ of formal power series by factoring with certain ideals. Sometimes one gets "double rings", i.e. rings with two possibilities for the multiplication for a given addition. If R has an identity, then the group of units with respect to $\circ$ has interesting structures. At least, the ideals and the usual radicals of the constructed near-rings can be described in a nice manner.

1. Introduction

For a commutative ring R with identity 1_R, the formal power series over R, denoted by $R[[x]]$, provide a lot of colour and beauty to mathematics. In particular, if $R_0[[x]]$ denotes the power series of the form

$$(1) \qquad r_1x + r_2x^2 + \ldots + r_nx^n + \ldots$$

then the structure $(R_0[[x]], +, \cdot, \circ)$, where $\circ$ denotes the composition, has the properties that 1) $(R_0[[x]], +, \cdot)$ is a ring, 2) $(R_0[[x]], +, \circ)$ is a right distributive near-ring with identity x, and 3) $(f \cdot g) \circ h = (f \circ h) \cdot \cdot (g \circ h)$. In other words, $(R_0[[x]], +, \cdot, \circ)$ is a tri-operational algebra, or a composition ring [1, 9, 10]. If R is a commutative ring with identity 1_R, then the sets

$$(2) \qquad I_k := \{x^k \cdot a(x) \mid a(x) \in R_0[[x]]\}, \quad k = 1, 2, 3, \ldots$$

are composition-ring-ideals, i.e., the I_k''s are near-ring-ideals of $(R_0[[x]], +, \circ)$ and also ring-ideals of $(R_0[[x]], +, \cdot)$, [7]. If R is a field of characteristic $\neq 2$, then all composition-ideals are of the form (2), so $R_0[[x]]$ has the remarkable property that every near-ring-ideal is also a ring-ideal, [6].

When Gonshor [5] was developing the concept of an abstract affine near-ring, he was the first, who constructed near-rings from a ring R and an R-module M. This has proven to be extremely useful in diverse situations [2]. Other near-ring constructions from a ring R and an R-module M can be found in [3] and [4]. In this work we describe a new way to construct near-rings in such a way which is motivated by the study of the quotient structures of $R_0[[x]]$ by the above ideals I_1, I_2. By factoring with I_3 one can construct near-rings from commutative rings R and R-algebras A. These quotient structures sometimes yield "double rings", i.e., $\cdot$ and $\circ$ may each be used with $+$ to get a ring. If R has an identity, then the group of units with respect to $\circ$ has interesting structures and the ideals in such near-rings have a particularly nice description.

2. Background and development

If we factor $R_0[[x]]$ by I_1 we get by the isomorphism $\tau\colon R_0[[x]]/I_1 \to R$ with $\tau(rx + I_1) = r$ a double ring, namely:

Proposition 1. *For a commutative ring R with identity 1_R, we have*

1) $(R_0[[x]]/I_1, +, \cdot)$ *is the zero-ring;*
2) $(R_0[[x]]/I_1, +, \circ)$ *is a ring isomorphic to* $(R, +, \cdot)$*;*
3) *the group of units in* $(R_0[[x]]/I_1, +, \circ)$ *is isomorphic to the group of units of* $(R, +, \cdot)$.

Factoring $R_0[[x]]$ with I_2 and the isomorphism $\tau(ax+bx^2+I_2) = (b,a)$ lead us to the following definition of $\circ$ on $R \times R$:

$$(b,a)\circ(d,c) = (ad + c^2 b, ac).$$

Together with the componentwise addition we get a near-ring $(R\times R, +, \circ)$. We extend this to: Let R be a commutative ring and let M be a left R-module. On the set $M \times R$ define $+$ componentwise and

$$(a,x)\circ(b,y) = (xb + y^2 a, xy). \tag{3}$$

We get

Theorem 2. *For a commutative ring R and a left R-module M, the structure $(M \times R, +, \circ)$ is a right distributive zero-symmetric near-ring. If R has identity 1_R, then $(0, 1_R)$ is the identity for $M \times R$. $(M \times R, +, \circ)$ is a ring iff $r + r \in (0 : M)$ for all r in R. R boolean implies $\circ$ commutative. The group of units of $M \times R$ with respect to $\circ$ and the identity $(0, 1_R)$ is isomorphic to the semidirect product $M^+ \times {}_\theta\mathcal{U}(R)$ where $\mathcal{U}(R)$ denotes the group of units of the ring R and $[a,x]$ the elements of $M \times U(R)$ with $[a,x]\circ[b,y] = [a + x^{-1}b, xy]$.*

