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PROJECTIONS, SKEWNESS AND RELATED CONSTANTS IN REAL NORMED SPACES

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Abstract: In real normed spaces, the notion of skewness was introduced by Fitzpatrick and Reznick. The radial projection constant had been already studied several years before and its relations with some projection constants had been pointed out. Here we introduce and study a modified version of skewness and we continue the study of the above notions. We compare all these constants and we establish several relations, some of them depending on properties of the underlying space.

1. Introduction

Let X be a normed space over the real field $\mathbb{R}$. We denote by S the unit sphere of $X : S = \{x \in X;\ \|x\| = 1\}$. Also, we set for x, y in

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X:

$$\tau(x,y) = \lim_{t\to 0+} \frac{\|x+ty\|-\|x\|}{t} = \inf_{t>0} \frac{\|x+ty\|-\|x\|}{t}$$

and so

$$-\tau(x,-y) = \lim_{t\to 0-} \frac{\|x+ty\|-\|x\|}{t} = \sup_{t<0} \frac{\|x+ty\|-\|x\|}{t}.$$

Now define the multivalued map $J : X \to X^*$ in the following way. For $x \in X$ denote by $J(x)$ the nonempty set

$$J(x) = \{f \in X^*;\ \|f\| = \|x\|;\ f(x) = \|x\|^2\}$$

(X^* denoting the topological dual of X). For any x we have:

$$\|x\|\ \tau(x,y) = \sup\{f(y);\ f \in J(x)\}.$$

The space X is *smooth* if and only if $J(x)$ is a singleton for any $x \in X$, or equivalently, if and only if $\tau(x,y) = -\tau(x,-y)$ for any pair x,y; in this case $\tau(x,.)$ is linear in its second argument. We say that x is *orthogonal* to y, and we write $x \perp y$, when $\|x+ty\| \geq \|x\|$ for all $t \in \mathbb{R}$. Note that the following is true:

(1.1) $$x\perp y \Leftrightarrow \pm x \perp \pm y \Leftrightarrow -\tau(x,-y) \leq 0 \leq \tau(x,y).$$

In particular, if X is smooth, then we have

(1.2) $$x\perp y \Leftrightarrow \tau(x,y) = 0 \Leftrightarrow f(y) = 0 \text{ for } f \in J(x).$$

We shall write $x \perp M$ when $x \perp m$ for all $m \in M$. We denote by $[M]$ the linear span of M ($[y]$ = linear span of y).

We shall write: X is (H), when the norm of X derives from an inner product; in this case, $\tau(x,y)$ reduces to the inner product when $x,y \in S$. We recall that when $\dim(X) \geq 3$, then X is (H) if and only if orthogonality is symmetric (i.e., $x\perp y$ implies $y \perp x$).

The space X is said to be *uniformly nonsquare* (abbreviated: (UNS)) when $\sup\{\min(\|x+y\|, \|x-y\|);\ x,y \in S\} < 2$. Recall that (UNS) spaces are reflexive.

The notion of *skewness* for a normed space was introduced in [8] to describe the "asymmetry" of the norm:

(1.3) $$s(X) = \sup\{s(x,y);\ x,y \in S\}$$

where

(1.4) $$s(x,y) = \tau(x,y) - \tau(y,x).$$

Note that

(1.5) $$s(x,y) = -s(y,x) = s(-x,-y) \text{ for any pair } x,y.$$

For any space X, $0 \leq s(X) \leq 2$. Moreover, the extreme values 0 and 2 characterize - respectively – (H) spaces and spaces which are not (UNS) (see [8]).

We recall the definitions of some other constants that we shall compare with $s(X)$. The *radial map* T from X onto its unit ball, is the radial projection onto the unit ball defined by

$$T(x) = \begin{cases} x & \text{if} \quad \|x\| \leq 1 \\ x/\|x\| & \text{if} \quad \|x\| > 1. \end{cases}$$

The *radial constant* of X is the number

$$k(X) = \sup\{\frac{\|Tx - Ty\|}{\|x - y\|};\ x, y \in X;\ x \neq y\} \in [1,2].$$

