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STRUCTURAL THEOREMS FOR THE S–ASYMPTOTIC AND QUASI–ASYMPTOTIC OF DISTRIBUTIONS

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Abstract: The equivalence of various definitions of the S-asymptotic and of the quasiasymptotic are proved. The use of them are given in the comments after the main theorems.

1. Introduction

In the last twenty years many aspects of the asymptotic behaviour of distributions have been elaborated and used. Between them the important are the quasiasymptotic and the S-asymptotic which play a special role in investigations of the asymptotic behaviour of generalized functions' integral transforms. The quasiasymptotic has been elaborated in [12] and for the S-asymptotic it is done in [6]. In the both cited books and in the references given there, one can find applications of these notions. For the application of the quasiasymptotic expansion we refer to [6], [12] and [13].

Our aim is to give several equivalent definitions for these two notions of the asymptotic behaviour of distributions. They are useful in many different cases. We shall point out some of them in the comments of Theorems 1 and 3.

All the results of the paper are given in the one-dimensional case. They can be proved also in the many - dimensional case by using a cone Γ instead of the interval $[0,\infty)$, but the assertions in this case have more complicated forms.

2. Notation

$\mathcal{D}'$ is the space of Schwartz distributions $\mathcal{S}'_a$ is the space of tempered distributions with support in $[a,\infty)$, $a\in\mathbb{R}$; $\mathcal{S}'_{[0,\infty)} := \mathcal{S}'_+$. The Laplace transform of $T\in\mathcal{S}'_+$ is defined by

$$\tilde{T}(z)=\mathcal{L}[T](z)=\langle T(t),\, e^{izt}\rangle\,,\quad z\in\mathbb{R}+i\mathbb{R}_+\quad(\mathbb{R}_+=(0,\infty))\,.$$

The class of distributions f_α, $\alpha\in\mathbb{R}$, belonging to $\mathcal{S}'_+$, is defined by:

$$f_\alpha(t)=\begin{cases}\theta(t)t^{\alpha-1}/\Gamma(\alpha), & \alpha>0,\\ f^{(m)}_{\alpha+m}(t), & \alpha\le 0,\ \alpha+m>0,\ n\in\mathbb{N},\end{cases}\tag{2.1}$$

where $\theta(t)=1$, $t\ge 0$; $\theta(t)=0$, $t<0$. We denote: $f^{(-m)}=f_m*f$, $m\in\mathbb{R}$ ($*$ is the sign of convolution); $f_{-n}=\delta^{(n)}$, $n\in\mathbb{N}$, $f_0=\delta$. There holds $f_\alpha*f_\beta=f_{\alpha+\beta}$, $\alpha,\beta\in\mathbb{R}$. Let α_n, $n\in\mathbb{N}$, be positive numbers such that $\alpha_n\to 0$, $n\to\infty$, and let

$$\delta_n\in C^\infty,\operatorname{supp}\delta_n\subset[-\alpha_n,\alpha_n]\,,\quad \delta_n\ge 0\,,\quad \int_{-\infty}^{\infty}\delta_n(x)dx=1,\quad n\in\mathbb{N}\,.$$

Then, the sequence $\{\delta_n\}$ is called a *δ-sequence* ([1], p. 75). If $\varphi\in\mathcal{D}$, then $\delta_n*\varphi\to\varphi$ in $\mathcal{D}$; hence $\{\delta_n*\varphi;\ n\in\mathbb{N}\}$ is a bounded set in $\mathcal{D}$.

In the sequel, we shall use the class of slowly varying functions. A function $L\in\mathcal{L}^1_{loc}$ is a *slowly varying* one if $L(x)>0$, $x>0$, and if

$$\lim_{u\to\infty}\frac{L(ut)}{L(u)}=1\,,\ \text{for every } t>0\,.\tag{2.2}$$

We know ([9], Chapter 1.5) that if $L_2(x)\to\infty$, $x\to\infty$ and L_1,L_2 are slowly varying functions, then $L_1\circ L_2$ is slowly varying one, as well. Hence, for $x\in\mathbb{R}$

$$(2.3)\qquad \lim_{h\to\infty} = \frac{L(x+h)}{L(h)} = \lim_{u\to\infty}\frac{L(\ell nut)}{L(\ell nu)} = 1\,, \quad t>0\,,\ x>-\infty\,.$$

If x belongs to a compact set, then the limit (2.3) is uniform in x ([9], Chapter 1.2).