Proof. The proof of each item is direct except that about the group of units. If $(0,1) = (a,x)\circ(b,y) = (xb + y^2a, xy)$, then $y = x^{-1}$ and from $0 = xb + y^2 a$, we get $b = x^{-1}(-y^2 a) = -x^{-3}a$. Hence $(a,x)^{-1} = (-x^{-3}a, x^{-1})$. Let $K = \{(a,1) \mid a \in M\}$. We shall see that K is a normal subgroup of $\mathcal{U}(M \times R)$, the units of the near-ring $M \times R$. For $(a,1),(b,1) \in K$, $(a,1)\circ(b,1) = (a+b,1)$, so K is isomorphic to M^+. Now $(a,x)\circ(b,1)\circ(a,x)^{-1} = (xb+a,x)\circ(-x^{-3}a, x^{-1}) = (-x^{-2}a + x^{-2}(xb+a), 1) = (x^{-1}b, 1)$. Thus K is normal in $\mathcal{U}(M\times R)$. Let $L = \{(0,x) \mid x \in \mathcal{U}(R)\}$. Then $(0,x)^{-1} = (0,x^{-1}) \in L$ and $(0,x)\circ(0,y) = (0,xy) \in L$ if $(0,y)\in L$ also. Hence, $L \cong \mathcal{U}(R)$. Certainly $L \cap K = \{(0,1)\}$. For $(a,x) \in M \times \mathcal{U}(R)$, we have $(x^{-2}a, 1)\circ(0,x) = (a,x)$, so $K \circ L = M \times \mathcal{U}(R)$ and $\mathcal{U}(M\times R) \cong M \times {}_\theta\mathcal{U}(R)$. But what is our $\theta\colon U(R) \to \operatorname{Aut} M$? Looking at $(a,x) = (x^{-2}a,1)\circ(0,x)$, we define the map $T(a,x) = [x^{-2}a, x]$. Then $T[(a,x)\circ(b,y)] = [x^{-1}y^{-2}b + x^{-2}a, xy]$ and $T(a,x)\circ T(b,y) = [x^{-2}a,x]\circ[y^{-2}b,y]$. So, if we define on $M \times \mathcal{U}(R)$ the product $[a,x]\circ[b,y] = [a + x^{-1}b, xy]$, then T becomes an isomorphism onto $(M \times {}_\theta\mathcal{U}(R), \circ)$ where $[a,x]\circ[b,y] = [a + \theta(x)b, xy] = [a + x^{-1}b, xy]$ and $\theta(x)b = x^{-1}b$. $\Diamond$

Corollary 3. *Let R be a commutative ring with identity 1_R, and let A be a commutative R-algebra with identity 1_A. On $A \times R$, define $(a,b) + (c,d) = (a+c, b+d)$, $(a,b)\cdot(c,d) = (bd, 0)$, where $bd = bd1_A$, and $(a,b)\circ(c,d) = (bc + d^2a, bd)$. Then $(A\times R, +, \cdot, \circ)$ is a composition ring* *where* $(A \times R)^3 = 0$ *with respect to* $\circ$. $A \times R$ *is a* double ring, *i.e., both* $(A \times R, +, \cdot)$ *and* $(A \times R, +, \circ)$ *are rings iff* $r + r \in (0 : A)$ *for all* r *in* R. *The following 3 assertions are equivalent:* (i) R *boolean,* (ii) $\circ$ *left distributive over* $\cdot$, (iii) $\circ$ *commutative.*

