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WELL – CHAINEDNESS CHARACTERIZATIONS OF CONNECTED RELATORS

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Abstract: We show that the properties of uniform, proximal and topological connectedness of a relator (generalized uniformity) can be characterized in terms of the well-chainedness of some of its natural symmetrizations.

1. Introduction

In [8] , Mrówka and Pervin proved that a uniformity is uniformly (proximally) connected if and only if it is well-chained.

Moreover, in [5], Levine proved that a uniformity $\mathcal{U}$ on X is well-chained if and only if X^2 is the only equivalence in $\mathcal{U}$

Now, as some natural extensions of the above results, we prove that a relator (i.e., a generalized uniformity) $\mathcal{R}$ is uniformly, resp. proximally connected if and only if the relators $\mathcal{R}\nabla\mathcal{R}^{-1}$ ($\mathcal{R} \odot \mathcal{R}^{-1}$) resp. $\mathcal{R} \vee \mathcal{R}^{-1}$ ($\mathcal{R} \circ \mathcal{R}^{-1}$) are well-chained. Moreover, a relator $\mathcal{R}$ on X is uniformly, proximally and topologically connected if and only if X^2 is the only equivalence in, $\mathcal{R}^*$, $\mathcal{R}^\#$ and $\hat{\mathcal{R}}$, respectively.

The necessary prerequisites concerning relators, which are possibly unfamiliar to the reader, are briefly laid out in the next three preparatory sections.

2. A few basic facts on relators

If $\mathcal{R}$ is a nonvoid family of reflexive relations on a set X, then $\mathcal{R}$ is called a *relator* on X, and the pair $(\mathcal{R}) = (X, \mathcal{R})$ is called a *relator space* [10].

If A and B are sets and x is a point in a relator space $X(\mathcal{R})$, then we write

(i) $B \in \operatorname{Int}_{\mathcal{R}}(A)$ $(B \in \operatorname{Cl}_{\mathcal{R}}(A))$ if $R(B) \subset A$ $(R(B) \cap A \neq \emptyset)$ for some (all) $R \in \mathcal{R}$;

(ii) $x \in \operatorname{int}_{\mathcal{R}}$ $(x \in \operatorname{cl}_{\mathcal{R}}(A))$ if $\{x\} \in \operatorname{Int}_{\mathcal{R}}(A)$ $(\{x\} \in \operatorname{Cl}_{\mathcal{R}}(A))$.

If $\mathcal{R}$ is a relator on X, then the relators

$$\mathcal{R}^* = \{S \subset X^2 : \exists R \in \mathcal{R} : R \subset S\},$$
$$\mathcal{R}^\# = \{S \subset X^2 : \forall A \subset X : \exists R \in \mathcal{R} : R(A) \subset S(A)\}$$

and

$$\hat{\mathcal{R}} = \{S \subset X^2 : \forall x \subset X : \exists R \in \mathcal{R} : R(x) \subset S(x)\}$$

are called the *uniform*, *proximal* and *topological refinements* of $\mathcal{R}$, respectively.

Namely, for instance, $\mathcal{R}^\#$ and $\hat{\mathcal{R}}$ are the largest relators on X such that $\operatorname{Int}_{\mathcal{R}^\#} = \operatorname{Int}_{\mathcal{R}}$ and $\operatorname{int}_{\hat{\mathcal{R}}} = \operatorname{int}_{\mathcal{R}}$, respectively.

Finally, if $\mathcal{R}$ is a relator, then the relator $\mathcal{R}^{-1} = \{R^{-1} : R \in \mathcal{R}\}$ is called the *inverse* of $\mathcal{R}$, and the relators

$$\mathcal{R} \nabla \mathcal{R}^{-1} = \{R \cup R^{-1} : R \in \mathcal{R}\} \ (\mathcal{R} \odot \mathcal{R}^{-1} = \{R \circ R^{-1} : R \in \mathcal{R}\})$$

and

$$\mathcal{R} \vee \mathcal{R}^{-1} = \{R \cup S^{-1} : R, S \in \mathcal{R}\} \ (\mathcal{R} \circ \mathcal{R}^{-1} = \{R \circ S^{-1} : R, S \in \mathcal{R}\})$$

are called the *strong* and *proper union (composition) symmetrizations* of $\mathcal{R}$, respectively.

