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ON UNITAL EXTENSIONS OF NEAR-RINGS AND THEIR RADICALS

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Abstract: Not every near-ring can be embedded as an ideal in a near-ring with an identity. A necessary and sufficient condition on a near-ring N for such an extension $\overline{N}$ to exist is known. The construction of $\overline{N}$ is not canonical in the sense that the quotient $\overline{N}/N$ is not fixed for a given N. We modify this extension to one (resembling the Dorroh extension of rings) for which the quotient is always fixed. For radicals with hereditary semisimple classes, the radical of N and the radical of this extension coincide if and only if the ring of integers has zero radical.

1. Introduction

Not every near-ring has a unital extension. Betsch [1] gave an example of such a near-ring on a non-commutative group and asks whether such near-rings on commutative groups exist. We provides such examples in section 1 below. Subsequently Betsch gives a necessary and sufficient condition on a near-ring N to have a unital extension $\overline{N}$. He also gives an explicit description of this near-ring $\overline{N}$. In section 1 we provide an alternative construction of the near-ring $\overline{N}$. This construction, which generalizes the well-known Dorroh extension of a ring, has the advantage that it makes it easy to compare the radicals of N and its unital extension (section 2).

1. Unital extensions of near-rings

All near-rings considered are 0-symmetric and right distributive near-rings.

Example 1.1. *There exists near-rings with commutative underlying groups which are never left ideals nor right ideals in a near-ring with an identity:*

Let G be any group which contains an element $e \neq 0$ with order not 2. Let N be the near-ring on G with multiplication defined by:

$$nm = \begin{cases} n & \text{if } m \neq 0 \\ 0 & \text{if } m = 0. \end{cases}$$

Let $\overline{N}$ be a near-ring with an identity 1 such that $N \subseteq \overline{N}$. If N is a right ideal in $\overline{N}$, then $e(e+1) \in N$. Thus $e(e+1) = (e(e+1))(-e) = e(e-e) = 0$. If N is a left ideal in $\overline{N}$, then $e(e+1) - e = e(e+1) - e1 \in N$. Thus $e(e+1) - e = (e(e+1) - e)(-e) = e(e-e) - e = -e$ and whence $e(e+1) = 0$. Hence, if N is either a left or a right ideal of $\overline{N}$, then $e(e+1) = 0$. Consequently, since $e + e \neq 0$, we have $0 = 0e = (e(e+1))e = e(e+e) = e$. But this contradicts the choice of $e \neq 0$. ◇

In [1], Betsch has given a necessary and sufficient condition on a near-ring to have a unital extension. This condition on a near-ring N is:

(BC) There exists a faithful N-group Γ (hence N is considered as a subnear-ring of $M_0(\Gamma)$) such that:

(i) The mapping $x \longrightarrow -1 + x + 1$ of $M_0(\Gamma)$ into itself induces an automorphism of N (1 is the identity map on Γ).

(ii) For all $n, m \in N$ and $a \in \mathbb{Z}$ ($\mathbb{Z}$ the integers), $n(m + a1) \in N$ (the cyclic subgroup of $M_0(\Gamma)$ generated by 1 is considered as an $\mathbb{Z}$ -module).

The near-ring $\overline{N}$ is a subnear-ring of $M_0(\Gamma)$ and is given by $\overline{N} = \{n + a1 | n \in N, a \in \mathbb{Z}\}$. This near-ring $\overline{N}$ is not canonical in the sense that for a near-ring N satisfying the condition (BC), $\overline{N}/N$ need not be fixed. It can be verified that $\overline{N}/N$ is always either one of the rings $\mathbb{Z}$

(integers) or $\mathbb{Z}_a$ (integers mod a) for some $a \geq 1$. When comparing the radicals of N and $\overline{N}$, it is useful to know the radical of $\overline{N}/N$. Since this quotient is not fixed, it is not always straightforward to compare the respective radicals. In order to fix the quotient, we propose a slightly modified construction, denoted by $D(N)$, such that for any near-ring N satisfying the condition (BC), $D(N)/N \cong \mathbb{Z}$. Furthermore, if N is a ring, the faithful N-group Γ can be chosen such that $D(N)$ is the usual unital extension of N (i.e. the Dorroh extension of N, cf [3]). Although this may not be the most ecconomical embedding, this construction enables us to give an easy criterion for comparing the radicals of N and $D(N)$ (Theorem 2.1 below).

Theorem 1.2. *Let N be near-ring which satisfies the condition* (BC). *Then there exists a unital extension $D(N)$ of N such that $D(N)/N \cong$ $\cong \mathbb{Z}$($\mathbb{Z}$ is the ring of integers.)*

Proof. Let Γ be the faithful N-group provided by our assumption BC on N (hence $N \hookrightarrow M_0(\Gamma)$). On the cartesian product $N \times \mathbb{Z}$ define addition and multiplication by:

$$(n, a) + (m, b) = (n + a1 + m - a1, a + b)$$

$$(n, a)(m, b) = ((n + a1)(m + b1) - (ab)1, ab)$$

At the outset, we must verify that these operations are well defined. Since $n \longrightarrow -1+n+1$ is an automorphism of N (1 is the identity map on Γ), it follows that $a1 + m - a1 \in N$ for all $a \in \mathbb{Z}, m \in N$. Furthermore, $(n + a1)(m + b1) - (ab)1 = n(m + b1) + a1(m + b1) - (ab)1$. The first term is in N from the second part of the condition (BC); hence we only concern ourselves with the last two terms.