Analogously we get by the isomorphism $\tau\colon R_0[[x]]/I_3 \to R$ with $\tau(ax + bx^2 + cx^3 + I_3) = (c, b, a)$:

Theorem 4. *Let* R *be a commutative ring with identity* 1_R. *Let* A *be a commutative* R*-algebra with identity* 1_A. *On* $S = A \times A \times R$ *define* $+$ *componentwise, define* $\cdot$ *by*

$$(c, b, a) \cdot (f, e, d) = (ae + db, ad, 0),$$

where $ad = ad1_A$, *and define* $\circ$ *by*

$$(c, b, a) \circ (f, e, d) = (af + 2dbe + d^3c, ae + d^2b, ad).$$

Then $(S, +, \cdot, \circ)$ *is a composition ring with identity* $1_S = (0, 0, 1_R)$ *and* $S^4 = 0$ *with respect to* $\circ$. *The following three assertions are equivalent:* (i) R *boolean,* (ii) $\circ$ *left distributive over* $\cdot$, (iii) $\circ$ *commutative. They imply that* S *is a double ring.*

Let $\mathcal{U}(S)$ *denotes the group of units of* $(S, \circ)$. *Then* $\mathcal{U}(S) = \{(c, b, u) \mid c, b \in A, u \in \mathcal{U}(R))\}$ *and* $(c, b, u)^{-1} = (2u^{-5}b^2 - u^{-4}c, -u^{-3}b, u^{-1})$.

Let $K = \{(c, b, 1) \mid c, b \in A\}$, *and let* $L = \{(0, 0, u) \mid u \in \mathcal{U}(R)\}$, *where* $\mathcal{U}(R)$ *denotes the units of* $(R, \cdot)$. *Then* K *is a normal subgroup of* $\mathcal{U}(S)$, *and* $K \cong A^+ \oplus^F A^+$ *where*

$$(a, b) \oplus^F (c, d) = (a + c + 2bd, b + d);$$

i.e., $F\colon A \times A \to A$ *defined by* $F(b, d) = 2bd$ *is a factor set. Also* $L \cong \mathcal{U}(R)$ *and* $\mathcal{U}(S)$ *is a semidirect product of* K *by* L. *In fact,*

$$\mathcal{U}(S) \cong (A^+ \oplus^F A^+) \times_\theta \mathcal{U}(R)$$

where

$$[(a, b), u] \cdot [(c, d), v] = [(a, b) \oplus^F \theta(u)(c, d), uv]$$

and

$$\theta(u)(c, d) = (u^{-2}c, u^{-1}d).$$

Proof. Again, the proof of the first paragraph of the theorem is direct, but requires careful calculations. We shall turn our attention to proving those items concerning the group of units. One easily sees that $\mathcal{U}(S) = \{(c, b, u) \mid c, b \in A, u \in U(R)\}$, and that $(c, b, u)^{-1} = (2u^{-5}b^2 - u^{-4}c, -u^{-3}b, u^{-1})$. Focusing on the third coordinates, one easily gets that K is a normal subgroup of $\mathcal{U}(S)$, and we easily see that $K =$

$= A^+ \oplus {}^F A^+$ as described in the theorem. So K is an abelian group, and is isomorphic to an extension of A^+ by A^+, but is not a direct sum of A^+ with A^+ ([11]).

One easily gets $L \cong \mathcal{U}(R)$ and $L \cap K = \{(0,0,1)\}$, and from

$$(u^{-3}c, u^{-2}b, 1) \circ (0,0,u) = (c,b,u),$$

that $K \circ L = \mathcal{U}(S)$. Thus $U(S)$ is a semidirect product of K by L. So there is a homomorphism $\theta\colon U(R) \to \mathrm{Aut}(A^+ \oplus A^+)$ such that

$$\mathcal{U}(S) \cong (A^+ \oplus {}^F A^+) \times {}_\theta\mathcal{U}(R).$$

We need to define θ. Define $T\colon \mathcal{U}(S) \to (A^+ \oplus {}^F A^+) \times {}_\theta\mathcal{U}(R)$ by $T(c,b,u) = [u^{-3}c, u^{-2}b, u]$. One gets that T is an isomorphism if $\theta(u)(x,y) = (u^{-2}x, u^{-1}y)$. This defines our required θ. $\Diamond$