3. Proximally and topologically open sets

Definition 3.1. A subset A of a relator space $X(\mathcal{R})$ is called

(i) *proximally open (closed)* if $A \in \operatorname{Int}_{\mathcal{R}}(A)$ $(X \setminus A \notin \operatorname{Cl}_{\mathcal{R}}(A))$;

(ii) *topologically open (closed)* if $A \subset \operatorname{int}_{\mathcal{R}}(A)$ ($\operatorname{cl}_{\mathcal{R}}(A) \subset A$);
(iii) *proximally (topologically) clopen* if it is both proximally (topologically) closed and open.

Remark 3.2. It is easy to see that a proximally open (closed) set is topologically open (closed). Moreover, a set is proximally (topologically) closed if and only if its complement is proximally (topologically) open.

Definition 3.3. A relator $\mathcal{R}$ on X, or a relator space $X(\mathcal{R})$, is called *proximally symmetric* if $\mathcal{R}^{-1} \subset \mathcal{R}^{\#}$.

Remark 3.4. In [12], it has been proved that a relator $\mathcal{R}$ is proximally symmetric if and only if the relation $\operatorname{Cl}_{\mathcal{R}}$ is symmetric. Hence, it is clear that a subset of a proximally symmetric relator space is proximally closed if and only if its complement is proximally closed.

Therefore, as an immediate consequence of the second assertions of Remarks 3.2 and 3.4, we can also state

Proposition 3.5. *A subset of a proximally symmetric relator space is proximally open (closed) if it is proximally clopen.*

4. Connected and well-chained relators

Definition 4.1. A relator $\mathcal{R}$ on a set X is called *connected (well-chained)* if

$$A^2 \cup (X \setminus A)^2 \notin \mathcal{R} \qquad \left(X^2 = \bigcup_{n=1}^{\infty} R^n \right)$$

for all $\emptyset \neq A \subsetneq X$ ($R \in \mathcal{R}$). Moreover, $\mathcal{R}$ is called *uniformly*, *proximally* and *topologically connected (well-chained)* if the relators $\mathcal{R}^*$, $\mathcal{R}^{\#}$ and $\hat{\mathcal{R}}$ are connected (well-chained), respectively.

The appropriateness of these unusual definitions and the validity of the next fundamental theorems have been established in [2] and [3].

Theorem 4.2. *A relator $\mathcal{R}$ on X is proximally (topologically) connected if no proper nonvoid subset of $X(\mathcal{R})$ is proximally (topologically) clopen.*

Remark 4.3. In [2], it has also been proved that a uniformly directed relator is proximally connected if and only if it is uniformly connected.

Theorem 4.4. *A relator $\mathcal{R}$ on X is well-chained (topologically well-chained) if and only if no proper nonvoid subset of $X(\mathcal{R})$ is proximally (topologically) open.*

Remark 4.5. Hence, it is easy to infer that a relator is well-chained if and only if it is uniformly (proximally) well-chained.

From Th. 4.2 and 4.4, by Prop. 3.5, it is clear that we also have

Theorem 4.6. *A proximally symmetric relator is well-chained if and only if it is proximally connected.*

Remark 4.7. By Th. 4.2 and 4.4, it is also clear that a well-chained (topologically well-chained) relator is always proximally (topologically) connected.

5. Well-chainedness characterizations of uniform connectedness

To easily get the main result of this section, we need only prove

Proposition 5.1. *If $\mathcal{R}$ is a relator on X and $A \subset X$, then the following assertions are equivalent:*

(i) $E_A = A^2 \cup (X \setminus A)^2 \in \mathcal{R}^*$;

(ii) *A is a proximally open subset of $X(\mathcal{R} \nabla \mathcal{R}^{-1})$ ($X(\mathcal{R} \odot \mathcal{R}^{-1})$).*

Proof. If (i) holds, then there exists an $R \in \mathcal{R}$ such that $R \subset E_A$. Hence, since E_A is an equivalence on X such that $E_A(A) = A$, it is clear that

$$(R \cup R^{-1})(A) \subset A \quad \text{and} \quad (R \circ R^{-1})(A) \subset A .$$

Therefore, (ii) also holds.

