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COMMUTATIVITY THEOREM FOR s-UNITAL RINGS

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Abstract: The main result of this paper is a commutativity theorem for associative rings satisfying the polynomial identity $x^t[x^n, y]y^s = \pm[x, y^m]$ (see Th. 1).

1. Introduction

Throughout the present paper R will represent an associative ring (with or without unity 1), $Z(R)$ the center of R, $N(R)$ the set of all nilpotent elements of R, $N'(R)$ the set of all zero divisors of R, and $C(R)$ the commutator ideal of R. A ring R is called left (resp. right) s-unital if $x \in Rx$ (resp. $x \in xR$) for every x in R. Further, R is called s-unital if R is both left and right s-unital, that is $x \in xR \cap Rx$, for every x in R. If R is s-unital (resp. left or right s-unital), then for any finite subset F of R, there exists an element e in R such that $ex = xe = x$ (resp. $ex = x$ or $xe = x$), for every x in F.

For any x, y in R, we write as usual $[x, y] = xy - yx$. For a positive integer n, we consider the following property of a ring R

$Q(n)$: For all x, y in R, $n[x,y] = 0$ implies $[x,y] = 0$.

Obviously, every n-torsion free ring R has the property $Q(n)$ and every ring R has the property $Q(1)$. If a ring R has the property $Q(n)$, then R has the property $Q(m)$ for any factor m of n.

In a recent paper [2] we considered (one-sided) s-unital rings R satisfying

(P): There exists non-negative integers m, n, s and t, $m > 0$ or $n > 0$, and $s \neq t$ for $m = n = 1$ such that $x^t[x^n, y] = \pm x^s[x, y^m]$, for all x, y in R, or $x^t[x^n, y] = \pm[x, y^m]x^s$, for all x, y in R.

Now, our objective is to investigate the commutativity of a ring R which satisfies the polynomial identity

(1) $$x^t[x^n, y]y^s = \pm[x, y^m],$$

for some given non-negative integers m, n, s and t. Since we, as in the case that R has a unity 1, under $x^t y$, resp. xy^s, for $t = 0$, resp. $s = 0$, understand y, resp. x, the above identity take sence also when some of the exponents becomes zero. For $m = n = 0$, or $m = n = 1$ and $s = t = 0$, any ring R satisfies the identity (1), and thus, in this case, she cannot contribute to the commutativity of a ring. Hence, we can exclude the above mentioned values of non-negative integers m, n, s and t. For the remained values we will prove here three theorems. The main result of the present paper is the following

Theorem 1. *Let m, n, s and t be fixed non-negative integers such that $m > 0$ or $n > 0$, and $s > 0$ or $t > 0$ if $m = 1$, $n = 1$. If R is a ring which satisfies the polynomial identity (1), then R is commutative provided that one of the following additional conditions is fulfilled:*

(a) *$m = 0$, and R is an s-unital (resp. a left s-unital for $s = 0$, or a right s-unital for $t = 0$) ring with property $Q(n)$;*

(b) *$n = 0$, and R is a left or right s-unital ring with the property $Q(m)$;*

(c) *$m = 1$, $n \geq 1$, or $m > 1$, $n = 1$ and $s = t = 0$;*

(d) *$m > 1$, $n > 1$, and R is a left or right s-unital ring with the property $Q(m)$;*

(e) *$m > 1$, $n = 1$, $s + t > 0$, and R is a left or right s-unital ring (with the property $Q(m \mp 1)$ for $t = 0$).*

2. Preparation for the proof

In the preparation for the proof of the above theorem, we start by stating without proof the following well-known lemmas.

Lemma 1. ([4, Lemma]). *Let R be a ring with 1, and let f be a polynomial function of two variables such that $f(x+1,y) = f(x,y)$ for all $x, y \in R$. If there exists a positive integer n such that $x^n f(x,y) = 0$ for all $x, y \in R$, then $f(x,y) = 0$ for all $x, y \in R$.*
Lemma 2. ([9, Lemma 3]). *Let x and y be elements in a ring R. If $[x,[x,y]] = 0$, then $[x^k, y] = kx^{k-1}[x,y]$ for all integers $k \geq 1$.*
Lemma 3. ([14, Lemma]). *Let R be a left (resp. right) s-unital ring. If for each pair of elements x and y in R, there exists a positive integer $k = k(x,y)$ and an element $e = e(x,y)$ of R such that $x^k e = x^k$ and $y^k e = y^k$ (resp. $ex^k = x^k$ and $ey^k = y^k$), then R is an s-unital ring.*