One might be tempted to develop the same theory for the composition ring of polynomials over R, namely $(R[x], +, \cdot, \circ)$ with

$$I_K = \{x^k \cdot a \mid a \in R[x]\}.$$

But these I_K''s are not right ideals of $(R[x], +, \circ)$, so one must consider the polynomials with constant term equal to 0, namely $(R_0[x], +, \cdot, \circ)$. Let

$$J_K := \{x^k \cdot a \mid a \in R_0[x]\}.$$

Then $J_K = I_K \cap R_0[x]$. Thus we easily see that the J_K''s are ideals in $(R_0[x], +, \cdot, \circ)$ and that

$$(R_0[x]/J_K, +, \cdot, \circ) \cong (R_0[[x]]/I_K, +, \cdot, \circ).$$

3. Ideals in the near-ring $M \times R$

Our aim is to describe all ideals of $M \times R$. First we start with a submodule N of M. Then

$$I := (N : M) = \{r \in R \mid rM \subseteq N\}$$

is an ideal of R. With this ideal we get

Proposition 5. *Let N be a submodule of M and let J be an ideal of R with $J \subseteq I = (N : M)$. Then $N \times J$ is an ideal of $M \times R$.*

Now we start with an ideal I of R and construct the submodule $N = IM$, the submodule generated by all im, where $i \in I$ and $m \in M$. Since $I \subseteq (N : M)$, we have

Corollary 6. *For an ideal I of R and submodule $N = IM$ of M, we get the ideal $N \times I$ of $M \times R$.*

Theorem 7. *Let K be an ideal of $M \times R$. Define*

$$I(K) = \{i \in R \mid (m,i) \in K \text{ for some } m \in M\},$$

$$N(K) = \{n \in M \mid (n,r) \in K \text{ for some } r \in R\}.$$

Then $I(K)$ is an ideal of R, $N(K)$ is a submodule of M, $I(K) \subseteq (N(K) : M)$, and so $N(K) \times I(K)$ is an ideal of $M \times R$ with $K \subseteq N(K) \times I(K) \subseteq N(K) \times (N(K) : M)$.

Proof. For $i, j \in I(K)$, if $(m,i), (n,j) \in K$, then $(m,i) - (n,j) = (m - n, i - j) \in K$, so $i - j \in I(K)$. For $r \in R$, we have $(0,r) \in M \times R$, so $(m,i) \circ (0,r) = (r^2 m, ir) \in K$, yielding $ir \in I(K)$. Thus $I(K)$ is an ideal of R. For $m, n \in N(K)$, the above argument shows $m - n \in N(K)$. But since $(0,r) \in M \times R$, we get $(0,r) \circ (m,i) = (rm, ri) \in K$, so $rm \in N(K)$, making $N(K)$ a submodule. Certainly $K \subseteq N(K) \times I(K)$. We need only show that $I(K) \subseteq (N(K) : M)$, and then apply Prop. 5 to get that $N(K) \times I(K)$ is an ideal of $M \times R$. Take $i \in I(K)$ and $m \in M$. We proceed to show that $im \in N(K)$. For $(x,i) \in K$, and $(m,1) \in M \times R$, we have $(x,i) \circ (m,1) = (im + x, i) \in K$, yielding $x, im + x \in N(K)$, thus $im \in N(K)$. $\Diamond$

Let K be an ideal of $M \times R$. If $(n,i) \in K$, define

$$N_i(K) := \{m \in M \mid (m,i) \in K\},$$

and

$$I_n(K) := \{j \in R \mid (n,j) \in K\}.$$

By a straightforward computation we get

Proposition 8. *For an ideal K of $M \times R$, and $(n,i) \in K$, $N_0(K)$ is a submodule of M, $I_0(K)$ is an ideal of R, $N_i(K) = n + N_0(K)$, and $I_n(K) = i + I_0(K)$.*