3. S-asymptotic in $\mathcal{D}'$

The S-asymptotic of distributions has been appeared as a useful notion in the theory of generalized functions and its applications [6]. We shall repeat the definition of it:

Definition 1. Let T belong to $\mathcal{D}'$ (to $\mathcal{S}'$) and $c(h)$, $h \geq h_0$ be a positive measurable function. It is said that T has the S-asymptotic at infinity related to c, with the limit $g \neq 0$ in $\mathcal{D}'$ (in $\mathcal{S}'$) if for every $\phi \in$ $\in \mathcal{D}(\phi \in \mathcal{S})$

$$(3.1)\qquad \lim_{h\to\infty}\left\langle \frac{T(x+h)}{c(h)}, \phi(x)\right\rangle = \langle g(x), \varphi(x)\rangle\,.$$

We shall write in short $T \sim c.g$, $h \to \infty$, in $\mathcal{D}'$ (in $\mathcal{S}'$). Since

$$(3.2)\qquad (T*\phi)(h) = \langle T(x+h), \check{\phi}(x)\rangle\,, \quad h\in\mathbb{R}\,, \quad \check{\phi}(x) = \phi(-x)\,,$$

the S-asymptotic can be defined in an equivalent form: If for every $\phi \in \mathcal{D}$ ($\phi \in \mathcal{S}$) there exists $C_\phi \in \mathbb{R}$ such that

$$(3.3)\qquad \lim_{h\to\infty}\frac{(T*\phi)(h)}{c(h)} = C_\phi\,,\ \text{where } C_\phi \neq 0 \text{ for some } \varphi\in\mathcal{D}\,.$$

Relation (3.1) implies (3.3) in a trivial way. By the equality of the weak and the strong sequential convergence in $\mathcal{D}'$, (3.3) implies (3.1).

We know ([6], p. 85) that c and g from Def. 1 have the form:

$$c(h) = \exp(\alpha h)L(\exp h)\,, \quad h > h_0\,, \quad g(x) = M\exp(\alpha x)\,, \quad x\in\mathbb{R}\,,$$

where M and α belong to $\mathbb{R}$, $M \neq 0$.

Theorem 1. *Let $T \in \mathcal{D}'$ and $c(h) = \exp(\alpha h)L(\exp h)$, $h \geq h_0$. The following conditions are equivalent:*

(a)

$$\lim_{h\to\infty}\frac{T(.+h)}{c(h)} = M\exp(\alpha.)\,,\ \textit{in } \mathcal{D}'\,, \quad M\neq 0\,.$$

(b) *For every $\phi \in \mathcal{D}$,*

$$\lim_{h\to\infty} \frac{(T*\phi)(.+h)}{c(h)} \to M_\phi \exp(\alpha .) \ in\ \mathcal{D}',$$

where

$$M_\phi = M\int_{\mathbb{R}} e^{\alpha t}\phi(t)dt, \quad M \neq 0.$$

(c) *For a δ-sequence $\{\delta_n\}$ there exists a sequence $\{M_n\}$ from $\mathbb{R}$, such that $M_n \to M \neq 0$, $n\to\infty$, and*

$$\lim_{h\to\infty} \frac{(T*\delta_n)(.+h)}{c(h)} = M_n \exp(\alpha .), \tag{3.4}$$

$$in\ \mathcal{D}',\ uniformly\ for\ n\in\mathbb{N}.$$

(d) *There are functions F_i, $i = 0,\dots,n$, which are continuous on (a,∞), $a\in\mathbb{R}$, such that for every $i = 0,\dots,n$,*

$$\frac{F_i(x+h)}{\exp(\alpha h)L(\exp h)} \to c_i \exp(\alpha x), \quad h\to\infty,$$

uniformly for $x\in(a,b)$, $a<b<\infty$, $\sum_{i=0}^{n} c_i\alpha^i = M \neq 0$, and the restriction of T on (a,∞) is of the form $T = \sum_{i=0}^{n} D^iF_i$.