Recall (see e.g. [9]) that

(1.6) $$k(X) = \sup\{\frac{1}{\|tx + y\|};\ x, y \in S;\ x \perp y, t \in \mathbb{R}\}$$

or also:

$$k(X) = \sup\{\frac{\|y\|}{\|x - y\|};\ y \neq 0;\ x \perp y\} = \sup\{\frac{1}{d(x,[y])};\ x \perp y;\ x, y \in S\}$$

where $d(x,A) = \inf\{\|x - a\|;\ a \in A\}$.
The extreme values of $k(X)$ (1 and 2) characterize – respectively – spaces where orthogonality is symmetric, and spaces which are not (UNS) (see [9] and the references there). For the sake of completeness, we recall that some results related to $k(X)$ were already given by Gurariĭ in two not too known papers (see [13] and [14]).
Remark. *Let* $x, y \in S, x \perp y, \lambda \neq 0$;*then* $\|y + \lambda x\| \geq |\lambda| \cdot \|x\|$. This shows that in (1.6), to obtain $\inf \|y + \lambda x\|$ it is enough to consider $\lambda \in [-1,1]$.

The radial constant is connected with the projection constants onto subspaces of X (see e.g. [2]). These and other similar relations will be studied in some detail here.

The present paper is organized in the following way. In Section 2 we indicate some general properties of the functional τ. In Section 3 we define new constants of skewness and we compare them with $s(X)$. In Section 4 we compare the radial constant with the constants of skewness. Section 5 deals with projection constants. Finally, in Section 6, we give some estimates for these constants in uniformly convex and uniformly smooth spaces.

Several results in sections 3,4 and 5 rely upon smoothness properties of X. Our modified measures of skewness are useful to obtain relations with constants which are related to orthogonal pairs. Relations among the constants we indicated and projections will be obtained by using what we shall call, according to [11], "polar" projections.

2. Some properties of the functional τ.

The following properties of τ are well known. For $x, y \in X, \mu \in \mathbb{R}$ and $\lambda > 0$ we have:

(2.1) $\tau(x, \mu x + \lambda y) = \mu\|x\| + \lambda\tau(x, y)$ (this is true also for $\lambda = 0$)

(2.2) $$\tau(\lambda x, y) = \tau(x, y)$$

(2.3) $$|\tau(x, y)| \leq \|y\|.$$

We indicate a few more properties, which will be used later: though very simple, probably they are not so well known.

Lemma 2.1. *Let $x, y \in S$. Then $\tau(x, y) = 1$ implies $\tau(y, x) = 1$.*

Proof. Let $x, y \in S, 1 = \tau(x, y) = \inf_{t>0} \frac{\|x+ty\|-\|x\|}{t}$. Therefore for all $t > 0 : 1 \leq \frac{1+|t|-1}{t} = 1$, which implies $\|x + ty\| = 1 + t$. Thus, for $t > 0, \|y + tx\| = t\|\frac{y}{t} + x\| = 1 + t$, which implies $\tau(y, x) = = \lim_{t\to 0^+} \frac{\|y+tx\|-\|y\|}{t} = 1$. ◇

Lemma 2.2. *A space X is smooth if and only if the following condition holds:*

(2.4) $$x \perp y \text{ if and only if } \tau(x, y) = 0.$$

If in addition orthogonality is symmetric, then smoothness is also equivalent to:

(2.5) *if* $x, y \in S$, *then* $x \perp y$ *implies* $\tau(x, y) = \tau(y, x)$.

Proof. Of course, if X is smooth, then $x \perp y$ implies $\tau(x,y) = 0$, and moreover, $\tau(x,y) = \tau(y,x) = 0$ if orthogonality is symmetric, so we have to prove the converse statements ("if" parts).

(2.4) $\Rightarrow$ X smooth: we prove the contrapositive. If X is not smooth, then there is a pair x, y with $x, y \in S$ and $-\tau(x,-y) = \lambda < \tau(x,y)$; then $-\tau(x, -y + \lambda x) = 0$, so $x \perp y - \lambda x$. But we have $\tau(x, y - \lambda x) > 0$, so (2.4) does not hold.