An especially important role in proving all results of this paper play the following two results. The first is due to T. P. Kezlan [7, Th.] and H. E. Bell [3, Th. 1] (also see [12, Prop. 2]), and the second was proved by W. Streb [13, Hauptsatz 3].
Theorem KB. *Let f be a polynomial in non-commuting indeterminates $x_1, \ldots, x_n$ with (relatively prime) integral coefficients. Then the following are equivalent:*

1) *For any ring R satisfying the polynomial identity $f = 0$, $C(R)$ is a nil ideal;*
2) *every semi-prime ring R satisfying $f = 0$ is commutative;*
3) *for every prime p, $(GF(p))_2$ fails to satisfy $f = 0$.*

Theorem S. *Let R satisfy a polynomial identity of the form $[x,y] = p(x,y)$, where $p(X,Y) \in \mathbb{Z}\langle X,Y\rangle$, the ring of polynomials in two non-commuting indeterminates over the ring $\mathbb{Z}$ of integers, has the following properties:*

(i) *$p(X,Y)$ is the kernel of the natural homomorphism from $\mathbb{Z}\langle X,Y\rangle$ to $\mathbb{Z}[X,Y]$, the ring of polynomial in two commuting indeterminates;*
(ii) *each monomial of $p(X,Y)$ has total degree at least 3;*
(iii) *each monomial of $p(X,Y)$ has X-degree at least 2, or each monomial of $p(X,Y)$ has Y-degree at least 2.*

Then R is commutative.

Now, we need the following Lemma which enables us to reduce the proof of Th. 1 to ring R with unity 1 (if R is left or right s-unital).
Lemma 4. *Let m, n, s and t be fixed non-negative integers such that $m > 0$ or $n > 0$, and $s > 0$ or $t > 0$ if $m = n = 1$. If a ring R satisfies (1), then R is s-unital in all of the following cases:*

(a′) *$m = 0$, and R is a left s-unital ring for $s = 0$, or a right s-unital ring for $t = 0$;*

(b′) *$n = 0$, and R is a left or right s-unital ring;*

(c′) *$m > 1$, $n > 1$ (or $n = 1$, $s+t > 0$) and R is a left or right s-unital ring.*

Proof. Let x and y be arbitrary elements in R. If R is a left (resp. right)s-unital ring, then we can choose an element e (resp. f) in R such that $ex = x$ and $ey = y$ (resp. $xf = x$ and $yf = y$).

Case (a′): For $m = 0$ the identity (1) reduces to

$$x^t[x^n, y]y^s = 0 \quad \text{for all} \quad x, y \in R. \tag{2}$$

If $s = 0$ and R is left s-unital, then by (2), for $x = e$, we get $y = ye^n$, and thus, R is s-unital. For $t = 0$, and R a right s-unital ring, from (3) we derive $x^n = fx^n$ and $y^n = fy^n$, which by Lemma 3, means that R is also left s-unital.

Case (b′): For $n = 0$, the identity (1) becomes

$$[x, y^m] = 0 \quad \text{for all} \quad x, y \in R. \tag{3}$$

Hence, by (3), $x = xe^m$ (resp. $x = f^m x$) if R is left (resp. right) s-unital and thus, R is s-unital.

Case (c′): If R is left s-unital, then by (1), $x = xe^m - x^{n+t}ex^s + {}$ $+ x^{n+s+t} \in xR$, since $m > 1$ and $n > 1$ (or $n = 1$ and $s+t > 0$). Hence, R is s-unital.

Similarly, one can see that R is s-unital if R is right s-unital. ◊

Further, we prove that, for the ring in Th. 1, $C(R) \subseteq N(R)$. In fact, we prove the following lemma:

Lemma 5. *Let m, n, s and t be fixed non-negative integers such that $m > 0$ or $n > 0$, and $s > 0$ or $t > 0$ if $m = n = 1$. If R satisfies the polynomial identity (1), then the commutator ideal $C(R)$ of R is a nil ideal, i.e. $C(R) \subseteq N(R)$.*

Proof. In view of Th. KB, it suffices to prove that, for every prime p, there exist x, y in the full ring $(GF(p))_2$ of 2×2 matrices over Galois field $GF(p)$ which fail to satisfy the identity (1). Actually, we can take

$$x = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad y = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \text{for} \quad m = 0,$$

and

$$x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad y = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \text{for other cases.} \quad \Diamond$$