(e) *For a δ-sequence $\{\delta_n\}$,*

$$\lim_{h\to\infty} \frac{(T*\delta_n)(h)}{c(h)} = p_n, \quad n\in\mathbb{N},\ where\ p_n\neq 0\ for\ some\ n,$$

and for every $\phi\in\mathcal{D}$,

$$\sup_{k\geq 0}\left|\frac{(T*\phi)(h)}{c(h)}\right| < \infty.$$

Proof. (a) $\Rightarrow$ (b). Using the properties of convolution, for every $\psi\in\mathcal{D}$ we have

$$\begin{aligned}\lim_{h\to\infty}\left\langle \frac{(T*\phi)(x+h)}{c(h)}, \psi(x)\right\rangle &= \lim_{h\to\infty}\left\langle \frac{T(x+h)}{c(h)}, (\check{\phi}*\psi)(x)\right\rangle \\ &= \langle Me^{\alpha x}, (\check{\phi}*\psi)(x)\rangle = M\langle (e^{\alpha\cdot}*\phi)(x), \psi(x)\rangle \\ &= M\int_{\mathbb{R}} e^{-\alpha t}\phi(t)dt\langle \exp(\alpha x), \psi(x)\rangle.\end{aligned} \tag{3.5}$$

(b) $\Rightarrow$ (a). This can be derived as a consequence of Th. XXIII in [8], Ch. VI, but we shall give a direct proof which is more elementary. It is proved in [7] that any $\psi\in\mathcal{D}$, can be written in the form $\psi =$

$= \psi_1 * \theta_1 + \cdots + \psi_k * \theta_k$, where ψ_i, and θ_i, $i = 1, 2, \ldots, k$, are from $\mathcal{D}$. This implies

$$\lim_{h\to\infty} \left\langle \frac{T(x+h)}{c(h)}, \psi(x) \right\rangle =$$

$$(3.6) \qquad = \lim_{h\to\infty} \sum_{i=1}^{k} \left\langle \frac{(T * \check{\psi}_i)(x+h)}{c(h)}, \theta_i(x) \right\rangle =$$

$$= \sum_{i=1}^{k} M_{\psi_i} \langle e^{\alpha x}, \theta_i(x) \rangle$$

Since in $\mathcal{D}'$ the weak and the strong sequential convergence are equal, there exists a $g \in \mathcal{D}'$ such that limit (3.6) is equal to $\langle g, \phi \rangle$. But we know that g has the form $g(x) = Me^{\alpha x}$, $x \in \mathbb{R}$, $M \neq 0$.

(a) $\Rightarrow$ (c). Let $\{\delta_n\}$ be a δ-sequence. For any $\phi \in \mathcal{D}$, $\{\delta_n * \phi;\ n \in \mathbb{N}\}$ is a bounded set in $\mathcal{D}$. We have

$$\lim_{h\to\infty} \left\langle \frac{(T * \delta_n)(x+h)}{c(h)}, \phi(x) \right\rangle =$$

$$(3.7) \qquad = \lim_{h\to\infty} \left\langle \frac{T(x+h)}{c(h)}, (\check{\delta}_n * \phi)(x) \right\rangle =$$

$$= \langle Me^{\alpha x}, (\check{\delta}_n * \phi)(x) \rangle = \langle M_n e^{\alpha x}, \phi(x) \rangle .$$

By using the equivalence of the strong and the weak sequential convergences in $\mathcal{D}'$, it follows that relation (3.7) implies (3.4) and (c).

(c) $\Rightarrow$ (a). Let $\phi \in \mathcal{D}$ and

$$a_{n,h} = \left\langle \frac{(T * \delta_n)(x+h)}{c(h)}, \phi(x) \right\rangle = \frac{(T * (\check{\delta}_n * \phi))(h)}{c(h)}, \ n \in \mathbb{N},\ h > 0 .$$

We have $a_{n,h} \to a_n$, $h \to \infty$, uniformly for $n \in \mathbb{N}$, where

$$a_n = M_n \langle \exp(\alpha x), \phi(x) \rangle ,\ n \in \mathbb{N},$$

$$a_n \to a = M \langle \exp(\alpha x), \phi(x) \rangle ,\ n \to \infty .$$

Also $a_{n,h} \to a_h$, $n \to \infty$, where $a_h = \langle \frac{T(x+h)}{c(h)}, \phi(x) \rangle$, $h > 0$. This implies $a_h \to a$, $h \to \infty$, what is in fact (a).

(d) $\Rightarrow$ (a). If two distributions T_1 and T_2 are equal on some interval (a, ∞) and T_1 has the S-asymptotic related to c with the limit g, then the same holds for T_2, as well. Hence, we can take $T = 0$ on $(-\infty, a)$. For a $\varphi \in \mathcal{D}$, $\operatorname{supp} \varphi \subset K$ and h large enough, we have

$$\left\langle \frac{T(x+h)}{\exp(\alpha h)L(\exp h)}, \varphi(x)\right\rangle =$$

$$= \sum_{i=0}^{n} \int_K \frac{F_i(x+h)}{\exp(\alpha h)L(h)}(-1)^i \varphi^{(i)}(x)dx \to$$

$$\to \sum_{i=0}^{n} \alpha^i c_i \langle e^{\alpha x}, \varphi(x)\rangle\,,\ h \to \infty\,.$$

