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AN INDIVIDUAL ERGODIC THEOREM FOR RANDOM FUZZY SETS

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Abstract: Random variables with values in the space of fuzzy subsets of a Banach space are studied and an individual ergodic theorem for such variables is proved. To do it the representation of the random fuzzy set by the system of random sets and an embedding theorem is used.

1. Introduction

Limit theorems for random fuzzy sets (or fuzzy-valued random variables) have recently been studied by several authors in two directions. The first one uses the Kwakernaak's definition of fuzzy random variables and has been investigated by Boswell and Taylor [7], Kruse [11], Miyakoshi and Shimbo [12] and others. Another concept of fuzzy random variables (as a generalization of random sets) was defined by Puri and Ralescu [15]. The limit properties of such random variables were studied by Klement, Puri and Ralescu [10], Inoue [9], Ban [4], [5], and others.

This note is a contribution in the second direction. We extend the results of Ban [4] using a different type of convergence for random fuzzy sets. Our results are established for Banach spaces of type p, $p > 1$.

2. Preliminaries

A Banach space $\mathcal{X}$ is of *type p* if there exists a constant $K > 0$ such that

$$E\left\|\sum_{i=1}^{n} f_i\right\|^p \leq K.\sum_{i=1}^{n} E\left\|f_i\right\|^p$$

holds for any independent $\mathcal{X}$-valued random elements $f_1, \ldots, f_n$ with $Ef_i = 0$, where $\|.\|$ is the norm in $\mathcal{X}$. A Banach space $\mathcal{X}$ is said to have the *Radon-Nikodym property* if for each finite measure space $(\Omega, \mathcal{A}, \mu)$ and each μ-continuous $\mathcal{X}$-valued measure $m : \mathcal{A} \to \mathcal{X}$ of bounded variation, there exists a Bochner integrable function $f : \Omega \to \to \mathcal{X}$ such that

$$m(A) = \int_A f d\mu \quad \text{for all} \quad A \in \mathcal{A}.$$

Throughout this paper, let $(\Omega, \mathcal{A}, P, T)$ be a dynamical system with a measure-preserving mapping $T : \Omega \to \Omega$ and a probability measure P. Let $\mathcal{X}$ be a Banach space of type $p, p > 1$ having the Radon-Nikodym property. It is well-known that the set $\mathbb{K}(\mathcal{X})$ of nonempty compact subsets of $\mathcal{X}$ is a complete separable metric space with respect to the *Hausdorff distance* h defined by

$$h(A, B) = \max\left\{\sup_{a\in A}\inf_{b\in B}\|a-b\|,\ \sup_{b\in B}\inf_{a\in A}\|a-b\|\right\} \text{ for } A, B \in \mathbb{K}(\mathcal{X}).$$

The space $\mathbb{K}(\mathcal{X})$ has a linear structure induced by the *Minkowski addition* and *scalar multiplication*:

$$A + B = \{a + b : a \in A, b \in B\} \quad \text{and} \quad \alpha.A = \{\alpha.a : a \in A\}$$

for $A, B \in \mathbb{K}(\mathcal{X})$ and $\alpha \in R$.

By $\operatorname{co}\mathbb{K}(\mathcal{X})$ we denote the space of all nonempty compact convex subsets of $\mathcal{X}$. For every $A \in \mathbb{K}(\mathcal{X})$, the number $|A|$ is defined by $|A| = \sup_{x\in A}\|x\|$ and the *inner radius* $r(A)$ is given by

$$r(A) = \sup_{a \in \mathrm{co}\, A} \inf \{R \geq 0 : \exists a_1, \dots a_n \in A,$$
$$a \in \overline{\mathrm{co}}\{a_1, \dots, a_n\} \text{ and } ||a - a_i|| \leq R\}$$

where $\overline{\mathrm{co}} B$ is the closed convex hull of B, $B \subseteq \mathcal{X}$.

The following property of the Hausdorff distance is very important from our point of view: there exists a constant $K > 0$ such that

$$h(A_1 + \cdots + A_n, \mathrm{co}\,(A_1 + \cdots + A_n)) \leq K^{1/p} \left[r^p(A_1) + \cdots + r^p(A_n)\right]^{1/p}$$

for every $A_1 + \cdots + A_n \in \mathbb{K}(\mathcal{X})$. The above property of the Hausdorff distance is a generalization of the Shapley-Folkman-Starr theorem, and was proved in [14].

