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A CONTINUOUS AND A DISCRETE VARIANT OF WIRTINGER'S INEQUALITY

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Dedicated to Professor Dr. Dr. h. c. mult. Edmund Hlawka on occasion of his 75th birthday

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Abstract. We prove: If f is a real-valued continuously differentiable function with period 2π and $\int_0^{2\pi} f(x)dx = 0$, then

$$\frac{6}{\pi} \max_{0\le x\le 2\pi} f(x)^2 \le \int_0^{2\pi} f'(x)^2 dx,$$

and, if $z_1, \dots, z_n (n \ge 2)$ are complex numbers with $\sum_{k=1}^n z_k = 0$, then

$$\frac{12n}{n^2-1} \max_{1\le k\le n} |z_k|^2 \le \sum_{k=1}^n |z_{k+1} - z_k|^2,$$

where $z_{n+1} = z_1$. The constants $6/\pi$ and $12n/(n^2-1)$ are best possible.

1. Introduction

In 1916 a remarkable result of W. Wirtinger, which compares the integral of a square of a function with that of the square of its first derivative, was published in W.Blaschke's book "Kreis und Kugel" [2, p. 105]:

Theorem A. *Let f be a real-valued function with period 2π and $\int_0^{2\pi} f(x)dx = 0$. If $f' \in L^2$, then*

$$(1.1) \qquad \int_0^{2\pi} f(x)^2 dx \le \int_0^{2\pi} f'(x)^2 dx$$

with equality holding if and only if

$$f(x) = A\cos(x) + B\sin(x) \qquad (A, B \in \mathbb{R}).$$

The following discrete analogue of Wirtinger's inequality was proved for the first time in 1950 by I.J. Schoenberg [11].

Theorem B. *If $z_1, \dots, z_n (n \ge 2)$ are complex numbers with $\sum_{k=1}^n z_k = 0$, then*

$$(1.2) \qquad 4\sin^2 \frac{\pi}{n} \sum_{k=1}^n |z_k|^2 \le \sum_{k=1}^n |z_{k+1} - z_k|^2,$$

where $z_{n+1} = z_1$. Equality holds in (1.2) if and only if $z_k = A\cos\frac{2\pi k}{n} + B\sin\frac{2\pi k}{n}$, $(k = 1, \dots, n;\ A, B \in \mathbb{C})$.

Theorem A and Theorem B have evoked the attention of many mathematicians and in the past years different proofs, intriguing extensions and refinements as well as many related results were discovered [1 – 13]; see in particular [1], [8, pp. 141 – 154] and the references therein.

The aim of this paper is to present variants of inequalities (1.1) and (1.2). More precisely we shall answer the questions: What is the best possible constant α such that

$$\alpha \max_{0 \le x \le 2\pi} f(x)^2 \le \int_0^{2\pi} f'(x)^2 dx$$

holds for all real-valued functions $f \in C^1$ fulfilling the conditions of Theorem A; and what is the best possible constant β_n such that

$$\beta_n \max_{1 \le k \le n} |z_k|^2 \le \sum_{k=1}^n |z_{k+1} - z_k|^2$$

is valid for all complex numbers $z_1, \dots, z_n$ satisfying the assumptions of Theorem B? Furthermore in both inequalities we determine all cases of equality.

2. The continuous case

In this section we establish a counterpart of Wirtinger's inequality (1.1).

Theorem 1. *If f is a real-valued continuously differentiable function with period 2π and $\int_0^{2\pi} f(x)dx = 0$, then*

$$\frac{6}{\pi} \max_{0\le x\le 2\pi} f(x)^2 \le \int_0^{2\pi} f'(x)^2 dx. \tag{2.1}$$

Equality holds in (2.1) *if and only if*

$$f(x) = c\,[\,3(\frac{x-\pi}{\pi})^2 - 1\,] \quad (0 \le x \le 2\pi)$$

where c is a real constant.