For $s = t = 0$, and $m > 1$, $n = 1$, the ring R in Th. 1 is commutative by Herstein's criterion [5, Th. 18] and also by Th. S, and for $m = 1$

and $n \geq 1$, by Th. S, R is commutative for arbitrary non-negative integers s and t (such that $s > 0$ or $t > 0$ if $n = 1$). In all remaining cases, the ring R in Th. 1 is s-unital by Lemma 4. Hence, for these cases, in view of [6, Prop. 1], we can and will assume that R has unity 1. Under this assumption, as the next step in the proof of Th. 1, we have

Lemma 6. *For the ring R in* Th. 1, *all nilpotent elements are central, i.e.* $N(R) \subseteq Z(R)$.

Proof. Take an arbitrary element a in $N(R)$. Then there exists a positive integer p such that

(4) $$a^k \in Z(R) \quad \text{for all integers} \quad k \geq p, \; p \text{ minimal.}$$

If $p = 1$, then $a \in Z(R)$. Suppose that $p > 1$, and set $b = a^{p-1}$. By (4), we have

(5) $$b^k \in Z(R), \quad \text{and} \quad b^k[x,b] = [x,b]b^k = 0 \quad \text{for all } x \in R \quad \text{and all integers} \quad k > 1\,.$$

1) Let $m = 0$ and suppose that R has the property $Q(n)$. Set $1 + b$ for x in (2). In view of (5) and the invertibility of $(1 + b)^t$, we get $n[b,y]y^s = 0$ for all $y \in R$, hence, by Lemma 1, $n[b,y] = 0$ for all $y \in R$. In view of the property $Q(n)$, this yields $[b,y] = 0$ for all $y \in R$, i.e. $a^{p-1} \in Z(R)$, which contradicts to the minimality of p in (4).

2) Let now $n = 0$ and suppose that R has the property $Q(m)$. Set $1 + b$ for y in (3). Then, in view of (5), $m[x,b] = 0$, for all $x \in R$, hence by $Q(m)$, $[x,b] = 0$ for all $x \in R$, i.e. $a^{p-1} \in Z(R)$, which is a contradiction.

3) Let $m > 1$, $n > 1$ and suppose that R has the property $Q(n)$. Then by (5), for $x = b$, the identity (1) gives $[b, y^m] = +b^t[b^n, y]y^s = 0$ for all $y \in R$. Therefore, setting $1 + b$ for x in (1), we get $(1 + b)^t[(1 + b)^n, y]y^s = 0$ for all $y \in R$. In view of (5) and the invertibility of $(1 + b)^t$, this implies $n[b,y]y^s = 0$ for all $y \in R$, hence, by Lemma 1 and the property $Q(n)$, $[b,y] = 0$ for all $y \in R$, i.e. $a^{p-1} \in Z(R)$, and this is a contradiction.

4) Let, finally, $m > 1$, $n = 1$ and $s + t > 0$. If $t > 0$, then setting $1 + b$ for x in (1), we get, in account of (5), $[b,y]y^s = -tb[b,y]y^s + [b,y^m] = -tb[b,y]y^s + b^t[b,y]y^s$, i.e. $b[b,y]y^s = -tb^2[b,y]y^s + b^{t+1}[b,y]y^s = 0$. Hence, $[b,y]y^s = -tb[b,y]y^s + b^t[b,y]y^s = 0$ for all $y \in R$. According to Lemma 1, this yields $[b,y] = 0$ for all $y \in R$, i.e. $a^{p-1} \in Z(R)$. If $t = 0$, then from (1), for $y = 1 + b$, we get $[x,b](1 + sb) = \pm m[x,b]$, i.e. $(m \mp 1)[x,b] = \pm s[x,b]b$, or, by (5), $(m \mp 1)[x,b]b = 0$, i.e. $(m \mp 1)[xb,b] = 0$. In view of $Q(m \mp 1)$, this yields $[x,b]b = 0$, i.e. $(m \mp 1)[x,b] = 0$

for all $x \in R$, and thus, $[x, b] = 0$ for all $x \in R$, i.e. $a^{p-1} \in Z(R)$, a contradiction. ◊

By Lemmas 5 and 6, for the ring R in Th. 1, we have

$$C(R) \subseteq N(R) \subseteq Z(R)\,, \tag{6}$$

hence, especially,

$$[x,[x,y]] = 0 \quad \text{for all} \quad x, y \in R\,. \tag{7}$$

In view of (7) and Lemma 2, the identity (1) can be rewritten in the form

$$nx^{n+t-1}[x,y]y^s = \pm m[x,y]y^{m-1} \quad \text{for all} \quad x, y \in R. \tag{1'}$$

By an argument similar to Lemma 1, it is easily to see, that for a ring R with unity 1 satisfying the identity (1′), and any $x, y \in R$,

$$m[x,y] = 0 \quad \text{if and only if} \quad n[x,y] = 0\,. \tag{8}$$

Especially, for such a ring R, the properties $Q(m)$ and $Q(n)$ are equivalent.