A *random set* is a Borel measurable function $F : (\Omega, \mathcal{A}) \to \mathbb{K}(\mathcal{X})$ (in the sense of measurability in metric spaces). A function $f : \Omega \to \mathcal{X}$ is called a *selection of* F if $f(\omega) \in F(\omega)$ a.e. Denote by

$$S_F = \left\{ f \in L^1(\Omega, \mathcal{A}, P; \mathcal{X}) : f \quad \text{is a selection of} \quad F \right\} .$$

Now, for a random compact set F, we define $EF = \{Ef : f \in S_F\}$; (Aumann [2]). Note that, in general, EF may be empty, but if $E|F| < \infty$ then $EF \in \mathbb{K}(\mathcal{X})$. A random compact set F is called *integrably bounded* if there is a nonnegative real-valued integrable function $\xi : \Omega \to R$ such that $||x|| \leq \xi(\omega)$ for all x and ω with $x \in F(\omega)$. The space of all integrably bounded random sets is denoted by $\mathcal{L}(\Omega, \mathcal{X})$.

3. Random fuzzy sets

A *fuzzy subset of* $\mathcal{X}$ is a function $u : \mathcal{X} \to [0,1]$. For each fuzzy subset u, set

$$L_\alpha u = \{x \in \mathcal{X} : u(x) \geq \alpha\}, \quad \alpha \in (0,1] .$$

Let $\mathrm{supp}\, u$ denote the *support* of u, i.e. the closure of the set $\{x \in \mathcal{X} : u(x) > 0\}$. Let $\mathcal{F}(\mathcal{X})$ denote the set of all fuzzy subsets $u : \mathcal{X} \to [0,1]$ such that $L_\alpha u \in \mathbb{K}(\mathcal{X})$ for every $\alpha \in (0,1]$, $\mathrm{supp}\, u \in \mathbb{K}(\mathcal{X})$, and u is upper semicontinuous. Similarly denote by $\mathrm{co}\, \mathcal{F}(\mathcal{X})$ the subspace of $\mathcal{F}(\mathcal{X})$ consisting of those u for which $L_\alpha u$ is compact convex for every $\alpha > 0$. Let $\mathrm{co}\, u$ denote the fuzzy subset such that $\overline{\mathrm{co}} L_\alpha u = L_\alpha(\mathrm{co}\, u)$, $\alpha \in [0,1]$.

The *linear structure* in $\mathcal{F}(\mathcal{X})$ is defined as follows: for u, v in $\mathcal{F}(\mathcal{X})$, $\alpha \in R$ and $x \in \mathcal{X}$ let

$$(u+v)(x) = \sup_{x=y+z} \min[u(y), v(z)]$$

and

$$(\alpha.u)(x) = \begin{cases} u(\alpha^{-1}x) & \text{if } \alpha \neq 0 \\ I_{\{0\}}(x) & \text{if } \alpha = 0 \end{cases}$$

where $I_{\{0\}}$ is the indicator function of the set $\{0\}$.

If $u, v \in \mathcal{F}(\mathcal{X})$, define the *distances* between u and v by

$$d(u,v) = \sup_{\alpha>0} h(L_\alpha u\,, L_\alpha v)$$

and

$$d_1(u,v) = \int_0^1 h(L_\alpha u\,, L_\alpha v)\, d\alpha\,.$$

It is well known that $(\mathcal{F}(\mathcal{X}), d)$ is a complete metric space.

Now, a *random fuzzy set* can be defined as a Borel measurable function $X : (\Omega, \mathcal{A}) \to (\mathcal{F}(\mathcal{X}), d)$. It is easy to show that, for every $\alpha \in (0,1]$, the random set $X^\alpha : \Omega \to \mathbb{K}(\mathcal{X})$ defined by $X^\alpha(\omega) = \{x \in \mathcal{X} : X(\omega)(x) \geq \alpha\}$ is a measurable random compact set. The functions X^α are called *α-cuts* of X. A random fuzzy set X is called *integrably bounded* if X^α is integrably bounded for all $\alpha \in (0,1]$. The space of all integrably bounded fuzzy variables is denoted by $FV(\Omega, \mathcal{F}(\mathcal{X}))$.