Now define

$$\mathbf{I}(N_0(K)) := \{i \in R \mid N_0(K) \times \{i\} \subseteq K\}$$

and

$$\mathbf{N}(I_0(K)) = \{m \in M \mid \{m\} \times I_0(K) \subseteq K\}.$$

Proposition 9. *For an ideal K of $M \times R$, $\mathbf{I}(N_0(K))$ is an ideal of R and $\mathbf{N}(I_0(K))$ is a submodule of M. Furthermore, $\mathbf{I}(N_0(K)) = I_0(K)$ and $\mathbf{N}(I_0(K)) = N_0(K)$, so $\mathbf{N}$ and $\mathbf{I}$ are inverse operations.*

Proof. It is direct to show that $\mathbf{I}(N_0(K)) = I_0(K)$ and $\mathbf{N}(I_0(K)) = N_0(K)$. Now apply Prop. 8. $\Diamond$

Proposition 10. *If $N_0(K) = N(K)$, then $I_0(K) = I(K)$, and conversely.*

Proof. $N_0(K) \subseteq N(K)$ and $I_0(K) \subseteq I(K)$ always hold. If $N_0(K) = N(K)$, let $i \in I(K)$ and let $m \in M$ be such that $(m,i) \in K$.

So $i \in I_m(K) = i + I_0(K)$. Also $m \in N_i(K) = m + N_0(K)$. But $m \in N(K) = N_0(K)$, forcing $N_i(K) = N_0(K) = N(K)$. Thus, $m \in N_0(K)$, and we have $(m,0),(m,i) \in K$, so $(0,i) \in K$ and $i \in I_0(K)$. So $I_0(K) = I(K)$.

If $I_0(K) = I(K)$, we take $m \in N(K)$. There is an i such that $(m,i) \in K$, so $m \in N_i(K)$ and $i \in I_m(K) = i + I_0(K) = i + I(K) = I(K) = I_0(K)$. This gives $(0,i),(m,i),(m,0) \in K$, so $m \in N_0(K)$, and $N_0(K) = N(K)$. ◊

Since $I_0(K) \subseteq (N_0(K) : M)$, by Prop. 5 $N_0(K) \times I_0(K)$ is an ideal of $M \times R$. Together with Th. 7 we get

Theorem 11. *If K is an ideal of $M \times R$, then*

$$N_0(K) \times I_0(K) \subseteq K \subseteq N(K) \times I(K). \tag{4}$$

Remark 1. *If we have equality in (4) on one side then also on the other side.*

Proof. Let be $K = N_0(K) \times I_0(K)$. If $n \in N(K)$, then $(n,r) \in N_0(K) \times I_0(K)$ for some $r \in R$, hence $n \in N_0(K)$ and $N(K) = N_0(K)$. By Prop. 10 we get $I(K) = I_0(K)$ and $K = N(K) \times I(K)$. If $K = N(K) \times I(K)$ and $r \in I(K)$, then also $(0,r) \in K$, since $N(K)$ is a submodule of M, hence $r \in I_0(K)$ and $I_0(K) = I(K)$. Again by Prop. 10 $N_0(K) = N(K)$ and K is also $N_0(K) \times I_0(K)$. ◊

Remark 2. *In (4) we get equality iff $K = N \times J$ with J an ideal of R with $J \subseteq (N : M)$ and N a submodule of M.*

Proof. If $K = N_0(K) \times I_0(K)$ or $K = N(K) \times I(K)$ then by Theorem 7 and Remark 1, K is of the required form. On the other side, if $K = N \times J$, then by Prop. 5 K is an ideal of $M \times R$ with $N(K) = N$ (for $n \in N$ and $j \in J \trianglelefteq R$ we get $(n,j) \in K$, hence $n \in N(K)$; for $n \in N(K)$ there is some $r \in R$ with $(n,j) \in N \times J$, hence $n \in N$) and $I(K) = J$ (for $j \in J \trianglelefteq R$ $jm \in N$ for all $m \in M$, hence $(jm,j) \in N \times J$ and $j \in I(K)$). So $K = N \times J = N(K) \times I(K)$ and by Remark 1 also $K = N_0(K) \times I_0(K)$. ◊