(a) $\Rightarrow$ (d). This follows from the structural theorem (see [10]), but we can give a direct proof. Since T has the S-asymptotic related to $c(h) = \exp(\alpha h)L(\exp h)$, $h > 0$, the set $\left\{\frac{T(x+h)}{c(h)};\ h \geq 0\right\}$ is bounded in $\mathcal{D}'$. For every $\varphi \in \mathcal{D}$, $\left\{\frac{T(x+h)}{c(h)} * \varphi(x); h > 0\right\}$ is a bounded set of regularizeds of distributions ([8], II, Ch. VI, §4). Let us denote by Ω an open neighbourhood of zero in $\mathbb{R}$ which is relatively compact and denote its closure by K. The proof of Th. XXII, Ch. VI, in [8], implies that there exists $m \in \mathbb{N}$ such that the linear mappings

$$(\alpha, \beta) \to \frac{T(.+h)}{c(h)} * \alpha * \beta\,, \quad h > 0\,,$$

are equicontinuous mappings of $\mathcal{D}_\Omega^m \times \mathcal{D}_\Omega^m$ into $\mathcal{L}_{(a,b)}^\infty$. Since

$$F(x,h) = \left(\frac{T(.+h)}{c(h)} * \alpha * \beta\right)(x) =$$

$$= \frac{(T * \alpha * \beta)(x+h)}{c(h)}\,, \quad x \in \mathbb{R}\,, \quad h > 0\,,$$

it follows that $F(.,h)$ is a continuous function for every $h > 0$. Also we have that the family $\{F(.,h); h > 0\}$, is uniformly bounded on the interval $[a,b]$. $\mathcal{D}_\Omega$ is a dense subset of $\mathcal{D}_\Omega^m$. The set of functions $\left\{\frac{T(.+h)}{c(h)} * \psi * \phi\,, h > 0\right\}$ converges in $\mathcal{L}_{(a,b)}^\infty$ for $\psi\,, \phi \in \mathcal{D}_\Omega$. Now, we use the Banach-Steinhaus Theorem to prove that for every $\alpha, \beta \in \mathcal{D}_\Omega^m$, $\frac{T(.+h)}{c(h)} * \alpha * \beta \to C_{\alpha,\beta} \exp(\alpha .)$, $h \to \infty$, in $\mathcal{L}_{(a,b)}$. Now, by using relation (VI, 6: 23) in [8], we have

$$D_x^{4k}(\gamma E * \gamma E * T)(x) - 2D_x^{2k}(\gamma E * \xi * T)(x) +$$
$$+ (\xi * \xi * T)(x) = T(x)\,, \quad x \in \mathbb{R}\,,$$

where E is a solution of $D^{4k}E = \delta$, ((II, 3; 19) in [8]); $\gamma\xi \in \mathcal{D}_\Omega$ and k is large enough so that $\gamma E \in \mathcal{D}_\Omega^m$. In this case $m = 4k$, $E_{4k} \equiv \gamma E * \gamma E * T$, $E_{2k} = -2\gamma E * \xi * T$; $E_0 = \xi * \xi * T$; $E_i = 0$, $i \neq 4k, 2k, 0$. Thus, (d) is proved.

(a) $\Rightarrow$ (e). It is trivial.

(e) $\Rightarrow$ (a). First, we shall prove that the set $G = \{\delta_n(. + x), n \in \in \mathbb{N}, x \in \mathbb{R}\}$ is dense in $\mathcal{D}$. Suppose that $T \in \mathcal{D}'$ and

$$\langle T, \delta_n(. + x)\rangle = (T * \check{\delta}_n)(-x) = 0, \quad n \in \mathbb{N}, \quad x \in \mathbb{R}.$$

Then for any $\varphi \in \mathcal{D}$, $\langle T * \check{\delta}_n, \varphi\rangle = 0$ and

$$\langle T, \varphi\rangle = \lim_{n\to\infty} \langle T, \delta_n * \varphi\rangle = \lim_{n\to\infty} \langle T * \check{\delta}_n, \varphi\rangle = 0.$$

This implies that $T = 0$ and hence, the set G is dense in $\mathcal{D}$. The Banach-Steinhaus Theorem and (b) imply (a). $\Diamond$

Comments for Theorem 1. 1. The following statement implies (a) but it is not equivalent to (a):

(f) *For a δ sequence $\{\delta_n\}$ there is a sequence M_n, from $\mathbb{R}$, such that $M_n \to M \neq 0$, $n \to \infty$, and*

$$\lim_{h\to\infty} \frac{(T * \delta_n)(h)}{c(h)} \to M_n, \quad n \in \mathbb{N},$$

where the limit is uniform for $n \in \mathbb{N}$.

