Mathematica Pannonica
4/1 (1993), 103 – 111

ABSOLUTE END POINTS AND THEIR MAPPING PROPERTIES

Janusz J. Charatonik

Mathematical Institute, Wrocław University, pl. Grunwaldzki 2/4, 50-384 Wrocław, Poland

Paweł Krupski

Mathematical Institute, Wrocław University, pl. Grunwaldzki 2/4, 50-384 Wrocław, Poland

Janusz R. Prajs

Institute of Mathematics, Pedagogical University, ul. Oleska 48, 54–951 Opole, Poland

Received December 1991

AMS Subject Classification: 54 C 10, 54 F 15

Keywords: Absolute end point, arc-like, confluent, continuum, end point, irreducible, local absolute end point, mapping, monotone, open.

Abstract: The concept of an absolute end point of an irreducible continuum is used to introduce the new concept of a local absolute end point of a continuum. Some characterizations of these concepts are obtained and their mapping properties are studied, especially for open, atomic, monotone and related mappings.

1. Introduction

Rosenholtz defined a concept of an absolute end point of an arc-like continuum and showed that such a point of an arc-like continuum is characterized by any of some four equivalent conditions given in ([8], Prop. 1.3, p. 1309). It has been recently observed in [3] that each of these conditions is also equivalent to each of two other conditions and that the scope of application of the definition can be extended to arbitrary irreducible continua, not necessarily arc-like ones (see below, Th. 3.1).

The extended notion of an absolute end point is applied in the present paper to define a local absolute end point of an arbitrary continuum. Some necessary and sufficient conditions are obtained under which a given point of a continuum is either a local absolute end point or an absolute end point.

The concept of an absolute end point is related to that of an end point introduced by Bing in Section 5 of [1], p. 660–662 (see [8], Remark, p. 1310) which is known to be invariant under confluent (thus under open as well as under monotone) mappings (see [2], Lemma, p. 172). It is known that an arc-like continuum is invariant under monotone ([1], Th. 3. p. 654) and under open mappings ([7], Th. 1.0, p. 259; compare also [6], Th. 1, p. 77), and that an irreducible continuum is preserved under monotone mappings (see e.g. [5], §48, I. Th. 3, p. 192). Therefore, it seems to be quite natural to know whether such mappings also preserve (local) absolute end points. This question is discussed in Section 4. Theorems of this sort are widely known to be useful in the study of various mapping properties of topological spaces in continua theory.

2. Preliminaries

All spaces considered in this paper are assumed to be metric and nondegenerate, and all mappings are continuous. For all undefined herein notions the reader is referred to books [5] and [9] and to Rosenholtz's paper [7]. Following Kuratowski [5], §49, I. p. 227, we say that a space X is *locally connected at a point* p of X provided that for each open set U of X containing p, the point p lies in the interior of a connected subset of U. Note that some other authors use name "connected im kleinen" in the same sense. By a *composant* of a point y in a continuum X we mean the union of all proper subcontinua of X containing y. A *simple triod* means the union of three arcs (called *arms*) emanating from a common end point and mutually disjoint outside the point. The other end points of the arcs are called *ends* of the triod. A continuum is called *irreducible* if it is irreducible between some pair of its points.

A mapping $f : X \to Y$ from a continuum X onto a continuum Y is said to be:

(a) *monotone*, provided that the inverse image of each subcontinuum of Y is connected;

(b) *interior at a point* $p \in X$, provided for every open neighbourhood U of p in X the point $f(p)$ is in the interior of $f(U)$;

(c) *open*, if for each open subset of X its image under f is an open subset of Y;

(d) *confluent*, if for each subcontinuum Q of Y each component of $f^{-1}(Q)$ is mapped onto Q by f;

(e) *light*, if $\dim f^{-1}(y) = 0$ for each $y \in Y$;

(f) *atomic*, provided that for each subcontinuum K of X either $f(K)$ is degenerate, or $f^{-1}(f(K)) = K$.

We shall collect several known properties of mappings of continua which will be needed in the sequel. For the proof of the first of them see [5]. §48. I, Th. 3, p. 192.

Proposition 2.1. *If a continuum X is irreducible between points p and q and a surjection $f : X \to Y$ is monotone, then the continuum Y is irreducible from $f(p)$ to $f(q)$.*

The next two propositions are immediate consequences of the definitions.

Proposition 2.2. *A mapping is open if and only if it is interior at each point of its domain.*

Proposition 2.3. *Let a continuum X be locally connected at a point p. If a mapping f defined on X is interior at p, then $f(X)$ is locally connected at $f(p)$.*

Further, the following results are known.

Proposition 2.4. *Each atomic mapping of a continuum is monotone* (see [4], Th. 1, p. 49).