With the $N(K)$'s and the $I(K)$'s we can describe all ideals of $M \times R$ by:

Theorem 12. *For every ideal K of $M \times R$ we have*

1. $K = \cup_{(m,r)\in K}[N_r(K) \times I_m(K)]$;
2. If $(m,r) \in K$, then $(m,r) + N_0(K) \times I_0(K) =$
$$= N_r(K) \times I_m(K) = (m + N_0(K)) \times (r + I_m(K)).$$

Proof. We need only prove 2. Take $(m+n, r+j) \in N_r(K) \times I_m(K)$. Then $(m,r)+(n,j) \in (m,r)+N_0(K) \times I_0(K)$. If $(m,r)+(n,j) \in \in (m,r)+N_0(K) \times I_0(K)$, then $(m+n, r+j) \in N_r(K) \times I_m(K)$. ◊

Proposition 13. *Let $A = N_0(K) \times I_0(K)$ for an ideal K of $M \times \times R$. Then $M \times R, K, K/A, N(K)/N_0(K)$ and $I(K)/I_0(K)$ are all left R-modules, where $r(m,x) = (rm, rx)$, $r[(m,x)+A] = (rm,rx)+A$, $r(x+N_0(K)) = rx + N_0(K)$, and $r(m+I_0(K)) = rm + I_0(K)$.*

Proof. One only needs to be careful that the action of r on $(m,x)+A$ is well defined, and the rest is routine. ◊

Corollary 14. *The ring R is isomorphic to a subring of the near-ring $M \times R$.*

Proof. $R' = \{(0,r) \mid r \in R\}$ is isomorphic to R. ◊

Theorem 15. *Let $A = N_0(K) \times I_0(K)$, and suppose K is an ideal of $M \times R$ with $A \subset K \subset N(K) \times I(K)$. Define*

$$\pi_M : \frac{K}{A} \to \frac{N(K)}{N_0(K)}, \quad \text{and} \quad \pi_R : \frac{K}{A} \to \frac{I(K)}{I_0(K)}$$

by $\pi_M[(m,j)+A] = m + N_0(K)$, and $\pi_R[(m,j)+A] = j + I_0(K)$. Then π_M and π_R are R-isomorphisms.

Proof. One easily sees that π_M and π_R are well defined. ◊

Corollary 16. *$N(K)/N_0(K) \cong I(K)/I_0(K) \cong K/A$ as R-modules.*

Lemma 17. *If $I(K) = R$, or $I_0(K) = R$, then $K = M \times R$.*

Proof. Since $1 \in R = I(K)$, there is an $m \in M$ such that $(m,1) \in K$. But $(m,1)$ is a unit of $M \times R$, so $M \times R \subseteq K \subseteq M \times R$. ◊

Theorem 18. *If K is a maximal ideal of $M \times R$, then $K = M \times \times I_0(K) = M \times I(K)$ and $I_0(K)$ is a maximal ideal of R. Conversely, if J is a maximal ideal of R, and $K = M \times J$, then K is a maximal ideal of $M \times R$ and $I_0(K) = I(K) = J$.*

Proof. Let us start with J being a maximal ideal of R and $K = M \times J$. Certainly $I_0(K) = I(K) = J$. Suppose L is an ideal of $M \times R$ and

$$K = M \times J \subset L \subseteq M \times R.$$

Then $N_0(L) = N(L) = M$ and $J \subseteq I_0(L) = I(L) \subseteq R$. Since J is maximal, if $J \subset I_0(L)$, then $I_0(L) = R$, so $L = M \times R$ by the lemma. So, $J = I_0(L) = I(L) \subset R$, and $N_0(L) \times I_0(L) = L = N(L) \times I(L)$, or $M \times I_0(L) = L = M \times I(L) = M \times J = K$. Thus K is a maximal ideal of $M \times R$. From

