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H – INTEGRAL NEAR – RINGS

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Abstract: H - integral near - rings are intended to fill the wide gap between the disparate types of integral near - rings on one hand and near - rings with large annihilator ideals (zero - near - rings at the extreme end) on the other hand. If H is a subset of a near - ring N, N is said to be H - *integral* if H has no divisors of zero and $N^2 \subseteq H$. After preliminary results and some motivating examples are presented, we show that such a near - ring N "consists" of an ideal K with $K^2 = 0$ and an integral near - ring; if the latter is finite, N is a semidirect sum of these two parts. This gives rise to a construction method to obtain a large class of H - integral near - rings in an easy way. The last section considers distributively generated H - integral near - rings. In this case and if K has finite index, $N/_K$ is a finite field.

1. Basic facts

In this paper, we consider *left near - rings* $(N, +, .)$, hence $(N, +)$

is a group (not necessarily abelian), $(N, .)$ a semigroup and $n_1(n_2 + n_3) = n_1n_2 + n_1n_3$ for all $n_1, n_2, n_3 \in N$. See [4] or [5] for the general theory of near-rings. For $n \in N$ and $S \subseteq N$ we use the notations $nS := \{ns | s \in S\}$ and $S^2 := \{s_1s_2 | s_1, s_2 \in S\}$ throughout the paper. A subset S of N is called *integral*, if S has no non-zero divisors of zero. N_0, N_c denote the zero-symmetric (constant) parts of N, respectively.

Definition 1.1. If H is an integral subset of a near-ring N with $N^2 \subseteq H$ then N is called *H-integral.*

If N is H-integral with $H = \{0\}$ then N has zero multiplication and may be considered as "known" from the near-ring point of view. If, on the other extreme, $H = N$ then N is an integral near-ring. Again, this case is well-studied (cf. e.g. [5], section 9b2). Hence in the sequel we mainly restrict ourselves to the study of H-integral near-rings with $\{0\} \neq H \neq N$. Note that $0 \in H$ for each H-integral near-ring, as well as $H^2 \subseteq H$; however, H need not be closed under addition, even if $(N, .)$ is commutative (cf. [5], 29 and 30 on p. 411 for such cases with $(N, +) = S_3$, the symmetric group of order 6).

A near-ring N may be H-integral for more than one H. If N^2 is integral, for instance, then N is H-integral for each H between N^2 and N. More precisely we have

Proposition 1.2. *Let $\mathcal{H}$ be the set of all subsets H of N such that N is H-monogenic. Then $\cap\mathcal{H}$ and this is the smallest element of $\mathcal{H}$, while $\cup\mathcal{H}$ is the biggest one.*

Proof. $\mathcal{H}$ is clearly closed w.r.t. intersections, hence $\cap\mathcal{H}$ is the smallest element in $\mathcal{H}$. But $\mathcal{H}$ is also closed under unions: If $h_1 \in H_1 \in \mathcal{H}$, $h_2 \in H_2 \in \mathcal{H}$, $h_1h_2 = 0$ implies $(h_1h_1)h_2 = h_1(h_1h_2) = h_10 = 0$. Since $h_1h_1 \in N^2 \subseteq H_2$, we get either $h_2 = 0$ or $h_1h_1 = 0$ (but then $h_1 = 0$). So $\cup\mathcal{H}$ is the greatest element in $\mathcal{H}$. $\lozenge$

We now give two examples of H-integral near-rings.

Example 1.3. Let N_1 be an arbitrary, N_2 an integral near-ring. Define, in $N := N_1 \times N_2$, $(x, y) \cdot (x', y') := (0, yy')$, $+$ component-wise. Then N is $H := \{0\} \times N_2$-integral.

In other cases, however, N is not so simply composed of an integral and an arbitrary part, even if N is commutative:

Example 1.4. Let $(G, +)$ be a non-abelian group and $K \leq G$ such that $G/_K$ is cyclic of prime or infinite order. Let $x + K$ be a generator of $G/_K$. If $g_1, g_2 \in G$, there are integers n_1, n_2 such that $g_i \in n_ix + K$ $(i = 1, 2)$. Define $g_1 {}^* g_2 := (n_1n_2)x$. By [2], Th. 2.1, $(G, +, {}^*)$ is a commutative near-ring. It is straightforward to see that G is H-integral with $H = \langle x \rangle$. Note that if x is of composite order, G would not be $\langle x \rangle$-integral.