Let us prove that (f) $\Rightarrow$ (c) which is equivalent to (a). For every compact set $K \subset \mathbb{R}$ we have

$$\frac{(T * \delta_n)(x + h)}{c(h)} = \frac{(T * \delta_n)(x + h)}{c(x + h)} \frac{c(x + h)}{c(h)} \to M_n \exp(\alpha x), \quad h \to \infty,$$

uniformly for $x \in K$, because of $\frac{c(x+h)}{c(h)} \to \exp(\alpha x)$, $h \to \infty$, uniformly for $x \in K$ (see [6], p.82). Now, we shall prove that (a) does not imply (f). Assume $H \in C^0 \cap \mathcal{L}^1$ but is not bounded on $\mathbb{R}$; this function has the S-asymptotic equal to zero related to 1 ([6], p. 104). For T we take $1 + H(t)$. Then,

$$\lim_{h\to\infty} [(1 + H) * \delta_n](h) = \lim_{h\to\infty} \langle 1 + H(t + h), \check{\delta}_n(t)\rangle = \langle 1, \check{\delta}_n(t)\rangle = 1.$$

We have

$$\lim_{n\to\infty} [(1 + H) * \delta_n](h) = 1 + H(h),$$

but the function $1 + H(h)$ has no limit at all, when $h \to \infty$.

2. We proved in [5] that the conditions of Th. 1 are equivalent to the following ones:

(g) *The set $\{f(. + h)/c(h); h \in \mathbb{R}\}$ is bounded in $\mathcal{D}'$ and for some $\phi \in \mathcal{D}$, for which $\mathcal{F}(\phi e^{-\alpha\cdot})(\xi) = \mathcal{L}[\phi](-\xi + i\alpha) \neq 0$, $\xi \in \mathbb{R}$,*

$$\lim_{h\to\infty}(f*\phi)(h)/c(h) = M\int_R \phi(t)e^{-\alpha t}dt\,,\quad M\neq 0\,,$$

where for $\mathcal{F}(\phi)$ we used in [5] $\int_{\mathbb{R}} e^{-ixt}\phi(t)dt$.

We shall illustrate the usefulness of (g) by an example. Let $P(x) = \sum_{i=0}^{m} a_i x^i$ be a polynomial such that $P(-x+i\alpha)\neq 0$, $x\in\mathbb{R}$. We say that a solution of a differential equation $P(D)U = F$ is *c-bounded* if the set $\{U(.+h)/c(h);\ h\in\mathbb{R}\}$ is a bounded subset in $\mathcal{D}'$. Assertion (g) implies: *If F has the S-asymptotic related to c, then any c-bounded solution has also the S-asymptotic related to c.* Let us show that. We choose $\phi\in\mathcal{D}$ such that $\mathcal{L}[\phi](-x+i\alpha)\neq 0$, $x\in\mathbb{R}$. For $\psi = P(D)\phi$ we have $\mathcal{L}[\psi](-x+i\alpha)\neq 0$, $x\in\mathbb{R}$ and

$$(U*\psi)(h)/c(h) = ((P(D)U)*\phi)(h)/c(h)\,,\quad h\geq h_0>0\,.$$

By (g) the assertion follows.

Note that the given example can be easily transfered to the many-dimensional case which is in fact more interesting.

3. Many properties of distributions and subspaces of distributions in Schwartz's theory can be derived from the corresponding regularizeds. An application of (b) is given in the next example. For the proof that every $T\in\mathcal{D}'_{\mathcal{L}^p}$, $1\leq p<\infty$ has the S-asymptotic zero related to $c=1$, we use the known fact that $f=(T*\varphi)\in\mathcal{L}^p$ for every $\varphi\in\mathcal{D}$. Then, for $\varphi\in\mathcal{D}$, $\operatorname{supp}\varphi\subset[-a,a]$ we have

$$|\langle f(x+h),\varphi(x)\rangle| \leq \sup_{-a\leq x\leq a}|\varphi(x)|\int_{-a+h}^{a+h}|f(x)|dx\to 0\,,\quad h\to\infty\,.$$

In the sequential approach to the theory of distributions, a distribution T is defined by the class $[T*\delta_n]$ which corresponds to the fundamental sequence $\{T*\delta_n\}$, where $\{\delta_n\}$ is a δ-sequence ([1], p. 79). To define the S-asymptotic of a distribution in this approach, we can use (c) and (e) from Th. 1.