3. Proof of main result and some comments and supplements

Proof of Th. 1. Case (a): Let $m = 0$ and suppose that R has the property $Q(n)$. Then (1′), in view of Lemma 1 and the property $Q(n)$, implies

$$[x,y] = 0 \quad \text{for all} \quad x, y \in R\,.$$

Case (b): If $n = 0$, and R has the property $Q(m)$, then (1′), Lemma 1 and the property $Q(m)$ yield

$$[x,y] = 0 \quad \text{for all} \quad x, y \in R\,.$$

Case (c): The commutativity of R in this case, was established earlier.

Case (d): Let $m > 1$, $n > 1$ and R be a ring with unity having the property $Q(m)$. Since R also satisfies (1′), R has the property $Q(n)$ too. Now, set $1 + x$ for x in (1), and combine the identity (1) with obtained one. Then we get $(1+x)^t[(1+x)^n, y]y^s = x^t[x^n, y]y^s$ for all $x, y \in R$, hence, by Lemma 1, $(1+x)^t[(1+x)^n, y] = x^t[x^n, y]$ for all $x, y \in R$. The last identity implies

$$n[x,y] = f(x,y) \quad \text{for all} \quad x, y \in R, \tag{9}$$

where $f(X, Y)$ is a polynomial satisfying conditions of Th. S. But, the ring R satisfies the identity

(10) $k[x,y]=0$ for all $x,y\in R$, and $k=(2^{n+t}-2)m$.

Namely, setting in (1′), $2x$ for x and combining the identity (1′) with obtained one, we get $k[x,y]y^{m-1}=0$ for all $x,y\in R$, and $k=(2^{n+t}-2)m$, or, by Lemma 1, the identity (10). Now, by (10), there exists a minimal positive integer p such that

(11) $p[x,y]=0$ for all $x,y\in R$.

If $p=1$, then R is commutative. Otherwise, by $Q(n)$, n is relatively prime to p, hence, there exist integers n' and p' such that $1=nn'+pp'$, and thus, in view of (9) and (11),

$$[x,y]=n'f(x,y) \quad \text{for all} \quad x,y\in R.$$

Hence, R is commutative by Th. S.

Case (e): Let $m>1$, $n=1$, $s+t>0$, and let R be a ring with unity 1.

For $t>0$, we can derive (9) as in the case (d). Since now $n=1$, this means that R is commutative (see Th. S).

If $t=0$, then R has the property $Q(m\mp 1)$. In this case, the identity (1′), for $s=m-1$, in view of Lemma 1 and the property $Q(m\mp 1)$, gives

$$[x,y]=0 \quad \text{for all} \quad x,y\in R.$$

For $s\neq m-1$, setting $1+y$ for y in (1′), we get

(12) $(m\mp 1)[x,y]=g(x,y)$ for all $x,y\in R$,

where $g(x,y)$ is a polynomial satysfying the conditions of Th. S. Since now, $s+1\neq m$, from (1′) we can easily derive

(13) $k[x,y]=0$ for all $x,y\in R$, and $k=|2^{s+1}-2^m|n$.

Thus, there exists again a minimal positive integer p for which (11) is satisfied. But then, from (12) and (13) we get, similarly as in the foregoing case,

$$[x,y]=m'g(x,y) \quad \text{for all} \quad x,y\in R,$$

and this, in view of Th. S, yields the commutativity of R. ◊

The following results are immediate consequences of Th. 1.