Not all non-trivial distributive near-rings are H-integral, since some are nilpotent, a property which no H-integral near-ring with $H \neq \{0\}$ can have.

For a subset S of a near-ring N, we denote its annihilator $\{a \in \in N | Sa = 0\}$ by $(0 : S)$, while $[0 : S]$ denotes the two-sided annihilator $\{a \in (0 : S) | aS = 0\}$. Also, let $S^* := S \setminus \{0\}$. We now list a number of properties of H-integral near-rings, some being technical (but necessary), some seem to be of independent interest.

Theorem 1.5. *Let N be H-integral and N_0 its zero-symmetric part. Suppose $N_0^2 \neq \{0\}$.*

(1) *For each $h \in H^*$, $(0 : h) = (0 : N_0)$; hence $K := (0 : h)$ is the same for each non-zero $h \in H$, and K is an ideal in N.*
(2) *$H \cap K = (hN) \cap K = \{0\}$ for all $h \in H^*$.*
(3) *$K \subseteq N_0$.*
(4) *For each $n \in N$, $n \in K \iff n$ is nilpotent $\iff n^2 = 0$. Each nilpotent element is therefore zero-symmetric.*
(5) *For each $n, m \in N$, $nm = 0 \iff [(n \in K,\ m \in N_0)$ or $m \in \in K] \iff nm \in K$.*
(6) *N_0 has the IFP (insertion-of-factors property).*
(7) *If $x \in N \setminus K$, $xn \equiv xm (\mathrm{mod}\, K) \iff n \equiv m (\mathrm{mod}\, K)$.*
(8) *K is a prime ideal.*
(9) *$N/_K$ is an integral and prime near-ring which is N-isomorphic to hN for each $h \in H^*$.*
(10) *If $\mathcal{P}(N)$ and $\mathcal{N}(N)$ denote the prime and the nil radical of N then $\mathcal{P}(N) = \mathcal{N}(N) = K$.*
(11) *If $N = N_0$ has the* DCC *on N-subgroups, too, then K also coincides with all Jacobson-type radicals $J_v(N)$ $(v = 0, 1/2, 1, 2)$.*
(12) *If N is not integral, it is never $\mathcal{P}$-, $\mathcal{N}$-, ... , J_2-semisimple.*
(13) *For each $S \subseteq N$, $(0 : S) = K$ or $(0 : S) = N$. Hence each annihilator right ideal is in fact an ideal (N is "almost small"* [5], 9.11).
(14) *If N is planar then N is integral.*

Proof. (1): We first show that $(0 : h) \subseteq (0 : N_0)$. Take $k \in (0 : h)$ and $0 \neq mm' \in N_0^2$. Then for each $n_0 \in N_0$, $hkn_0 = 0$, whence $kn_0 = 0$, since both h and kn_0 are in H. So $kN_0 = 0$. Also, $(0k)(mm') =$

$= 0(km)m' = 0m' = 0$, and since $mm' \neq 0$ we get $0k = 0$. So $n_0kn_0k = 0k = 0$, and $N_0k = 0$ is shown. Conversely, let $k \in (0 : N_0)$. Then for each $n_0 \in N_0$ we get $kn_0kn_0 = 0$, so $kN_0 = 0$. Hence $hkmm' = 0$, from which we deduce that $hk = 0$.

(2): Since $hN \subseteq H$, we consider $k \in H \cap K$. If $H \neq \{0\}$, take $h \in H^*$. By (1), we can write K as $K = (0 : h)$, so $h^2 = 0$, hence $h = 0$.

(3): Follows from the proof of (1).

(4): By (3), $K \subseteq N_0$, and each $k \in K$ has $k^2 = 0$ by (1). Conversely, suppose that $n^r = 0$ some $r \in \mathbb{N}$. Then $n^{2^r} = 0$; hence it sufficies to show that if some $a \in N$ fulfills $a^2 = 0$, then $a \in K$. If $n_0a \neq 0$ for some $n_0 \in N_0$ then $n_0aan = 0$ for all $n \in N_0$, hence $aN_0 = \{0\}$. As in the proof of (1), we see that $N_0a = 0$, so anyhow $a \in K$.