Conversely, if (ii) holds, then there exists an $R \in \mathcal{R}$ such that

$$(R \cup R^{-1})(A) \subset A \quad ((R \circ R^{-1})(A) \subset A).$$

Hence (since R and R^{-1} are reflexive), it is clear that $R(A) \subset A$ and $R^{-1}(A) \subset A$. Moreover, it is easy to see that the latter inclusion implies that $R(X \setminus A) \subset X \setminus A$. Therefore, we obviously have $R \subset E_A$, and thus (i) also holds. ◊

Remark 5.2. Because of Prop. 3.5, we may write "proximally clopen" in place of "proximally open" in the above proposition.

Now, as an immediate consequence of Def. 4.1, Th. 4.4 and Prop. 5.1, we can also state

Theorem 5.3. *If $\mathcal{R}$ is a relator on X, then the following assertions are equivalent:*

(i) *$\mathcal{R}$ is uniformly connected;*

(ii) *$\mathcal{R} \nabla \mathcal{R}^{-1}$ ($\mathcal{R} \odot \mathcal{R}^{-1}$) is well-chained.*

Hence, because of the equalities $(\mathcal{R}^{\square})^* = \mathcal{R}^{\square}$, where $\square \in \{*, \#, \wedge\}$, it is clear that we also have

Corolary 5.4. *A relator $\mathcal{R}$ is uniformly, proximally, resp. topologically connected if and only if the relators*

$$\mathcal{R}^* \nabla (\mathcal{R}^*)^{-1}, \quad \mathcal{R}^\# \nabla (\mathcal{R}^\#)^{-1}, \quad \textit{resp. } \hat{\mathcal{R}} \nabla (\hat{\mathcal{R}})^{-1}$$
$$(\mathcal{R}^* \odot (\mathcal{R}^*)^{-1}, \quad \mathcal{R}^\# \odot (\mathcal{R}^\#)^{-1}, \quad \textit{resp. } \hat{\mathcal{R}} \odot (\hat{\mathcal{R}})^{-1})$$

are well-chained.

Remark 5.5. Because of Th. 4.6, we may write "proximally connected" in place of "well-chained" in the above assertions.

6. An application of Theorem 5.3

In [3], as an analogue of Levine's [5, Th. 2.2], we have also proved

Theorem 6.1. *If $\mathcal{R}$ is a relator on X, then the following assertions are equivalent:*

(i) *$\mathcal{R}$ is well-chained;*

(ii) *X^2 is the only preorder in $\mathcal{R}^*$.*

Hence, because of the equalities $(\mathcal{R}^\square)^* = \mathcal{R}^\square$, where $\square \in \{\#, \wedge\}$ and Remark 4.5, it is clear that we also have

Corolary 6.2. *A relator $\mathcal{R}$ on X is well-chained (topologically well-chained) if and only if X^2 is the only preorder in $\mathcal{R}^\#$ $(\hat{\mathcal{R}})$.*

Moreover, from Th. 6.1, by using Th. 5.3, we can also easily derive

Theorem 6.3. *If $\mathcal{R}$ is a relator on X, then the following assertions are equivalent:*

(i) *$\mathcal{R}$ is uniformly connected;*

(ii) *X^2 is the only equivalence in $\mathcal{R}^*$.*

Proof. If (i) holds, then by Cor. 5.4, $\mathcal{R}^* \nabla (\mathcal{R}^*)^{-1}$ is well-chained. Therefore, by Th. 6.1, X^2 is the only preorder in $(\mathcal{R}^* \nabla (\mathcal{R}^*)^{-1})^*$. Moreover, if E is an equivalence in $\mathcal{R}^*$, then since $E = E \cup E^{-1}$, E is a preorder in $\mathcal{R}^* \nabla (\mathcal{R}^*)^{-1}$. Therefore, by the above observation, $E = X^2$. Consequently, (ii) also holds.

To prove the converse implication, note that if $A \subset X$, then $E_A = = A^2 \cup (X \setminus A)^2$ is an equivalence on X. Therefore, if (ii) holds, then $E_A \in \mathcal{R}^*$ implies $E_A = X^2$, i.e., $A = \emptyset$ or $A = X$. Consequently, (i) also holds. ◊

From Th. 6.3, because of the equalities $(\mathcal{R}^\square)^* = \mathcal{R}^\square$, where $\square \in \in \{\#, \wedge\}$, it is clear that we also have

Corolary 6.4. *A relator $\mathcal{R}$ is proximally (topologically) connected if and only if X^2 is the only equivalence in $\mathcal{R}^\#$ $(\hat{\mathcal{R}})$.*