$$N_0(K) \times I_0(K) \subseteq K \subseteq N(K) \times I(K) \subseteq M \times R,$$

either $N(K) \times I(K) = M \times R$ or $K = N(K) \times I(K)$. The former implies $I(K) = R$ and $K = M \times R$, so we must have $K = N(K) \times I(K)$. Thus $N_0(K) = N(K)$ and $I_0(K) = I(K)$, forcing $I(K) \subset R$. Thus

$$K = N(K) \times I(K) \subseteq M \times I(K) \subset M \times R.$$

But $M \times I(K)$ is an ideal, and K is maximal, so $N(K) = M$ and $K = M \times I(K)$. If J is an ideal of R and $I_0(K) = I(K) \subset J \subset R$, then $M \times J$ is an ideal of $M \times R$ and $K = M \times I(K) \subset M \times J \subset M \times R$, a contradiction. Thus $I_0(K) = I(K)$ is a maximal ideal of R. ◊

4. Some radicals for the near-ring $M \times R$

With the material in Section 3, we can easily compute some of the more popular radicals for $M \times R$ if we can compute the corresponding radical for R. Since

$$\mathcal{P}(N) \subseteq \mathcal{N}(N) \subseteq J_0(N) \subseteq J_{1/2}(N) \subseteq J_1(N) \subseteq J_2(N)$$

for any near-ring N, it is tempting to start with $\mathcal{P}(M \times R)$, but since $\mathcal{N}(M \times R)$ is so easy, we'll start there.

For $(a, x) \in M \times R$, $(a, x)^n = (x^{n-1}a + x^n a + \ldots + x^{2(n-1]}a, x^n)$, so (a, x) is nilpotent in $M \times R$ iff x is nilpotent in R. $\mathcal{N}(M \times R)$ is the sum of all nil ideals of $M \times R$. If I is a nil ideal of R, then $M \times I$ is a nil ideal of $M \times R$, so

Theorem 19. $\mathcal{N}(M \times R) = M \times \mathcal{N}(R)$.

Turning now to the $J_\nu(M \times R)$, we have an identity in $M \times R$, so $J_1(M \times R) = J_2(M \times R)$ and every left ideal K of $M \times R$ is modular. From [8, Th. 5.17], $J_1(M \times R)$ is the intersection of all 1-modular left ideals of $M \times R$, and by definition $J_{1/2}(M \times R)$ is the intersection of all 0-modular left ideals of $M \times R$. Then by [8, Th. 5.27], we have $J_0(M \times R)$ as the unique maximal ideal of $M \times R$ contained in $J_{1/2}(M \times R)$. So we see that the ν-modular left ideals of $M \times R$ are significant, for $\nu \in \{0, 1\}$. The 0-modular left ideals of $M \times R$ are exactly the maximal left ideals K of $M \times R$. By a similar argument as in Prop. 5 we see that, for any left ideal K of $M \times R$, $M \times I(K)$ is an ideal of $M \times R$ containing K. So the maximal left ideals of $M \times R$ are again exactly the ideals $M \times J$, where J is maximal in R. We get therefore

Theorem 20. $J_0(M \times R) = J_{1/2}(M \times R) = M \times J(R)$, *where* $J(R)$ *denotes the Jacobson radical of* R.

Proof. $J(R)$ is the intersection of all maximal ideals of R, since R is a commutative ring with 1. By Th. 18, the maximal ideals of $M \times R$ are the $M \times J$, where J is a maximal ideal of R. Now apply the above remark to definition $J_{1/2}(M \times R)$ and then Th. 5.27 of [8]. ◊

Theorem 21. $J_1(M \times R) = J_2(M \times R) = M \times J(R)$.

Proof. The 2-modular left ideals of $M \times R$ are just the maximal ideals $M \times J$, J maximal in R. $\Diamond$

There remains to compute the prime radical $\mathcal{P}(M \times R)$. From [8, Th. 6.27], we know that $\mathcal{P}(M \times R)$ is a nil ideal which contains the sum of all the nilpotent ideals.