(5): If $nm = 0$, take an arbitrary $n_0' \in N_0$. Then $nnmn_0' = 0$, so either $n^2 = 0$ and hence $n \in K$ by (4), or $n^2 \neq 0$, then $mN_0 = 0$. In the first case, write $m = m_0 + m_c \in N_0 + N_c$, $0 = nm = nm_0 + nm_c = nm_0 + m_c$. Now $nm_0 \in H \cap K = \{0\}$, so $m_c = 0$ and $m \in N_0$. In the second case, take $ab \in N_0^2$, $ab \neq 0$. Then for each $c \in N$ we get $cmab = 0$ and hence $N_0m = 0$. This shows that $m \in (0 : N) = K$. Conversely, suppose that ($n \in K$, $m \in N_0$) or $m \in K$. In both cases, $nm \in H \cap K$ (since K is an ideal), so $nm = 0$ by (2). Finally, the second equivalence follows from (2), too.

(6) If $nm = 0$ then $n \in K$, or $m \in K$ by (5). Hence $nxm = 0$ for all $x \in N_0$, since $nxm \in H \cap K$.

(7) $xn \equiv xm (\operatorname{mod} K) \Rightarrow x(n - m) = xn - xm \equiv 0 \Rightarrow n - m \in K$ by (5) Conversely, $n - m \in K \Rightarrow xn - xm = x(n - m) \in K$, since K is an ideal

(8) Let I, J be ideals of N with $I \cdot J \subseteq K$. Then $I \cdot J \subseteq H \cap K$, so $I \cdot J = \{0\}$ Suppose $I \subseteq K$, and take $i \in I \setminus K$. For each $j \in J$, $ij = 0 = i0$, by (7), $j \in K$ hence $J \subseteq K$.

(9): If $h \in H^*$, $\phi : N \to hN$, $n \to hn$ is an N-epimorphism with kernel $(0 : h) = K$ $N/_K$ is integral by (5) and prime by (8).

(10): The intersection $\mathcal{P}(N)$ of all prime ideals of N is contained in K by (8). Conversely, if P is a prime ideal then $K \subseteq P$ because of $K \cdot K = \{0\} \subseteq P$ Hence $K = \mathcal{P}(N)$. By (4) and (5), K contains all nil ideals, and hence also their sum $\mathcal{N}(N)$. On the other hand, K itself is nil and hence $K = \mathcal{N}(N)$.

(11): Follows from [5], 5.61, while

(12): is a consequence of (10) and the fact that $\mathcal{P}(N) \subseteq J_2(N)$ always holds.

(13): If $n \in K$ then $nN_0 = \{0\}$ by (1) and (3). Hence $N_0 \subseteq (0 : n)$. If $n' = n'_0 + n'_c \in (0 : n)$ then $0 = nn' = nn'_0 + nn'_c = 0 + n'_c$. Hence $(0 : n) = N_0$. If, on the other hand, $n \notin K$ then $a \in (0 : n)$ implies $na = 0$, consequently $a \in K$ by (5), so $(0 : n) \subseteq K$. But also $nK = 0$ by (5), so $(0 : n) = K$. So all $(0 : n)$ are either $= K$ or $= N$, and the same applies to all $(0 : S)$.

(14): A planar near - ring N fulfills $N^2 = N$ by [5], 8.102. Hence $H = N$, and N is integral. $\Diamond$

Although for all $h_1, h_2 \in H^*$, the near-rings h_1N and h_2N are integral and N - isomorphic, they are not necessarily equal ([5], no. 37 on p. 411), nor are they always near - integral domains ([7], no. 74 on p. 112).