(2.5) $\Rightarrow$ X smooth (when orthogonality is symmetric): we prove the contrapositive. If $\dim(X) \geq 3$, then X must be (H) so we have nothing to prove; thus we assume $\dim(X) = 2$. Let us assume that X is not smooth; therefore (see (2.4)) there is a pair x, $y \in S, x \perp y$ (so $y \perp x$) such that $f_x(y) = \lambda \neq 0$ for some $f_x \in J(x)$; we can also assume $\lambda < 0$ (eventually, we change y into $-y$). Also, there exist $f \in J(x)$ and $g \in J(y)$ such that $f(y) = g(x) = 0$ (so $f \neq g$). Let $f_x = \alpha f + \beta g$, thus $\alpha = 1$ and $\beta = \lambda$. Take $z = \lambda x - y$ so $f_x(z) = 0$. Then also $z \perp x$, so we have: $1 = \|y\| \leq \|y - \lambda x\| \leq \|y - \lambda x + \lambda x\| = 1$. Therefore the value of the convex function of $t : F(t) = \|y + tx\|$ is 1 for $0 \leq t \leq -\lambda$, and so (set $a = 1/t$) $\|x + ay\| = a$ for $a \geq -1/\lambda$. By taking $\lambda_0 > -1/\lambda$ we obtain also, for $t \in R : \|x + \lambda_0 y\| = \lambda_0 \|y\| \leq \|\lambda_0 y + x + tx\|$, so $x + \lambda_0 y \perp x$ and then $x \perp x + \lambda_0 y$. Now we have: $\tau(x + \lambda_0 y, x) = \lim_{t\to 0^+} \frac{\|x+\lambda_0 y + tx\| - \lambda_0}{t} = \lim_{t \to 0^+} \frac{(1+t)\|x + (\lambda_0/(1+t))y\| - \lambda_0}{t}$; since $\lambda_0/(1+t) > -1/\lambda$ for t small enough, we obtain $\tau(x + \lambda_0 y, x) = \lim_{t \to 0^+} \frac{(1+t)(\lambda_0/(1+t)) - \lambda_0}{t} = 0$. But then, since $\frac{x+\lambda_0 y}{\lambda_0} \in S$ and $\frac{x + \lambda_0 y}{\lambda_0} \perp x$, (2.5) would imply also $\tau(x, \frac{x+\lambda_0 y}{\lambda_0}) = \tau(\frac{x+\lambda_0 y}{\lambda_0}, x) = 0$, against $f(\frac{x+\lambda_0 y}{\lambda_0}) = \frac{1}{\lambda_0} > 0$; this contradiction proves that (2.5) cannot hold when X is not smooth, which concludes the proof. $\diamondsuit$

For a result similar to the second part of Lemma 2.2, see Lemma 2 in [15].

3. Types of skewness

We introduce the following new constant:

(3.1) $$s_1(X) = \sup\{s(x,y);\ x, y \in S;\ x \perp y\}.$$

We give first some indications about the range of $s_1(X)$.

Proposition 3.1. *We always have* $0 \le s_1(X) \le s(X) \le 2$, *and these estimates are sharp.*

Proof. The estimates $s_1(X) \le s(X) \le 2$ are trivial. Now take a pair $x_0, y_0 \in S$ such that $x_0 \perp y_0$ and $y_0 \perp x_0$ (this is always possible: see [3]). But of course for any pair x, y we have either $\tau(x,y) - \tau(y,x) \ge 0$ or $\tau(y,x) - \tau(x,y) \ge 0$, so $s_1(X) \ge \max\{s(x_0 y_0), s(y_0, x_0)\} \ge 0$. If X is (H) then clearly $s_1(X) = 0$. If $X = \mathbb{R}^2$ with the norm given by: $\|(x,y)\| = \max\{|x|,|y|\}$, then it is easy to prove that $s_1(X) = 2$. In fact, it is enough to consider the following elements: $x = (1,1)$ and $y = (-1, a)$, with $0 < a < 1$, to prove that: $x, y \in S, x \perp y, \tau(x,y) = a, \tau(y,x) = -1$ and so $s_1(X) \ge 1 + a$; thus we have $s_1(X) = 2$. ◊

Now consider $\mathbb{R}^2$ with the norm given by

$$\|(x,y)\| = \begin{cases} \max\{|x|,|y|\} & \text{if} \quad xy \ge 0 \\ |x| + |y| & \text{if} \quad xy < 0 \end{cases}$$

The unit ball of X is a hexagon. Now take $y = (1,0)$ and $x = (a,1)$ with $0 < a < 1$. We have $x \perp y$ and $y \perp x$ (in fact orthogonality is symmetric in this space). Moreover, $\tau(y,x) = a$ and $\tau(x,y) = 0$, so $s_1(X) \ge s(y,x) = a$: this shows that $s_1(X) \ge 1$. Moreover, symmetry of orthogonality implies $\tau(y,x) \ge 0$, so $s(x,y) \le 1$ for any pair x, y on S with $x \perp y$; thus $s_1(X) = 1$ for this space.