Proof. We may assume

$$\max_{0\le x\le 2\pi} f(x)^2 = f(x_0)^2 > 0, \quad 0 \le x_0 < 2\pi.$$

Then we have the following integral identity:

$$\int_{x_0}^{x_0+2\pi} \Big[\frac{f'(x)}{f(x_0)} - \frac{3}{\pi^2}(x - x_0 - \pi)\Big]^2 dx = \tag{2.2}$$

$$= \int_{x_0}^{x_0+2\pi} \Big[\frac{f'(x)}{f(x_0)}\Big]^2 dx - \frac{6}{\pi^2 f(x_0)} \int_{x_0}^{x_0+2\pi} f'(x)\;(x - x_0 - \pi)dx +$$

$$+ \frac{9}{\pi^4} \int_{x_0}^{x_0+2\pi} (x - x_0 - \pi)^2 dx = \frac{1}{f(x_0)^2} \int_{x_0}^{x_0+2\pi} f'(x)^2 dx - \frac{6}{\pi}\,,$$

where the third integral of (2.2) has been calculated by integration by parts and by using the assumptions $f(x_0) = f(x_0 + 2\pi)$ and $\int_{x_0}^{x_0+2\pi} f(x)dx = 0$.
Hence we obtain

$$\int_0^{2\pi} f'(x)^2 dx = \int_{x_0}^{x_0+2\pi} f'(x)^2 dx \ge \frac{6}{\pi} \max_{0\le x\le 2\pi} f(x)^2.$$

We discuss the cases of equality. Let $f(x) = c\,[3(\frac{x-\pi}{\pi})^2 - 1]$ $(0 \le$ $\le x \le 2\pi; c \in \mathbb{R})$. Simple calculations reveal that f^2 attains its maximum at 0 which implies

$$\int_0^{2\pi} f'(x)^2 dx = \frac{24c^2}{\pi} = \frac{6}{\pi} \max_{0\leq x\leq 2\pi} f(x)^2.$$

If equality holds in (2.1) then we obtain from the identity above:

$$f'(x) = \frac{3f(x_0)}{\pi^2}(x - x_0 - \pi) \quad (x_0 \leq x \leq x_0 + 2\pi)$$

which leads to

$$f(x) = \frac{3f(x_0)}{2\pi^2}(x - x_0 - \pi)^2 + c' \quad (c' \in \mathbb{R}).$$

Setting $x = x_0$ we get $c' = -\frac{1}{2}f(x_0)$; thus we have

$$f(x) = \frac{1}{2}f(x_0)[\frac{3}{\pi^2}(x - x_0 - \pi)^2 - 1] \qquad (x_0 \leq x \leq x_0 + 2\pi)$$

or

$$f(x) = \begin{cases} \frac{1}{2}f(x_0)[3(\frac{x-x_0+\pi}{\pi})^2 - 1], & 0 \leq x \leq x_0 \\ \frac{1}{2}f(x_0)[3(\frac{x-x_0-\pi}{\pi})^2 - 1], & x_0 \leq x \leq 2\pi. \end{cases}$$

Since f is differentiable at $x_0 \in [0, 2\pi)$ we conclude $x_0 = 0$; this yields

$$f(x) = \frac{1}{2}f(0)\,[3(\frac{x-\pi}{\pi})^2 - 1] \qquad (0 \leq x \leq 2\pi). \ \diamondsuit$$

3. The discrete case

Now we provide a variant of Schoenberg's inequality (1.2), respectively a discrete analogue of (2.1).

Theorem 2. *If* $z_1, \ldots, z_n$ $(n \geq 2)$ *are complex numbers with* $\sum_{k=1}^n z_k = 0$, *then*

$$\text{(3.1)} \qquad \frac{12n}{n^2-1} \max_{1\leq k\leq n} |z_k|^2 \leq \sum_{k=1}^n |z_{k+1} - z_k|^2,$$

where $z_{n+1} = z_1$. *Equality holds in* (3.1) *if and only if*

$$z_k = \begin{cases} c\left[1 + \frac{6(k-r)(k+n-r)}{n^2-1}\right], & 1 \leq k \leq r-1, \\ c\left[1 + \frac{6(k-r)(k-n-r)}{n^2-1}\right], & r \leq k \leq n, \end{cases}$$

where $r \in \{1, \dots, n\}$ and c is a complex constant.