7. Well-chainedness characterizations of proximal connectedness

As an analogue of Prop 5.1, we can also prove

Proposition 7.1. *If $\mathcal{R}$ is a relator on X and $A \subset X$, then the following assertions are equivalent:*

(i) *A is a proximally clopen subset of $X(\mathcal{R})$;*

(ii) *A is a proximally open subset of $X(\mathcal{R} \vee \mathcal{R}^{-1})$ $(X(\mathcal{R} \circ \mathcal{R}^{-1}))$.*

Proof. If (i) holds, then there exist $R, S \in \mathcal{R}$ such that $R(A) \subset A$ and $S(X \setminus A) \subset X \setminus A$. Moreover, the latter inclusion implies that $S^{-1}(A) \subset A$. Hence, it is clear that

$$(R \cup S^{-1})(A) = R(A) \cup S^{-1}(A) \subset A$$

and

$$(R \circ S^{-1})(A) = R(S^{-1}(A)) \subset A .$$

Therefore, (ii) also holds.

Conversely, if (ii) holds, then there exist $R, S \in \mathcal{R}$ such that

$$(R \cup S^{-1})(A) \subset A \quad ((R \circ S^{-1})(A) \subset A).$$

Hence (since R and S are reflexive), it is clear that $R(A) \subset A$ and $S^{-1}(A) \subset A$. Moreover, the latter inclusion implies that $S(X \setminus A) \subset \subset X \setminus A$. Therefore, (i) also holds. ◊

Remark 7.2. Because of Prop. 3.5, we may again write "proximally clopen" in place of "proximally open" in the above proposition.

Now, as an immediate consequence of Th. 4.2. and 4.4 and Prop. 7.1, we can also state

Theorem 7.3. *If $\mathcal{R}$ is a relator on X, then the following assertions are equivalent:*

(i) *$\mathcal{R}$ is proximally connected;*

(ii) *$\mathcal{R} \vee \mathcal{R}^{-1}$ $(\mathcal{R} \circ \mathcal{R}^{-1})$ is well-chained.*

Hence, because of the equalities $(\mathcal{R}^{\Box})^{\#} = \mathcal{R}^{\Box}$, where $\Box \in \{\#, \wedge\}$, it is clear that we also have

Corolary 7.4. *A relator $\mathcal{R}$ is proximally, resp. topologically connected if the relators $\mathcal{R}^{\#} \vee (\mathcal{R}^{\#})^{-1}$, resp. $\hat{\mathcal{R}} \vee (\hat{\mathcal{R}})^{-1}$ $(\mathcal{R}^{\#} \circ (\mathcal{R}^{\#})^{-1}$, resp. $\hat{\mathcal{R}} \circ (\hat{\mathcal{R}})^{-1})$ are well-chained.*

Remark 7.5. Because of Th. 4.6, we may again write "proximally connected" in place of "well-chained" in the above assertions.

8. An application of Theorem 7.3

The advantage of the relator $\mathcal{R} \vee R^{-1}$ over $\mathcal{R} \nabla \mathcal{R}^{-1}$ lies mainly in the next

Proposition 8.1. *If $\mathcal{R}$ is a relator on X and $\Box \in \{*, \#\}$, then*

$$(\mathcal{R} \vee \mathcal{R}^{-1})^{\Box} = \mathcal{R}^{\Box} \cap (\mathcal{R}^{\Box})^{-1}.$$

Proof. It is easy to see that

$$(\mathcal{R} \vee \mathcal{S})^{\Box} = \mathcal{R}^{\Box} \cap \mathcal{S}^{\Box},$$

whenever $\mathcal{S}$ is also a relator on X. Moreover, in [12], it has been proved that

$$(\mathcal{R}^{-1})^{\Box} = (\mathcal{R}^{\Box})^{-1}$$

is also true. $\Diamond$

Remark 8.2. Hence, because of the equalities $(\mathcal{R}^{\Box})^{\Box} = \mathcal{R}^{\Box}$, it is clear that

$$(\mathcal{R}^{\Box} \vee (\mathcal{R}^{\Box})^{-1})^{\Box} = (\mathcal{R} \vee \mathcal{R}^{-1})^{\Box} = (\mathcal{R}^{\Box} \cap (\mathcal{R}^{\Box})^{-1})^{\Box},$$

whenever $\Box \in \{*, \#\}$.