Corollary 1 ([8, Th.]). *Let m,t be fixed non-negative integers. Suppose that R satisfies the polynomial identity $x^t[x,y]=[x,y^m]$. Then*

a) *if R is left s-unital, then R is commutative except for $(m,t)=(1,0)$;*

b) *if R is right s-unital, then R is commutative except for $m = 1$, $t = 0$; and also $m = 0$, $t > 0$.*

Corollary 2 ([11, Th. 2.]). *Let $m \geq n \geq 1$ be fixed integers with $mn > 1$, and let R be an s-unital ring. Suppose that every commutator $[x, y]$ in R is $m!$-torsion free. If further, R satisfies the polynomial identity $[x^n, y] = [x, y^m]$, then R is commutative.*

Corollary 3 ([1, Lemma 2(2)]). *Let R be a ring with unity and $n > 1$ a fixed positive integer. If R is n-torsion free and satisfies the identity $[x^n, y] = [x, y^n]$, then R is commutative.*

Finally, as complements to Th. 1, we prove the following two theorems, which are similar to Th. 3, resp. Th. 4 in [2].

Theorem 2. *Let R be a left or right s-unital ring which satisfies (1) and has the property $Q(2)$. Suppose that one of the integers $m - s - 1$ and $n + t - 1$ is odd. If, moreover, R has one of the properties $Q(m)$, $Q(n)$, or especially, if $(m, n) = 2^r$ for some non-negative integer r, then R is commutative.*

Proof. If $m-s-1$, resp. $n+t-1$ is an odd integer, then from (1), for $-y$ instead of y, resp. for $-x$ instead of x, one gets $x^t[x^n, y]y^s = \pm[x, y^m]$. This, combined with (1), yields, in view of $Q(2)$,

$$(14) \qquad x^t[x^n, y]y^s = 0, \quad [x, y^m] = 0 \quad \text{for all} \quad x, y \in R.$$

In view of the second part of (14), we see as in the proof of case (b) in Th. 1, that R is s-unital, R has the property $C(R) \subseteq N(R)$ and that

$$(15) \qquad m[x, b] = 0 \quad \text{for all} \quad x \in R,$$

where b is defined as in the proof of Lemma 6. Now, by Lemma 1, from the first part of (14), one gets

$$(16) \qquad x^t[x^n, y] = 0 \quad \text{for all} \quad x, y \in R.$$

Setting $1 + b$ for x in (16), we arrive, in view of (5) and the invertibility of $(1 + b)^t$, at the identity

$$(17) \qquad n[b, y] = 0 \quad \text{for all} \quad y \in R.$$

If R has one of the properties $Q(m)$ and $Q(n)$, or, especially, if $(m, n) = 2^r$ for some non-negative integer r, then from (15) and (17) one can easily derive

$$[b, y] = 0 \quad \text{for all} \quad y \in R, \quad \text{i.e.} \quad a^{p-1} \in R.$$

This contradiction shows that $N(R) \subseteq Z(R)$, and thus R satisfies (6), hence also (7). Therefore, by Lemma 2, the identities in (14) can be rewritten in the form

$$nx^{n+t-1}[x,y]y^s = 0, \quad m[x,y]y^{m-1} = 0 \quad \text{for all} \quad x,y \in R,$$

hence, in view of Lemma 1,

$$n[x,y] = 0, \quad m[x,y] = 0 \quad \text{for all} \quad x,y \in R. \tag{18}$$

From (18), in view of $Q(2)$, follows the commutativity of R, since R has one of the properties $Q(m)$ and $Q(n)$, or especially, $(m,n) = 2^r$ for some non-negative integer r. ◊

Theorem 3. *Let R be a left or right s-unital ring which satisfies* (1). *Suppose that $s \neq m$, resp. $n+t > 1$, and R has the property $Q(k)$, where $k = |2^m - 2^s|$, resp. $k = 2^{n+t} - 2$. Then R is commutative, provided that R has one of the properties $Q(m)$ and $Q(n)$, or, especially, $(m,n) = 2^r \cdot r'$ for some non-negative integer r and some odd divisor r' of k.*

Proof. If $s \neq m$, resp. $n+t > 1$, and R has the property $Q(k)$ for $k = |2^m - 2^s|$, resp. $k = 2^{n+t} - 2$, then from (1), for $2y$ instead of y, resp. for $2x$ instead of x, in view of $Q(k)$, one derives (14). Since k is even, and R has the property $Q(k)$, then R has also the property $Q(2^r r')$ for every non-negative integer r and every odd divisor r' of k. Now, the proof is similar to the proof of Th. 2, and can be omitted. ◊

Remark 1. *If R is a right (resp. left) s-unital ring which satisfies the identity*

$$y^s[x^n,y]x^t = \pm[x,y^m],$$

then the opposite ring R' of R is left (resp. right) s-unital and satisfies the identity (1). Thus all previous results still true if one replaces "left (resp. right) s-unital" by "right (resp. left) s-unital" and the identity (1) by the identity (19).

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