The expected value EX of any random fuzzy set X is defined in such a way that the following properties are satisfied:

(i) $EX \in \mathcal{F}(\mathcal{X})$

(ii) $L_\alpha(EX) = \{x \in \mathcal{X} : (EX)(x) \geq \alpha\} = E(X^\alpha)$ for each $\alpha \in [0,1]$.

The proof of the existence and uniqueness of this integral for any integrably bounded random fuzzy set works in the same way as in [15].

Combining [13] and [10] the following lemma can be proved.

Lemma 1. *There exists a normed space $\mathcal{M}$ such that $(\operatorname{co}\mathcal{F}(\mathcal{X}), d_1)$ can be embedded isometrically into $\mathcal{M}$, i.e. there exists an isometry $j : (\operatorname{co}\mathcal{F}(\mathcal{X}), d_1) \to \mathcal{M}$ such that*

(1) *j preserves the linear structure in $\operatorname{co}\mathcal{F}(\mathcal{X})$,*

(2) *$E(j(X)) = j(E(X))$ for every $X \in FV(\Omega, co\mathcal{F}(\mathcal{X}))$ with $E|\operatorname{supp} X| < \infty$ (where $E(j(X))$ is the Bochner integral in $\mathcal{M}$).*

For the formulation of the individual ergodic theorem we need the notion of conditional expectation of the random fuzzy set X with values in $\operatorname{co}\mathcal{F}(\mathcal{X})$. Let $\mathcal{A}_0$ be a sub-σ-algebra of $\mathcal{A}$. An $\mathcal{A}_0$-measurable random fuzzy set Y is called the *conditional expectation of X relative to $\mathcal{A}_0$* if

$$\int_A X\,dP = \int_A Y\,dP \quad \text{for any} \quad A \in \mathcal{A}_0 .$$

The existence and uniqueness almost everywhere of the conditional expectation of any integrably bounded random fuzzy set was shown in [3]. We denote it by $E(X|\mathcal{A}_0)$.

Lemma 2. *Let $X \in FV(\Omega, \operatorname{co}\mathcal{F}(\mathcal{X}))$ with $E|\operatorname{supp} X| < \infty$. Let j be the isometry from Lemma 1, and let $\mathcal{A}_0$ be a sub-σ-algebra of $\mathcal{A}$. Then*

$$E(j \circ X|\mathcal{A}_0) = j \circ E(X|\mathcal{A}_0) \quad a.e.$$

Proof. Since $j \circ \int_A X\,dP = \int_A j \circ X\,dP$, (see [10]), we obtain for any $A \in \mathcal{A}_0$

$$\int_A E(j \circ X|\mathcal{A}_0)\,dP = \int_A j \circ X\,dP = j \circ \int_A X\,dP =$$

$$= j \circ \int_A E(X|\mathcal{A}_0)\,dP = \int_A j \circ E(X|\mathcal{A}_0)\,dP$$

and therefore

$$E(j \circ X|\mathcal{A}_0) = j \circ E(X|\mathcal{A}_0) . \quad \lozenge$$

The main result of this paper is the following theorem.

Theorem. *Let $X \in FV(\Omega, \mathcal{F}(\mathcal{X}))$ with $E|\operatorname{supp} X| < \infty$ and $\mathcal{A}_0 = \{A \in \mathcal{A} : A = T(A)\}$. Then*

$$d_1 - \lim_{n\to\infty} n^{-1} \sum_{i=1}^{n} X(T^i(\omega)) = E(\operatorname{co} X|\mathcal{A}_0) \quad a.e.$$