Now, by using the first parts of Remark 4.5 and Prop. 8.1, from Th. 7.3 we can also easily get

Theorem 8.3. *If $\mathcal{R}$ is a relator on X, then the following assertions are equivalent:*

(i) *$\mathcal{R}$ is proximally connected;*

(ii) *$\mathcal{R}^* \cap (\mathcal{R}^*)^{-1}$ is well-chained.*

Hence, because of the equalities $(\mathcal{R}^{\Box})^{\#} = \mathcal{R}^{\Box}$ and $(\mathcal{R}^{\Box})^{*} = \mathcal{R}^{\Box}$, where $\Box \in \{\#, \wedge\}$, it is clear that we also have

Corolary 8.4. *A relator $\mathcal{R}$ is proximally (topologically) connected if and only if the relator $\mathcal{R}^{\#} \cap (\mathcal{R}^{\#})^{-1}$ $(\hat{\mathcal{R}} \cap (\hat{\mathcal{R}})^{-1})$ is well-chained.*

Remark 8.5. Because of Th. 4.6, we may again write "proximally connected" in place of "well-chained" in the above assertions.

9. A few supplementary notes

Note 9.1. Analogously to Prop. 5.1 and 7.1, we can also prove that a subset A of a relator space $X(\mathcal{R})$ is proximally open if and only if the Davis-Pervin relation $R_A = A^2 \cup (X \setminus A) \times X$ belongs to $\mathcal{R}^*$. Therefore, as an important addition to Th. 6.1, we can also state that a relator $\mathcal{R}$ on X is well-chained if and only if X^2 is the only Davis-Pervin relation in $\mathcal{R}^*$.

Note 9.2. In addition to Prop. 5.1 and 5.7, it is also worth mentioning that

$$\mathcal{R} \odot \mathcal{R}^{-1} \subset (\mathcal{R} \nabla \mathcal{R}^{-1})^* \quad \text{and} \quad \mathcal{R} \circ \mathcal{R}^{-1} \subset (\mathcal{R} \vee \mathcal{R}^{-1})^* .$$

But, despite this and Prop. 5.1 and 5.7, the corresponding relators need not even have the same topologically open sets. Namely, for instance, if $X = \{1, 2, 3\}$ and $R, S \subset X^2$ such that

$$R(1) = \{1\}, \quad R(2) = \{2,3\}, \quad R(3) = \{2,3\},$$
$$S(1) = X, \quad S(2) = \{2\}, \quad S(3) = \{3\},$$

then $\mathcal{R} = \{R, S\}$ is a relator on X such that

$$\mathcal{T}_{\mathcal{R} \odot \mathcal{R}^{-1}} = \mathcal{T}_{\mathcal{R} \circ \mathcal{R}^{-1}} = \{\emptyset, \{1\}, \{2,3\}, X\},$$

but

$$\mathcal{T}_{\mathcal{R} \nabla \mathcal{R}^{-1}} = \mathcal{T}_{\mathcal{R} \vee \mathcal{R}^{-1}} = \{\emptyset, \{1\}, \{1,2\}, \{1,3\}, \{2,3\}, X\}.$$

Note 9.3. In a continuation of [12], the second author proved that a relator (topological relator) $\mathcal{R}$ is weakly symmetric if and only if the relator $\hat{\mathcal{R}} \cap (\hat{\mathcal{R}})^{-1}$ $(\hat{\mathcal{R}} \circ (\hat{\mathcal{R}})^{-1})$ is topologically equivalent to $\mathcal{R}$. Therefore, as an important addition to Cors. 8.4 and 7.4, we can also state that *a weakly symmetric relator (weakly symmetric topological relator) $\mathcal{R}$ is topologically connected if and only if the relator $\hat{\mathcal{R}} \cap (\hat{\mathcal{R}})^{-1}$ $(\hat{\mathcal{R}} \circ (\hat{\mathcal{R}})^{-1})$ is topologically connected.*

To feel the importance of the above statements, note that if $\mathcal{R}$ is a relator on X, then $\hat{\mathcal{R}} \cap (\hat{\mathcal{R}})^{-1}$ $((\hat{\mathcal{R}} \circ (\hat{\mathcal{R}})^{-1})^*)$ is the family of all semi-neighbourhood (neighbourhoods) of the diagonal of X^2.

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