Suppose $a > 0$ (a similar argument takes care of the case $a < 0$). Then

$$a1(m + b1) - (ab)1 = (m + b1) + \ldots + (m + b1) - (ab)1 = m + (b1+$$

$$+m - b1) + (2b1 + m - 2b1) + \ldots + ((ab)1 + m - (ab)1) + (ab)1 - (ab)1$$

which is in N.

It can be verified that $+$ defines a group structure on $N \times \mathbb{Z}$ with additive identity (0,0) and the additive inverse of (n, a) given by $(-a1 - n + a1, -a)$. Furthermore, the multiplication is associative and distributive over the addition, hence we have a near-ring which we denote by $D(N)$. Clearly $N \cong \{(n, 0) | n \in N\} \triangleleft D(N), D(N)/N \cong \mathbb{Z}$ and (0,1) is the multiplicative identity of $D(N)$. $\diamondsuit$

If R is a ring, then R satisfies condition (BC) with $\Gamma = D(R)^+$, where $D(R)$ here denotes the usual Dorroh extension of the ring R. In this case, the addition in the above construction simplifies to $(n,a) + +(m,b) = (n+a1+m-a1, a+b) = (n+m, a+b)$ and the multiplication becomes $(n,a)(m,b) = ((n+a1)(n+b1)-ab1, ab)+(nm+bn+am, ab)$. Hence the above construction coincides with Dorroh extension of the ring R for this choice of Γ.

A sufficient "internal" condition on a near-ring N which implies the condition (BC) is given by:

Proposition 1.3. *Let N be a near-ring which contains a left ideal L with $(L:N)_N = 0$ such that:*

1. *For any $N \in N, a \in \mathbb{Z}$, there exists an $p \in N$ such that $-ak+nk+ak-pk \in L$ for all $k \in N$.*
2. *For any $n, m \in N, a \in \mathbb{Z}$, there exists an $p \in N$ such that $n(mk+ak)-pk \in L$ for all $k \in N$.*

Then N satisfies condition (BC).

Proof. Since L is a left ideal of N with $(L:N)_N = 0, \Gamma := N/L$ is a faithfull N-group via $n(x+L) = nx+L$. Embed N in $M_0(\Gamma)$ by $\varphi : N \longrightarrow M_0(\Gamma)$ defined by $\varphi(n) = \varphi_n : \Gamma \longrightarrow \Gamma, \varphi_n(x+L) = nx+L$. Let $f : M_o(\Gamma) \longrightarrow M_0(\Gamma)$ be the function defined by $f(x) = -1+x+1$. By condition 1 above, f induces an automorphism of $N \cong \varphi(N)$. Moreover, condition 2 above yields the requirement (ii) of (BC). $\diamond$

The converse of the above proposition is not true: Consider any non-zero ring R with $R^2 = 0$.

2. The radical of the unital extension D(R).

Radical classes will be in the sense of Kurosh and Amitsur, cf [4] or Wiegandt [5]. The *semisimple class* of a radical $\mathcal{R}$ is the class $S\mathcal{R} = \{N | \mathcal{R}(N) = 0\}$. $S\mathcal{R}$ is *hereditary* if $I \triangleleft N \in S\mathcal{R}$ implies $I \in S\mathcal{R}$. As is well known, $S\mathcal{R}$ is hereditary if and only if $\mathcal{R}(I) \subseteq \mathcal{R}(N)$ for all near-rings N and $I \triangleleft N$. The variety of 0-symmetric near-rings contains many examples of radicals with hereditary semisimple classes, for example, J_2, J_3 and $\mathcal{G}$ (the Brown-McCoy radical class). Many more examples can be found in [4]. Some useful properties of a radical class $\mathcal{R}$ required here are:

(1) $\mathcal{R}(N/I) = 0$ implies $\mathcal{R}(N) \subseteq I$ for $I \triangleleft N$;

(2) $\mathcal{R}(\mathcal{R}(N)) = \mathcal{R}(N)$ for all N;
(3) $\mathcal{R}(N/\mathcal{R}(N)) = 0$ for all N.

Our final result generalizes the corresponding result from the variety of rings (cf De la Rosa and Heyman [2]), albeit with some restrictions. This is necessitated by the fact that, contrary to the case for rings, not every semisimple class of near-rings is necessarily hereditary and not every near-ring has a unital extension.

Theorem 2.1. *Let $\mathcal{R}$ be a radical class with a hereditary semisimple class. Then $\mathcal{R}(N) = \mathcal{R}(D(N))$ for all near-rings N which satisfy the condition* (BC) *if and only if $\mathcal{R}(\mathbb{Z}) = 0$.*

Proof. If $\mathcal{R}(\mathbb{Z}) = 0$ and $D(N)$ exists for the near-ring N, then $\mathcal{R}(D(N)/N) = \mathcal{R}(\mathbb{Z}) = 0$; hence $\mathcal{R}(D(N)) \subseteq \mathcal{R}(N)$. But $S\mathcal{R}$ hereditary implies $\mathcal{R}(N) \subseteq \mathcal{R}(D(N))$ which yields $\mathcal{R}(D(N)) = \mathcal{R}(N)$. Conversely, suppose $\mathcal{R}(D(N)) = \mathcal{R}(N)$ for all near-rings N which satisfy the condition (BC). In particular, since $\mathbb{Z}$ is a ring, so is $A := \mathcal{R}(\mathbb{Z})$ and $\mathcal{R}(D(A)) = \mathcal{R}(A) = \mathcal{R}(\mathcal{R}(\mathbb{Z})) = \mathcal{R}(\mathbb{Z}) = A$. Since $\mathbb{Z} \cong D(A)/A = = D(A)/\mathcal{R}(D(A))$, we have $\mathcal{R}(\mathbb{Z}) = \mathcal{R}(D(A)/\mathcal{R}(D(A))) = 0$. ◊

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