Proof. First, let us consider the "convex case" i.e. $X(\omega) = \operatorname{co} X(\omega)$. Let $j : \operatorname{co}\mathcal{F}(\mathcal{X}) \to \mathcal{M}$ be the embedding of the space $\operatorname{co}\mathcal{F}(\mathcal{X})$ into a Banach space $\mathcal{M}$. From the construction of the isometry j, it is easy to show that $||j(\operatorname{co} X)||_{\mathcal{M}} \le |\operatorname{supp} \operatorname{co} X|$, which means that

$$E||j(\operatorname{co} X)||_{\mathcal{M}} \le E|\operatorname{supp} \operatorname{co} X| \le E|\operatorname{supp} X| < \infty$$

since $||j(\operatorname{co} X)||_{\mathcal{M}}$ is measurable. Using the individual ergodic theorem in Banach spaces (Beck and Schwartz [6]) and Lemma 1, we have

$$\lim_{n\to\infty} n^{-1} \sum_{i=1}^{n} j \circ \operatorname{co} X(T^i(\omega)) = j \circ E(\operatorname{co} X|\mathcal{A}_0) \quad \text{a.e.}$$

From the properties of j, it follows that

$$\lim_{n\to\infty} n^{-1}\sum_{i=1}^{n} \operatorname{co} X(T^i(\omega)) = E(\operatorname{co} X|\mathcal{A}_0) \quad \text{a.e.}$$

Now,

$$d_1\left(n^{-1}\sum_{i=1}^{n} X(T^i(\omega)),\quad E(\operatorname{co} X|\mathcal{A}_0)\right) \le$$

$$\le d_1\left(n^{-1}\sum_{i=1}^{n} X(T^i(\omega)), n^{-1}\sum_{i=1}^{n} \operatorname{co} X(T^i(\omega))\right) +$$

$$+d_1\left(n^{-1}\sum_{i=1}^{n} \operatorname{co} X(T^i(\omega)),\quad E(\operatorname{co} X|\mathcal{A}_0)\right).$$

The second term converges almost everywhere to zero for $n \to \infty$.

We have to prove that

$$\lim_{n\to\infty} d_1\left(n^{-1}\sum_{i=1}^{n} X(T^i(\omega)), n^{-1}\sum_{i=1}^{n} \operatorname{co} X(T^i(\omega))\right) =$$

$$= \lim_{n\to\infty}\int_0^1 h\left(L_\alpha\left(n^{-1}\sum_{i=1}^{n} X(T^i(\omega))\right),\right.$$

$$\left. L_\alpha\left(n^{-1}\sum_{i=1}^{n} \operatorname{co} X(T^i(\omega))\right)\right) d\alpha = 0.$$

Denote by

$$f_n^\omega(\alpha) = h\left(L_\alpha\left(n^{-1}\sum_{i=1}^{n} X(T^i(\omega))\right), L_\alpha\left(n^{-1}\sum_{i=1}^{n} \operatorname{co} X(T^i(\omega))\right)\right).$$

It is well-known that

$$L_\alpha\left[n^{-1}\sum_{i=1}^{n} X(T^i(\omega))\right] = n^{-1}\sum_{i=1}^{n} L_\alpha[X(T^i(\omega))]$$

(see e.g. [13]) and thus for fixed $n \in \mathbb{N}$, $\omega \in \Omega$ and $\alpha \in (0,1]$ we have

$$f_n^\omega(\alpha) \le K^{1/p}\left[n^{-p}\left[r^p(L_\alpha(X(T^i(\omega)))) + \cdots + r^p(L_\alpha(X(T^n(\omega))))\right]\right] =$$

$$= K^{1/p}\left[n^{-p}\sum_{i=1}^{n} r^p(L_\alpha(X(T^i(\omega))))\right]^{1/p}.$$

The α-cuts $L_\alpha(X(T^i(.)))$ of random fuzzy sets $X(T^i(.))$ are compact-valued random sets and

$$r\left(L_\alpha(X(T^i(\omega)))\right) \leq 2.\left|L_\alpha(X(T^i(\omega)))\right| \leq 2.\left|\operatorname{supp} X(T^i(\omega))\right| .$$