Proof. Let $\max_{1\le k\le n}|z_k| = |z_r| > 0$. Using the assumptions $z_{n+1} = z_1$ and $\sum_{k=1}^n z_k = 0$ we obtain after several elementary (but tedious) calculations the following identity:

$$\sum_{k=1}^{r-1}\left|\frac{z_{k+1}-z_k}{nz_r} - \frac{12(k+n-r)-6(n-1)}{n(n^2-1)}\right|^2 + \tag{3.2}$$

$$+\sum_{k=r}^{n}\left|\frac{z_{k+1}-z_k}{nz_r} - \frac{12(k-r)-6(n-1)}{n(n^2-1)}\right|^2 =$$

$$= \sum_{k=1}^{n}\left|\frac{z_{k+1}-z_k}{nz_r}\right|^2 + \frac{36}{[n(n^2-1)]^2}\left\{\sum_{k=1}^{r-1}(2k+n-2r+1)^2 + \right.$$

$$\left. + \sum_{k=r}^{n}(2k-n-2r+1)^2\right\} - \frac{12}{n^2(n^2-1)}\mathrm{Re}\left\{\frac{1}{z_r}\sum_{k=1}^{r-1}(z_{k+1} - \right.$$

$$\left. - z_k)(2k+n-2r+1) + \frac{1}{z_r}\sum_{k=r}^{n}(z_{k+1}-z_k)(2k-n-2r+1)\right\} =$$

$$= \frac{1}{n^2|z_r|^2}\sum_{k=1}^{n}|z_{k+1}-z_k|^2 - \frac{12}{n(n^2-1)}$$

which implies

$$\sum_{k=1}^{n}|z_{k+1}-z_k|^2 \ge \frac{12n}{n^2-1}\max_{1\le k\le n}|z_k|^2.$$

It remains to discuss the cases of equality. Let $r \in \{1, \dots, n\}, c \in \mathbb{C}$ and let

$$z_k = \begin{cases} c\left[1 + \frac{6(k-r)(k+n-r)}{n^2-1}\right], & 1 \le k \le r-1, \\ c\left[1 + \frac{6(k-r)(k-n-r)}{n^2-1}\right], & r \le k \le n. \end{cases}$$

Then we have

$$\max_{1\le k\le n}|z_k| = |z_r| = |c|$$

which leads to

$$\sum_{k=1}^{n}|z_{k+1}-z_k|^2 = \frac{12n}{n^2-1}|c|^2 = \frac{12n}{n^2-1}\max_{1\le k\le n}|z_k|^2.$$

Now we assume that equality holds in (3.1). Then we conclude from (3.2):

$$(3.3)\qquad \frac{z_{k+1}-z_k}{nz_r} = \begin{cases} \frac{12(k+n-r)-6(n-1)}{n(n^2-1)}, & 1 \le k \le r-1, \\ \frac{12(k-r)-6(n-1)}{n(n^2-1)}, & r \le k \le n. \end{cases}$$

Let $1 \le k \le r$; because of $z_{n+1} = z_1$ we obtain from (3.3):

$$z_k - z_r = \sum_{j=r}^{n}(z_{j+1}-z_j) + \sum_{j=1}^{k-1}(z_{j+1}-z_j) = \frac{6(k-r)(k+n-r)}{n^2-1}z_r;$$

and if $r \le k \le n$, then (3.3) yields

$$z_k - z_r = \sum_{j=r}^{k-1}(z_{j+1}-z_j) = \frac{6(k-r)(k-n-r)}{n^2-1}z_r.$$

This completes the proof of Theorem 2. ◇

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