4. For the Abelian and Tauberian type theorems for integral transforms of distributions, the assertion (d) of Th. 1. may be useful (see, for example, [5] and [11]).

4. The quasiasymptotic at ∞

This notion is introduced by Zavialov and intensively studied by

Drožžinov, Vladimirov and Zavialov; see the monograph [12] and references there. By using it, the quoted authors have obtained comprehensive results in investigations of the Laplace transform and in applications in the quantum field theory. For the use of this notion in investigations of the Stieltjes transform we refer to [6].

Definition 2. It is said that $T \in \mathcal{S}'_+$ has the quasiasymptotic (quasi asymptotic behaviour) at infinity related to some positive measurable function $c(k)$, $k \geq k_0$ if there exists $g \in \mathcal{D}'$, $g \neq 0$ such that for every $\phi \in \mathcal{D}$,

$$\lim_{k\to\infty} \left\langle \frac{T(kx)}{c(k)}, \phi(x) \right\rangle = \langle g, \phi \rangle .$$

Let us recall some properties of the quasiasymptotic behaviour of a $T \in \mathcal{S}'_+$. The existence of the limit in Def. 2 implies that this limit also exists in $\mathcal{S}'$ and that $g(x) = Cf_{\alpha+1}(x)$, $x \in \mathbb{R}$, $C \neq 0$ and $c(k) = k^\alpha L(k)$, $k \geq k_0$. The most important result concerning the quasiasymptotic behaviour is the following Tauberian theorem of Drožžinov and Zavialov for the Laplace transform:

Theorem 2. [2]. *Let $T \in \mathcal{S}'_+$, $c(k) = k^\alpha L(k)$, $k \geq k_0$. The following statements are equivalent:*

(i) $$\frac{T(kx)}{c(k)} \to Cf_{\alpha+1}, k \to \infty \quad in \quad \mathcal{S}', C \neq 0;$$

(ii) (A) $\lim_{y\to 0+} \frac{y}{c(1/y)}\tilde{T}(iy) = C$, $C \neq 0$; (B) *there exist* $D > 0$, $m \in \mathbb{N}$ *and* $r_0 > 0$ *such that*

$$\left| \frac{r}{c(1/r)} \tilde{T}(re^{i\varphi}) \right| \leq \frac{D}{\sin^m \varphi}, \quad 0 < r \leq r_0, \quad 0 < \varphi < \pi .$$

In the next theorem we give several equivalent statements but for $T \in S'_\alpha$.

Theorem 3. *Let $T \in S'_a$, $a \in \mathbb{R}$ and $c(k) = k^\alpha L(k)$, $k \geq k_0$. The following statements are equivalent:*

(a) $$\lim_{k\to\infty} \frac{T(k.)}{c(k)} = Cf_{\alpha+1}, \quad in \quad \mathcal{D}', \quad C \neq 0$$

(b) $$\lim_{k\to\infty} \frac{T(k. + b)}{c(k)} = Cf_{\alpha+1}, \quad in \quad \mathcal{S}', \quad for\ some \quad b \in \mathbb{R};\ C \neq 0$$

(c) (A) $\lim_{y\to 0+} \frac{y}{c(1/y)}\tilde{T}(iy) = M \neq 0$; (B) *there exist* $D_1 > 0$, $m \in \mathbb{N}_0$ *and* $r_0 > 0$ *such that*

$$\left|\frac{r}{c(1/r)}\tilde{T}(re^{i\varphi})\right| \le \frac{D_1}{\sin^m \varphi}, \quad 0 < r \le r_0, \quad 0 < \varphi < \pi .$$

(d) *For every* $\phi \in \mathcal{D}$,

$$\lim_{k\to\infty} \frac{(T * \phi)}{c(k)} = M_\phi f_{\alpha+1}\,, \text{ in } \mathcal{D}';\ \text{where } M_\phi \neq 0 \text{ for some } \phi \in \mathcal{D}.$$

(e) *There exists* $\phi_0 \in \mathcal{D}$ *with the property* $\tilde{\phi}_0(0) \neq 0$ *such that*

$$\lim_{k\to\infty} \frac{(T * \phi_0)(k.)}{c(k)} = M_{\phi_0} f_{\alpha+1}\,, \quad \text{in} \quad \mathcal{D}', \quad M_{\phi_0} \neq 0 .$$