The condition $N_0^2 \neq \{0\}$ in Th. 1.5 is indispensable: Define on $N := \mathbb{Z} \times \mathbb{Z}$ (with componentwise addition) $(a, b) \cdot (c, d) := (0, 3\bar{b}c + d)$, where $\bar{b}$ denotes the remainder $\in \{0, 1, 2, \}$ of b after division by 3. N becomes so a near-ring with $N_0 = \mathbb{Z} \times \{0\}$, $N_c = \{0\} \times \mathbb{Z}$, $N^2 = N_c$. If we take $H := N_c \cup \{(1, 1)\}$, N can be checked to be H - integral. $((0, 0) : N) = N_0$, but $((0, 0) : (1, 1))$ also contains, for instance, the element $(-1, 3)$, since $(1, 1)(-1, 3) = (0, 3 \cdot 1 \cdot (-1) + 3) = (0, 0)$. Therefore we adapt for the rest of this paper the

Convention: All near-rings have $N_0^2 \neq \{0\}$. So all H - integral near-rings have $H \neq \{0\}$.

2. Decompositions and constructions

In (9) of Th. 1.5 we have seen that an H - integral near-ring N is an extension of K by hN (h any element of H^*). In fact, we often can get even more:

Theorem 2.1. *Let K be H - integral such that $N/_K$ is not (group-) isomorphic to one of its proper subgroups. Then $(N, +)$ is a semidirect sum of K and hN (h any element in H^*).*

Proof. All that remains to be shown after Th. 1.5 is that $N = hN + K$. By the first isomorphism theorem for groups, $(hN + K)/_K \cong hN/_{(hN \cap K)} = hN/_{\{0\}} \cong hN \cong N/_K$, hence $N/_K = (hN + K)/_K$, so $N = hN + K$ as desired. $\Diamond$

Note that the assumption on $N/_K$ in Th. 2.1 is trivially fulfilled if $N/_K$ is finite. This theorem has a lot of consequences. For that, call a near-ring N *almost constant* if N is constant or $0m = 0$, $nm = m$ for all $n \neq 0$.

Corollaries 2.2. *Let N be H - integral and $N/_K$ finite.*

(i) *For each $h \in H^*$, hN is (as a near-ring!) isomorphic to $N/_K$. Hence all h_iN ($h_i \in H^*$) are pairwise isomorphic near-rings.*

(ii) *N has no non - zero nilpotent elements iff N is integral.*

(iii) *If hN is not almost constant then $(N,+)$ is nilpotent iff $(K,+)$ is nilpotent.*

Proof. (i): Since $(N,+)$ is a semidirect sum of K and hN (for $h \in H^*$), the map $\phi : N \to hN$, $x = k + hn \to hn$ is a (well - defined) group epimorphism. For $x, x' \in N$ $x = k + hn$, $x' = k' + hn'$ ($k, k' \in K$, $n, n' \in N$) we get $xx' = (k+hn)(k'+hn') = (k+hn)k' + (k+hn)hn' - -hnhn' + hnhn' = k'' + hnhn'$ for a suitable $k'' \in K$ (because K is an ideal of N). Hence $\phi(xx') = \phi(x)\phi(x')$, $\mathrm{Ker}\,\phi = K$; and we are done.

(ii): If N has no non - zero nilpotent elements then $K = \{0\}$, so $N = hN \subseteq N^2 \subseteq H$, so N is integral. The converse is clear.

(iii): By [5], 9.45 and 9.51 $(hN,+)$ is nilpotent if $h \in H^*$. So by Th. 2.1 (or by [6], p. 382), $(N,+)$ is nilpotent iff $(K,+)$ is. ◊

Let us remark that (iii) cannot be improved: Take any group $(G,+)$ and define $g * g' := g'$ for all $g, g' \in G$. Then $(G,+,^*)$ is H - integral for $H = G$, and $hG = H = G$ for all $h \in H$, $K = \{0\}$. We also remark that the proof of (i) in Cor. 2.2 shows that for all $a, a' \in hN$ and $k, k' \in K$, $(k + a)(k' + a') \equiv aa' (\mathrm{mod}\, K)$.

Corollary 2.3. *Let N be H - integral, $h \in H^*$, hH a finite ideal of N. Then $N \cong K \oplus hN$ (the direct sum in the near-ring sense).*

Proof. hN is then normal, hence $(N,+) = K + hN$. Also, if $x = k + hn$, $x' = k' + hn'$ are "typical" elements of N, then $xx' = (k+hn)(k'+hn') = = (k+hn)k' + (k+hn)hn' = (k+hn)hn' = (k+hn)hn' - hnhn' + hnhn' = = hnhn' = kk' + hnhn'$ (since $(k + hn)hn' - hnhn' \in K \cap hN = \{0\}$). Hence the result. ◊

Now we show that the semidirect decomposition in Th. 2.1 is in some sense the only decomposition of that kind.