Proposition 2.5. *Open mappings of compact spaces are confluent* (see [9], Th. 7.5, p. 148)

Proposition 2.6. *Open mappings preserve arc-likeness of continua* (see [7], Th. 1.0, p. 259; compare also [6], Th. 1, p. 77).

3. Characterizations of absolute end points and of local absolute end points

A point p of a continuum X is called an *absolute end point* of X if $X \setminus \{p\}$ is a composant (of some point) in X. Observe that a continuum with an absolute end point must be nondegenerate. Th. 3.1 below is a part of a result in [3].

Theorem 3.1. *The following conditions on a continuum X and a point p of X are equivalent:*

(1) *p is an absolute end point of X;*

(2) *X is irreducible between p and some other point, and X is locally connected at p;*

(3) *X is irreducible, and if X is irreducible between points x and y, then either x or y is p.*

The next proposition will lead to a new characterization of absolute end points in the realm of irreducible continua (see Th. 3.4).

Proposition 3.2. *If a continuum X contains an absolute end point p, then for each nondegenerate subcontinuum K of X the condition $p \in K$ implies $p \in \operatorname{int} K$.*

Proof. By Th. 3.1 the continuum X is irreducible from p to some other point $q \in X$. Let a nondegenerate subcontinuum K of X contain p. Then there is a point x such that $x \in K \setminus \{p\} \subset X \setminus \{p\}$. Since $X \setminus \{p\}$ is a composant in X, there exists a proper subcontinuum C of X containing both x and q, with $C \subset X \setminus \{p\}$. Since $p \in K$, $q \in C$ and $x \in K \cap C$, we conclude $K \cup C = X$ by the irreducibility of X between p and q. So $X \setminus C \subset K$. Since $C \subset X \setminus \{p\}$, the difference $X \setminus C$ is a nonempty open subset of K containing the point p, and thus $p \in \operatorname{int} K$. ◊

A point p of a continuum X is called a *local absolute end point* of X provided there is a subcontinuum K of X such that $p \in \operatorname{int} K$ and p is an absolute end point of K.

Theorem 3.3. *The following conditions on a continuum X and a point p of X are equivalent:*

(a) *p is a local absolute end point of X;*

(b) *there is a subcontinuum K of X irreducible between p and some other point of X, such that K is locally connected at p and $p \in \operatorname{int} K$;*

(c) *for each nondegenerate subcontinuum K of X if $p \in K$ then $p \in \operatorname{int} K$;*

(d) *p is a local absolute end point of each nondegenerate subcontinuum Y of X containing p.*

Proof. Conditions (a) and (b) are equivalent by Th. 3.1. Condition (a) implies (c) by Prop. 3.2. Assume (c) is satisfied. Consider a subcontinuum Y of X with $p \in Y$. Let a subcontinuum K of Y be irreducible between the point p and some other point of Y. Then (c) obviously implies the local connectedness of K at p, and thus (b) is satisfied for $X = Y$. By the above mentioned equivalence of (a) and (b) (still for $X = Y$), we get (d), which obviously implies (a). ◊

Theorem 3.4. *A local absolute end point of a continuum X is an absolute end point of X if and only if X is irreducible.*

Proof. One implication is a consequence of Th. 3.1. To show the other one, assume that the continuum X is irreducible between points a and b, and let a point p be a local absolute end point of X. Suppose $a \neq p \neq b$. Denote by A and B subcontinua of X which are irreducible between p and a and between p and b, respectively. By the implication (a) $\Longrightarrow$ $\Longrightarrow$ (c) of Th. 3.3, the continua A and B are both locally connected at p. Thus $p \in \operatorname{int} A \cap \operatorname{int} B$, whence there is a point $q \in A \cap B \setminus \{p\}$. Since p is an absolute end point of each of the continua A and B, by the implication (2) $\Longrightarrow$ (1) of Th. 3.1, we have a subcontinuum A' of A and a subcontinuum B' of B such that $a, q \in A'$, $b, q \in B'$, and $p \in X \setminus \setminus (A' \cup B')$. Hence X is not irreducible between a and b, a contradiction. Consequently, either $a = p$ or $b = p$, and thus p is an absolute end point of X according to the implication (3) $\Longrightarrow$ (1) of Th. 3.1. ◊

As an immediate consequence of Th. 3.4 we have the following result.

Corollary 3.5. *If p is a local absolute end point of a continuum X, then p is an absolute end point of each irreducible nondegenerate subcontinuum of X containing p.*

Note that a similar heredity with respect to subcontinua was proved for arc-like continua as the final part of Th. 1.0 of [8], p. 1308.