(f) *For a* δ*-sequence* $\{\delta_n$ *there is a* $C \neq 0$ *such that*

$$\lim_{k\to\infty} \frac{(T * \delta_n)(k.)}{c(k)} = C f_{\alpha+1}\,, \quad \text{in} \quad \mathcal{D}', \quad \text{and uniformly for } n \in \mathbb{N}.$$

Proof. (a) $\Rightarrow$ (b). We start with the relation

$$\left\langle \frac{T(kx+b)}{c(k)}, \phi(x) \right\rangle = \left\langle \frac{T(kx)}{c(k)}, \phi\left(x - \frac{b}{k}\right) \right\rangle, \quad \phi \in \mathcal{D}.$$

The set $\{\phi(. - \frac{b}{m});\ m \ge 1\}$ is bounded in $\mathcal{D}$. By using the equivalence of the weak and strong sequential convergences in $\mathcal{D}'$ and the fact that

$$\phi\left(. - \frac{b}{k}\right) \to \phi, \quad k \to \infty, \quad \text{in} \ \mathcal{D},$$

we have

$$\lim_{k\to\infty} \left\langle \frac{T(kx+b)}{c(k)}, \phi(x) \right\rangle = \lim_{k\to\infty} \left\langle \frac{T(kx)}{c(k)}, \phi\left(x - \frac{b}{k}\right) \right\rangle =$$
$$= \left\langle C f_{\alpha+1}, \phi \right\rangle, \quad \phi \in \mathcal{D}.$$

By Th. 3 in [12], p. 60, this limit holds also for $\varphi \in \mathcal{S}$.

(b) $\Rightarrow$ (a). We have to repeat (a) $\Rightarrow$ (b) but starting from the relation

$$\left\langle \frac{T(kx)}{c(k)}, \phi(x) \right\rangle = \left\langle \frac{T(ku+b)}{c(k)}, \phi\left(u + \frac{b}{k}\right) \right\rangle, \quad \phi \in \mathcal{D}. \quad \Diamond$$

Remark. If $T \in \mathcal{S}'_a$, then $T(.+a) \in \mathcal{S}'_0 = \mathcal{S}'_+$. This implies: *If* $T \in \mathcal{E}'$, *then there exists an* $m \in \mathbb{N}_0$ *such that* T *has the quasiasymptotic related to* $k^{-m-1}L(k)$ *with the limit* $\delta^{(m)}$, $k \to \infty$. This statement has been proved for $T \in \mathcal{E}' \cap \mathcal{S}'_+$ (see [6], p. 32).

(b) $\Rightarrow$ (c). Since $T(.+a) \in \mathcal{S}'$ and satisfies (i) of Th. 2, it satisfies (ii), as well. The Laplace transform of $T(.+a)$ is $e^{-ia\cdot}\mathcal{L}[T]$; hence (A) and (B) in Th. 2 (ii) imply (A) and (B) in (c).

(c) $\Rightarrow$ (a). If (c) is true for a $T \in \mathcal{S}'_a$, it is true for $T(. + a) \in \mathcal{S}'_+$ and it implies Th. 2 (i) for $T(. + a)$; hence (a) is true for T.

(c) $\Rightarrow$ (d). For a $T \in \mathcal{S}'_\alpha$ we have $T * \phi \in \mathcal{S}'_b$, $b \in \mathbb{R}$. We shall show that $T * \phi$ satisfies (c). Then, it will follow that $T * \phi$ satisfies (a) and that T satisfies (d). First, we have

$$\lim_{y \to 0+} \frac{y}{c(1/y)} \tilde{T}(yi)\tilde{\phi}(iy) = \lim_{y \to 0+} \frac{y}{c(1/y)} \tilde{T}(yi)\tilde{\phi}(0) = M_{\phi(0)} .$$

Moreover, there exist $D_2 > 0$, $m \in \mathbb{N}_0$ and $r_0 > 0$ such that

$$\left| \frac{r}{c(1/r)} \tilde{T}(re^{i\varphi})\tilde{\phi}(re^{\varphi i}) \right| \leq$$

$$\leq \left| \frac{r}{c(1/r)} \tilde{T}(re^{i\varphi}) \right| \max_{\substack{0<r\leq r_0 \\ 0\leq\varphi<\pi}} \left| \tilde{\phi}(re^{\varphi i}) \right| \leq \frac{D_2}{\sin^m \varphi} .$$

(d) $\Rightarrow$ (e). Let $\phi_1 \in \mathcal{D}$ for which $M_{\phi_1} \neq (0)$ in (d). If $\tilde{\phi}_1(0) \neq 0$, then we take $\phi_0 = \phi_1$. If $\tilde{\phi}_1(0) = 0$, we take $\phi_0 = \phi_1 + \phi_2$ where ϕ_2 is an arbitrary element from $\mathcal{D}$ with the properties $\tilde{\phi}_2(0) \neq 0$ and $M_{\phi_2} = 0$.