Theorem 2.4. *Let N be H - integral, $h \in H^*$, A a nilpotent ideal of N, B an integral N - subgroup of N. If $(N,+)$ is a semidirect sum of A and B then $A = K$ and $(B,+,.) \cong (hN,+,.)$.*

Proof. By (10) of Th. 1.5, $A \subseteq K \subseteq N_0$. Conversely, if $k \in K$

then $k = a + b$ ($a \in A$, $b \in B$). Now $0 = ak = a^2 + ab = ab$, hence $baba = b0a = 0a = 0$. But $ba \in BN \subseteq H$, so $ba = 0$ as well. Hence $0 = bk = ba + b^2$, whence $b^2 = 0$, hence $b = 0$ and $A = K$. As in the proof of Cor. 2.2 (i), $N/_K \cong B$ (as near - rings). Since $N/_K \cong hN$ as well, we have the desired result. ◊

We turn to construction methods for H - integral near - rings. The first one comes from Th. 2.1 and contains both Examples 1.3 and 1.4 as special cases:

Construction Method 1. Take any near - ring N_1, an integral near - ring N_2, and a semidirect sum $(N, +)$ of $(N_1, +)$ (normal) and $(N_2, +)$. Define in $N : (n_1 + n_2) \cdot (n_1' + n_2') := n_2 n_2'$. Then $(N, +, .)$ is H - integral for each H such that $N_2 \subseteq H \subseteq \{n_1 + n_2 | n_1 \neq 0\}$.

A special case of this construction is supplied by a method due to G. Ferrero [1].

Construction Method 2. Let $(G, +)$ be a group which is a semidirect sum of the normal subgroup K and the finite subgroup A. Let Φ be a fixed - point - free group of automorphisms of A, and R a (complete) system of representatives of the orbits of A^* under Φ. If $x = k + a$, $x' = k' + a'$ are in G, define $x \cdot y = 0$ if $a = 0$ and $x \cdot y = \phi(a')$ if a is in the orbit of $r \in R$ and $f(r) = a$ with $f \in F$. Then $(G, +, .)$ is H - integral with $H = \{k + a | k \in K,\ a \in A^*\} \cup \{0\}$, $K = (0 : G)$, $G/_K \cong A$, R =set of all left identities of $(A, .)$.

Note that the Method 2 works because this construction gives an integral near - ring $(A, +, .)$ and $(k + a)(k' + a') = aa'$ as in Method 1. That R is the set of left identities of $(A, .)$ is straightforward.

3. Distributively generated H - integral near - rings

In this final section, we briefly discuss the special class of d.g. H - integral near - rings. Let N'' be the second commutator subgroup of $(N, +)$. We will use the following

Lemma 3.1. ("Itô's Theorem", [3]) *If a group G is the sum of two abelian subgroups, then $G'' = \{0\}$.*

Theorem 3.2. *Let N be a d.g. near - ring such that $K \neq N$ has finite index in N. Then $N/_K$ is a finite field. If, moreover, $(K, +)$ is abelian then $N'' = \{0\}$.*

Proof. Recall that Th. 2.1 is applicable; N is zero - symmetric because it is d.g. If $d = k + hn$ ($k \in K$, $h \in H^*$, $n \in N$) is distributive then by Theorem 1.5 (1), hn is distributive, too.So hN is again d.g., and by [5], 9.48 (d), hN (and the isomorphic copy $N/_K$) are fields. In particular, $(hN, +)$ is abelian. If $(K, +)$ is abelian too, we can apply Itô's Theorem 3.1. $\Diamond$

Surprisingly enough, it possible for $(N, +)$ to be non-nilpotent, even if N is "almost a ring": the near-ring N on p. 411 of [5], no. 29 is a distributive, commutative and anticommutative H-integral near-ring with $hN \cong GF(2)$, K cyclic of order 3 and $(N, +) =$ the non-nilpotent group S_3.

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