Lemma 22. *If J is a nilpotent ideal of R, then $M \times J$ is a nilpotent ideal of R.*

Proof. That $M \times J$ is an ideal of R follows from Prop. 5. Suppose $J^n = \{0\}$. If $(m_1, r_1)(m_2, r_2), \ldots, (m_{2n}, r_{2n}) \in M \times J$, then

$$[(m_1, r_1)\ldots(m_n, r_n)] \circ [(m_{n+1}, r_{n+1})\ldots(m_{2n}, r_{2n})] = (a, 0) \circ (b, 0) = (0, 0).$$

Hence $(M \times R)^{2n} = \{(0, 0)\}$. $\Diamond$

Corollary 23. *Let W denote the ideal of R which is the sum of all nilpotent ideals J of R. Then $M \times W \subseteq \mathcal{P}(M \times R)$.*

Proposition 24. *If an ideal K of $M \times R$ contains $M \times \{0\}$, then $K = M \times I(K)$.*

Proof. In this case $N(K) = M = N_0(K)$. $\Diamond$

Theorem 25. *An ideal K of $M \times R$ is prime iff it is of the form $K = M \times P$ for some prime ideal P of R.*

Proof. Since every prime ideal K of $M \times R$ contains $M \times W$, it has the form $M \times P$ for some ideal P of R. It remains to show that an ideal of this form is prime in $M \times R$ iff P is prime in R: Let P be prime. If A and B are ideals of $M \times R$ and $A \circ B \subseteq M \times P$, then $I(A)I(B) \subseteq P$, so we may assume that $I(A) \subseteq P$. But then $A \subseteq M \times I(A) \subseteq M \times P$. Now let $M \times P$ be prime. If then A and B are ideals of R with $AB \subseteq P$, then $(M \times A) \circ (M \times B) \subseteq M \times P$, so we may assume $M \times A \subseteq M \times P$ and we can conclude that $A \subseteq P$. This makes P prime. $\Diamond$

Corollary 26. $\mathcal{P}(M \times R) = M \times \mathcal{P}(R)$.

References

[1] ADLER, I.: Composition rings, *Duke Math. J.* **29** (1962), 607–625.

[2] CLAY, J.: The fibered product near-ring and near-ring modules for certain categories, *Proc. Edin. Math. Soc.* **23** (1980), 15–26.

[3] CLAY, J.: The near-ring of some one-dimensional noncommutative formal group laws, near-rings and near-fields, *Proc. of Conf. Univ. Tübingen*, 1985; *North-Holland Mathematics Studies* **137** (1987), 41–52.

[4] CLAY, J. and GRAINGER, G.: Endomorphism near-rings of odd generalized dihedral groups, *J. of Algebra* **127** (1989), 320–339.

[5] GONSHOR, H.: On abstract affine near-rings, *Pacific J. Math.* **14** (1964), 1237–1240.

[6] KAUTSCHITSCH, H.: Connections between the ring-, near-ring- and composition-ideals in the algebra of formal power series, *Proc. Colloq. Univ. Algebra (Esztergom*, 1977), *Colloquia Math. Soc. János Bolyai* **29** (1982), 453–458.

[7] KAUTSCHITSCH, H. and MLITZ, R.: Maximal ideals in composition rings of formal power series, *Contributions to General Algebra* **6** (1988), HPT Wien, 131–140.

[8] MELDRUM, J.: Near-ring and their links with groups, *Research Notes in Mathematics* **134**, Pitman, Boston, 1985.

[9] MENGER, K.: Tri-operational algebras, *Reports of a Math. Colloq., 2nd Series, Issue* 5-6, Notre Dame (1944), 3–10.

[10] NÖBAUER, W.: Über die Automorphismen von Kompositionsalgebren, *Acta Math. Acad. Sci. Hungar.* **26** (1975), 275–278.

[11] ROTMAN, J.: Theory of Groups, Allyn and Bacon, Boston, 1984.