4. Mapping properties of absolute end points

Let us accept the following definition. A mapping $f : X \to Y$ between continua X and Y is said to be *partially confluent at a point* $p \in X$ provided that for each nondegenerate subcontinuum Q of Y such that $f(p) \in Q$ the component of $f^{-1}(Q)$ containing the point p is nondegenerate. Note that every confluent mapping $f : X \to Y$ of a continuum X onto Y is obviously partially confluent at each point of its domain just by the definition and that if $f : X \to Y$ is partially confluent at $p \in X$ and $g : Y \to Z$ is partially confluent at $f(p) \in Y$, then the composition $gf : X \to Z$ is partially confluent at p.

The main result of this section of the paper is the following theorem.

Theorem 4.1. *Let a surjective mapping $f : X \to Y$ between continua X and Y be both interior at a point p of X and partially confluent at p. If p is a local absolute end point of X, then $f(p)$ is a local absolute end point of Y.*

Proof. To show that $f(p)$ is a local absolute end point of Y we shall use Th. 3.3. Consider a nondegenerate subcontinuum Q of Y containing the point $f(p)$. We must show $f(p) \in \operatorname{int} Q$. Take the component K of $f^{-1}(Q)$ with $p \in K$ and observe that K is nondegenerate by the assumption of partial confluence of f at p. Since p is a local absolute end point of X, we conclude $p \in \operatorname{int} K$ by Th. 3.3. Now, since f is interior at p, it follows that $f(p) \in \operatorname{int} f(\operatorname{int} K) \subset \operatorname{int} f(K) \subset \operatorname{int} Q$, and thereby the proof is completed. $\Diamond$

Our next result concerns atomic mappings.

Proposition 4.2. *Let a surjection $f : X \to Y$ between continua X and Y be atomic. Then f is interior at each absolute end point of X.*

Proof. Let an absolute end point p of X be given. Observe that since X is irreducible between the point p and some other point q in X by Th. 3.1, the continuum Y is irreducible between $f(p)$ and $f(q)$ by Props. 2.4 and 2.1. Given a neighbourhood U of p in X, let C be a nondegenerate continuum in U satisfying $p \in C$ and $f(q) \in Y \setminus f(C)$. Since $X \setminus \{p\}$ is a composant in the (irreducible) continuum X, there is a continuum $K \subset X \setminus \{p\}$ with $K \cap C \neq \emptyset$ and $q \in K$. Then $f(K)$ is nondegenerate. As f is atomic we have $f^{-1}(f(K)) = K$, and thus $f(p) \in Y \setminus f(K)$. Since $f(C) \cap f(K) \neq \emptyset$, and since Y is irreducible between $f(p) \in f(C)$ and $f(q) \in f(K)$, we have $Y = f(C) \cup f(K)$ and

$$f(p) \in Y \setminus f(K) = f(C) \setminus f(K) = \operatorname{int}(f(C) \setminus f(K)) \subset$$
$$\subset \operatorname{int} f(C) \subset \operatorname{int} f(U). \qquad \Diamond$$

The next corollary is an easy consequence of Ths. 3.4, 4.1 and Props. 2.1–2.6 and 4.2.

Corollary 4.3. *Assume p is a local absolute end point of a continuum X and $f : X \to Y$ is a surjection. Then:*

The point $f(p)$ is a local absolute end point of Y if

(i) *f is both confluent and interior at p, or*

(ii) *f is open.*

The point $f(p)$ is an absolute end point of Y in each of the following cases:

(iii) *f is open and X is arc-like;*
(iv) *f is both confluent and interior at p, and X and Y are irreducible;*
(v) *f is open and X and Y are irreducible;*
(vi) *f is both monotone and interior at p, and X is irreducible;*
(vii) *f is atomic and X is irreducible.*

Remark 4.4. The two assumptions made on the mapping f in Th. 4.1 are independent in the sense that neither of them implies the other one. To see this we consider the following two easy examples.

Example 4.5. *There is a light retraction of an arc which is interior at both end points of the domain and which maps one end point onto an interior point of the range.*

Proof. The (orthogonal) projection of the arc pr onto the subarc pq pictured in Fig. A is such a mapping. ◊

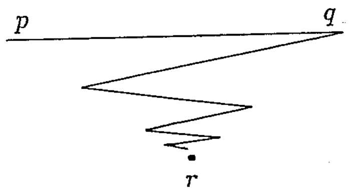

Fig. A

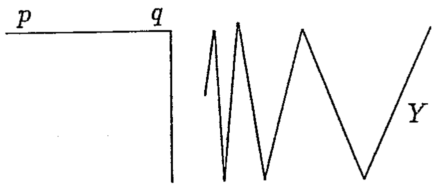

Fig. B

Example 4.6. *There are an arc-like continuum X with an absolute end point p and a monotone retraction f of X such that $f(p)$ is not an absolute end point of $Y = f(X)$.*

Proof. Take the one-point union X of the topologist's sine curve Y and an arc pq (a homeomorphic copy of X is illustrated in Fig. B). Then both $X = pq \cup Y$ and Y are arc-like continua. Let $f : X \to Y$ be the monotone retraction that shrinks the arc pq to the point q. Then p is an absolute end point of X, while $f(p) = q$ is not an absolute end point of Y. ◊

Note that the mapping f, being monotone, is partially confluent at p and, by its definition, is not interior at the point p.