(e) $\Rightarrow$ (a). The assumptions in (e) imply (a) and consequently (c) for $f * \phi_0$. Now, we have

$$\lim_{y \to 0+} \frac{y}{c(1/y)} \tilde{T}(iy)\tilde{\phi}_0(iy) = \lim_{y \to 0+} \frac{y}{c(1/y)} \tilde{T}(iy)\tilde{\phi}_0(i0) = M \neq 0 .$$

Taking care of the property $\tilde{\phi}_0(0) \neq 0$ we have

$$\left| \frac{r}{c(1/r)} \tilde{T}(re^{i\varphi}) \right| \leq \frac{\mathcal{D}'}{\sin^{m'} \varphi} ,$$

where $0 < r \leq r'_0$, $0 \leq \varphi < \pi$ for appropriate r'_0, $\mathcal{D}'$ and m'. This gives (c) for T and consequently (a).

(a) $\Rightarrow$ (f). Let $\phi \in \mathcal{D}$ and $n \in \mathbb{N}$. Then

$$\left\langle \frac{(T * \delta_n)(kx)}{c(k)}, \phi(x) \right\rangle = \frac{1}{kc(k)} \left\langle (T * \delta_n)(x), \phi\left(\frac{x}{k}\right) \right\rangle =$$

$$= \frac{1}{kc(k)} \left\langle T(x), \left(\check{\delta}_n * \phi\left(\frac{\cdot}{k}\right)\right)(x) \right\rangle =$$

$$= \frac{1}{c(k)} \left\langle T(kx), \int_{\mathbb{R}} \delta_n(-t)\phi\left(x - \frac{t}{k}\right) dt \right\rangle .$$

The set

$$\left\{ \int_{\mathbb{R}} \delta_n(-t)\phi\left(x - \frac{t}{m}\right) dt; \quad m, n \in \mathbb{N} \right\}$$

is bounded in $\mathcal{D}$ because of the properties of $\{\delta_n\}$. We have

$$\left\langle \frac{T(kx)}{c(k)}, \int_{\mathbb{R}} \delta_n(-t)\phi\left(x - \frac{t}{m}\right) dt \right\rangle \to$$

$$\to \left\langle C f_{\alpha+1}, \left(\int_{\mathbb{R}} \delta_n(-y) dy \right) \phi \right\rangle = \left\langle C f_{\alpha+1}, \phi \right\rangle, \quad k \to \infty,$$

uniformly for $n \in \mathbb{N}$. This limit is a consequence of the inequality

$$\left| \left\langle \frac{(T * \delta_n)(kx)}{c(k)}, \phi(x) \right\rangle - \left\langle C f_{\alpha+1}, \phi \right\rangle \right| \leq$$

$$\leq \left| \left\langle \frac{T(kx)}{c(k)}, \int_{\mathbb{R}} \delta_n(-t)\phi\left(x - \frac{t}{k}\right) dt \right\rangle - \right.$$

$$\left. - \left\langle C f_{\alpha+1}(x), \int_{\mathbb{R}} \delta_n(-t)\phi\left(x - \frac{t}{k}\right) dt \right\rangle \right| +$$

$$+ \left| \left\langle C f_{\alpha+1}(x), \int_{\mathbb{R}} \delta_n(-t)\phi\left(x - \frac{t}{k}\right) dt \right\rangle - \left\langle C f_{\alpha+1}, \phi \right\rangle \right|.$$

(f) $\Rightarrow$ (e). Since $\int\limits_{-\infty}^{\infty} \delta_n(x) = 1$, $n \in \mathbb{N}$, we can find an n_0 such that $\tilde{\delta}_{m_0}(0) \neq 0$. This implies that (e) holds. $\Diamond$

Comments for Theorem 3. Similar comments as for the S-asymptotic, hold for the quasiasymptotic, as well. Moreover, the use of the quasiasymptotic is more powerfull (see [12], [13], [6]). Note only that the results given in [12] for the convolution equations of elements from $\mathcal{S}'_+$ also hold for elements of $\mathcal{S}'_a$.

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