Since $E|\operatorname{supp} X| < \infty$, also $E\, r(L_\alpha(X \circ T^i)) < \infty$ for any $\alpha \in (0,1]$. Moreover, the constant K does not depend on n and thus

$$0 \leq \lim_{n\to\infty} f_n^\omega(\alpha) \leq K^{1/p} \lim_{n\to\infty} \left[n^{-p} \sum_{i=1}^{n} r^p(L_\alpha(X(T^i(\omega)))) \right]^{1/p} = 0$$

by the Birkhoff individual ergodic theorem and the assumption $p > 1$ (applying also the Marcinkiewicz-Zygmund strong law of large numbers). For fixed $\omega \in \Omega$, the functions $f_n^\omega(\alpha)$ are bounded, and we can apply the Lebesgue dominated convergence theorem. We obtain

$$\lim_{n\to\infty} d_1 \left(n^{-1} \sum_{i=1}^{n} X(T^i(\omega)) , n^{-1} \sum_{i=1}^{n} \operatorname{co} X(T^i(\omega)) \right) =$$

$$= \lim_{n\to\infty} \int_0^1 f_n^\omega(\alpha)\, d\alpha = \int_0^1 \lim_{n\to\infty} f_n^\omega(\alpha) d\alpha = 0 \quad \text{a.e.}$$

Thus $\lim\limits_{n\to\infty} d_1 \left(n^{-1} \sum\limits_{i=1}^{n} X(T^i(\omega)) , n^{-1} \sum\limits_{i=1}^{n} \operatorname{co} X(T^i(\omega)) \right) = 0$ a.e. and the proof is complete. $\Diamond$

Remark. The assumption of Radon-Nikodym property of the Banach space $\mathcal{X}$ can be deleted by another formulation of the main theorem. This property is needed to ensure the existence of conditional expectation of a random fuzzy set.

References

[1] ARROW, K. J. and HAHN, F. H.: General Competitive Analysis, Holden-Day, San Francisco, 1971.

[2] AUMANN, R. J.: Integrals of set-valued functions, *J. Math. Anal. Appl.* **12** (1965), 1-12.

[3] BAN, J.: Radon-Nikodym Theorem and Conditional Expectation of Fuzzy-valued Measures and Variables, *Fuzzy Sets and Systems* **34** (1990), 383–392.

[4] BAN, J.: Ergodic theorems for random compact sets and fuzzy random variables in Banach spaces, *Fuzzy Sets and Systems* **44** (1991), 71–82.

[5] BAN, J.: Martingale sequences of fuzzy random variables, *Proc. XI. Prague conference on information theory, statistical decision functions and random process, Prague, 1992*, 271–282.

[6] BECK, A. and SCHWARTZ, J. T.: A vector valued random ergodic theorem, *Proc. Amer. Math. Soc.* **8** (1957), 1049–1059.

[7] BOSWELL, S. B. and TAYLOR, M. S.: A central limit theorem for fuzzy random variables, *Fuzzy Sets and Systems* **24** (1987), 331–344.

[8] HIAI, F.: Radon-Nikodym Theorems for set-valued measures, *J. Multivariate Anal.* **8** (1978), 96–118.

[9] INOUE, H.: A strong law of large numbers for fuzzy random sets, *Fuzzy Sets and Systems* (to appear).

[10] KLEMENT, E. P., PURI, M. L. and RALESCU, D. A.: Limit theorems for fuzzy random variables, *Proc. R. Soc. Lond. A.* **407** (1986), 171–182.

[11] KRUSE, R.: The strong law of large numbers for fuzzy random variables, *Information Science* **28** (1982), 233–241.

[12] MIYAKOSHI, M. and SHIMBO, M.: Individual ergodic theorem for fuzzy random variables, *Fuzzy Sets and Systems* **13** (1984), 258–290.

[13] PURI, M. L. and RALESCU, D. A.: Differentials of fuzzy functions, *J. Math. Analysis and Appl.* **91** (1983), 552–558.

[14] PURI, M. L. and RALESCU, D. A.: Limit theorems for random compact sets in Banach spaces, *Math. Proc. Cambr. Phil. Soc.* **97** (1985), 151–158.

[15] PURI, M. L. and RALESCU, D. A.: Fuzzy Random Variables, *J. Math. Anal. Appl.* **114** (1986), 409–422.

[16] RÅDSTRÖM, H.: An embedding theorem for spaces of convex sets, *Proc. Amer. Math. Soc.* **3** (1952), 165–169.