Remark 4.7. Ex. 4.5 also shows that partial confluence of f at the point p is an essential assumption in Th. 4.1. Similarly, Ex. 4.6 shows that f being interior at p cannot be omitted in the assumptions of this theorem.

Remark 4.8. One cannot substitute "absolute end point" for "local absolute end point" both in the assumption and in the conclusion of Th. 4.1. Indeed, let f be the mapping from an arc pq onto a simple triod with end points a, b, c and the center d as pictured in Fig. C.

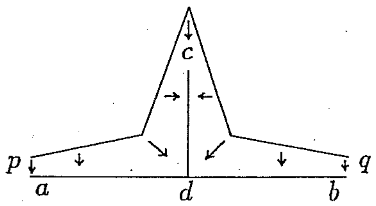

Fig. C

Then the mapping f is interior at p and partially confluent at p and the point p is an absolute end point of pq, while the image $f(p) = a$ is not an absolute end point of the triod.

Remark 4.9. The irreducibility of the range continuum Y is essential in parts (iv) and (v) of Cor. 4.3. The following example shows this.

Example 4.10. *There exists an open, light, surjective mapping $f : X \to Y$ of an irreducible continuum X with an absolute end point such that the range continuum Y is not irreducible (and thus contains no absolute end point).*

Proof. Let C be the standard Cantor ternary set in the unit interval $[0,1]$, and let p_i, q_i for $i \in \mathbb{N}$ be the end points of all components of $[0,1] \setminus C$ such that $p_i < q_i$ and that the sets $\{p_{3n} : n \in \mathbb{N}\}$, $\{p_{3n+1} : n \in \mathbb{N}\}$ and $\{p_{3n+2} : n \in \mathbb{N}\}$ are dense in C. Denote by T a simple triod with center r and ends a, b and c. Take the Cartesian product $C \times T$ and consider the quotient mapping $g : C \times T \to X_0$ which identifies for each $n \in \mathbb{N}$ all pairs of points of the form: (p_{3n}, a) with (q_{3n}, a), (p_{3n+1}, b) with (q_{3n+1}, b), and (p_{3n+2}, c) with (q_{3n+2}, c), and only these pairs (i.e., g is one-to-one out of them). Observe that for each $t_1, t_2 \in T$ the continuum X_0 is irreducible between $g(0, t_1)$ and $g(1, t_2)$. Further, put $r_0 = g(0, r)$, take an arc $r_0 s$ such that $r_0 s \cap X_0 = \{r_0\}$, and define $X = X_0 \cup r_0 s$. Note that X is an irreducible continuum having s as its absolute end point.

Finally consider an arbitrary homeomorphism $h : r_0 s \to ra \subset T$ such that $h(r_0) = r$ and a mapping $f : X \to T$ defined by $f(x) = t$ if $x = g(y, t)$ for some $y \in C$, and $f(x) = h(x)$ if $x \in r_0 s$. It is easy to see that the mapping f is open and light. Since its range T is not an irreducible continuum, $f(s)$ is not an absolute end point of T. The proof is completed. ◊

Remark 4.11. That the mapping f be both interior at p and monotone is an essential assumption in part (vi) of Cor. 4.3 because Exs. 4.5 and 4.6 show that even retractions of arc-like continua do not preserve absolute end points.

References

[1] BING, R. H.: Snake-like continua, *Duke Math. J.* **18** (1951), 653-663.

[2] CHARATONIK, J. J.: Generalized homogeneity and some characterizations of solenoids, *Bull. Polish Acad. Sci. Math.* **31** (1983), 171-174.

[3] CHARATONIK, J. J.: Absolute end points of irreducible continua, *Math. Bohemica* **118** (1993), 19–28.

[4] EMERYK, A. and HORBANOWICZ, Z.: On atomic mappings, *Colloq. Math.* **27** (1973), 49-55.

[5] KURATOWSKI, K.: Topology, Vol. II, Academic Press and PWN, 1968.

[6] ODINTSOV, A. A.: Open images of cyclicly snake-like continua, (Russian) *Vestnik Moskov. Univ. Ser I. Mat. Mekh.* No. 5 (1989), 76-78.

[7] ROSENHOLTZ, I.: Open maps of chainable continua *Proc. Amer. Math. Soc.* **42** (1974), 258-264.

[8] ROSENHOLTZ, I.: Absolute endpoints of chainable continua, *Proc. Amer. Math. Soc.* **103** (1988), 1305-1314.

[9] WHYBURN, G. T.: Analytic topology, *Amer. Math. Soc. Colloq. Publications* **28**, Providence RI, 1963.