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COMPLEX INTERPOLATION AND l_n^1 PROPERTY

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Abstract: We prove that, for a Banach space, the uniform non-l_n^1 property is preserved under Calderon's complex interpolation method.

Introduction

In this paper we prove that for a Banach space, the *uniform non*$-l_n^1$ property is preserved under the complex interpolation method introduced by Calderon [2].

Let X a complex Banach space and $B(X)$ its unit ball, that is $B(X) = \{x \in X : \|x\| \leq 1\}$. We say that X is *uniformly non*$-l_n^1$ if there exists some $\delta > 0$ such that, for any vectors $x_1, \ldots, x_n$ in $B(X)$ there exists a choice of signs $\varepsilon_1, ., \varepsilon_n (\varepsilon_i = 1$ or $-1)$ such that $\| \sum_{i=1}^n \varepsilon_i x_i \| < n(1-\delta)$ (see [4]).

We say that X is *B-convex* if it is *uniformly non*$-l_n^1$ for some integer n ([1],[4]). For $n = 2$, *uniformly non*$-l_n^1$ spaces are also called *uniform non-square.*

By (X_0, X_1), we denote an interpolation pair of complex Banach spaces and by $X_s, (0 < s < 1)$, the intermediate spaces obtained by Calderon's complex interpolation method. We will indicate, as usually, by $\|\cdot\|_0, \|\cdot\|_1, \|\cdot\|_s$, the norms in X_0, X_1, X_s respectively. We will use, also, some notation of Calderon's paper, in particular we will indicate by $F(X)$ the set of all functions $f : S \to X_0 + X_1$(with $S = \{z \in \in \mathrm{C}, 0 \le \mathrm{Re}\, z \le 1\}$), continuous in S, analytic in int$S$, with $f(j+it) \in \in X_j$ for $j = 0,1$ and with $f(j+it) \to 0$ as $|t| \to \infty$.

Main result

Theorem. *If X_0 or X_1 is uniformly non$-l_n^1$ then X_s is uniformly non$-l_n^1$ for every $s \in (0,1)$.*

Proof. Suppose that X_0 is uniformly non$-l_n^1$ and, by absurdity, that X_s is not uniformly non$-l_n^1$. This means that for every $\sigma > 0$ there exist $x_1, \ldots, x_n \in B(X_s)$ such that $\|\frac{1}{n}\sum_i \varepsilon_i x_i| \ge 1-\sigma$ for every choice of $\varepsilon_i = \pm 1$. For a fixed $\eta > 0$ there exist functions $f_i \in F(X)$ satisfying

a) $$f_i(s) = \frac{x_i}{1+\eta} = x_i';$$

b) $$\|f_i\| = \max_{j=0,1}(\sup_{t\in\mathbb{R}}\|f_i(j+it)\|_j) \le 1 (i = 1,2,\ldots,n).$$

For every choice of $\varepsilon_i = \pm 1$ we define

$$E_{\varepsilon_1\ldots\varepsilon_n} = \{t \in \mathbb{R} : \|\frac{1}{n}\sum_{i=1}^{n}\varepsilon_i f_i(it)\|_0 < 1-\delta\}$$

(where δ is taken from the definition of uniformly non$-l_n^1$ of X_0).

We will use the following inequality (see [2] p.117):

$$\lg(\|\frac{1}{n}\sum_i \varepsilon_i x_i'\|_s) \le \int_{-\infty}^{+\infty} \lg\|\frac{1}{n}\sum_{i=1}^{n}\varepsilon_i f_i(it)\|_0 \mu_0(s,t)dt +$$
$$+ \int_{-\infty}^{+\infty} \lg\|\frac{1}{n}\sum_{i=1}^{n}\varepsilon_i f_i(1+it)\|_1 \mu_1(s,t)dt$$

where $\mu_j(s,t)(j = 0,1)$ is the Poisson kernel for the strip. In our case we obtain:

$$\lg\frac{1-\sigma}{1+\eta} \le \int_{E_{\varepsilon_1\ldots\varepsilon_n}} \lg\|\frac{1}{n}\sum_{i=1}^{n}\varepsilon_i f_i(it)\|_0 \mu_0(s,t)dt +$$

$$+ \int_{E^c_{\varepsilon_1 \dots \varepsilon_n}} \lg \|\frac{1}{n} \sum_{i=1}^{n} \varepsilon_i f_i(it)\|_0 \mu_0(s,t)dt +$$

$$+ \int_{-\infty}^{+\infty} \lg \|\frac{1}{n} \sum_{i=1}^{n} \varepsilon_i f_i(1+it)\|_1 \mu_1(s,t)dt.$$

But since for every $t \in \mathbb{R}$: $\|\frac{1}{n}\sum_{i=1}^n \varepsilon_i f_i(j+it)\|_j \leq 1 \quad (j = 0,1)$, we obtain $\lg\frac{1-\sigma}{1+\eta} \leq (1-s)|E_{\varepsilon_1 \dots \varepsilon_n}|\lg(1-\delta)$ (where we set $|A| = \frac{1}{1-s}\int_A \mu_0(s,t)dt$) that is $\frac{1-\sigma}{1+\eta} \geq (1-\delta)\exp\{(1-s)|E_{\varepsilon_1 \dots \varepsilon_n}|\}$ and, being $\eta > 0$ arbitrary, we obtain $\sigma \leq 1 - (1-\delta)\exp\{(1-s)|E_{\varepsilon_1 \dots \varepsilon_n}|\}$.

If we choose $\sigma = 1 - (1-\delta)\exp(\frac{1-s}{2^{n+1}})$ we must have $|E_{\varepsilon_1 \dots \varepsilon_n}| \leq 1/2^{n+1}$. This implies that $|\bigcup E_{\varepsilon_1 \dots \varepsilon_n}| \leq 1/2$ that is $(\bigcup E_{\varepsilon_1 \dots \varepsilon_n})^c \neq \emptyset$ (where the union is taken over all choices of signs). But this is a contradiction since X_0 is uniformly non$-l^1_n$, so our theorem is proved. ◇

Corollary 1. *If X_0 or X_1 is B-convex, then X_s is B-convex for every $s \in (0,1)$.*

We also obtain the following result already proved in [3]:

Corollary 2. *If X_0 or X_1 is uniformly non-square then X_s is uniformly non-square.*

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