Remark 3.2. Note that Lemma 2.1 says that we have $s(x,y) < 2$ for every pair $x, y \in S$. This implies that $s(X)$ must be smaller than 2 when the norm has some property implying the continuity of the map J and the unit sphere has some kind of compactness. So, if X is smooth and $x, y \in S$, then $\tau(y,x) = s(y,x) < 1$ for $x \perp y$; thus $s_1(X) < 1$ in these spaces, under some assumptions of that type.

Lemma 3.3. *If* $s_1(X) = 0$, *then orthogonality is symmetric.*

Proof. Assume $s_1(X) = 0$. Let $x, y \in S$ and $x \perp y$, thus $0 \ge \tau(x,y) - \tau(y,x)$. Also, from $x \perp -y$, $\tau(x,-y) - \tau(-y,x) \le 0$. But also, $\tau(x,y) \ge 0$ and $\tau(x,-y) \ge 0$, and so $\tau(y,x) \ge 0, 0 \le \tau(-y,x) = \tau(y,-x)$. This implies $y \perp x$, which concludes the proof. ◊

Proposition 3.4. *Let* $\dim(X) \ge 3$; *then* $s_1(X) = 0$ *if and only if* X *is* (H). *If* $\dim(X) = 2$, *then* $s_1(X) = 0$ *if and only if orthogonality is symmetric and* X *is smooth.*

Proof. “If” part: The first statement is trivial; the second one is a consequence of Lemma 2.2.

"Only if" part: For $\dim(X) \geq 3$, the result follows from the above lemma. If $\dim(X) = 2$ and $s_1(X) = 0$, then orthogonality is symmetric (see Lemma 3.3); moreover for x, y in $S, x \perp y$, we have $s(x,y) \leq 0$ and $s(y,x) \leq 0$, so $\tau(x,y) - \tau(y,x) = 0$ and then smoothness of X follows from Lemma 2.2. ◇

We can also define the following constant.

$$
\begin{aligned}
s_2(X) &= \sup\{s(y,x);\ x,y \in S;\ x \perp y\} = \\
&= -\inf\{s(x,y);\ x,y \in S;\ x \perp y\}.
\end{aligned}
\tag{3.2}
$$

We have he following estimates.

Proposition 3.5. *For any space X we have $0 \leq s_2(X) \leq 1$ and these estimates are sharp.*

Proof. We obtain $0 \leq s_2(X)$ by considering a biorthogonal pair x, y in S. The inequality $s_2(X) \leq 1$ follows from the definition since $s(y,x) \leq$ $\leq \tau(y,x)$ when $x \perp y$. Moreover, we have $s_2(X) = 0$ if X is (H) and $s_2(X) = 1$ in the example of the hexagon (see after Proposition 3.1). ◇

Proposition 3.6. *We have $s_1(X) = s_2(X)$ (thus $s_1(X) \leq 1$) in the following cases:*

(i) *X is smooth;*

(ii) *orthogonality is symmetric.*

Proof. Let X be smooth, then $s(x,y) = -s(-x,y) = s(y,-x)$. Since $x \perp y$ is equivalent to $-x \perp y$, we easily obtain from this $s_1(X) = s_2(X)$. When orthogonality is symmetric, equality follows immediately from the definitions of $s_1(X)$ and $s_2(X)$. ◇

Remark 3.7. As we recalled in the introduction, the extreme values of $s(X)$, 0 and 2, characterize two important classes of spaces. Our Prop. 3.4 indicates the situations for which we have $s_1(X) = 0$. It is possible to have X smooth, thus $s_1(X) \leq 1$ (see Propositions 3.5 and 3.6) and X not (UNS); compare this result with Cor. 4.2.

We could raise the following questions:

Questions 3.8. For what spaces we have $s_1(X) = 1$? Note that the condition X (UNS) does not imply $s_1(X) < 1$ or $s_2(X) < 1$ (see again the hexagon; see also Cor. 6.2). Moreover, X (UNS) implies $s_1(X) \leq s(X) < 2$. We do not know if X (UNS) implies $s_1(X) \leq 1$. Note that, for Propositions 3.4 and 3.6, when $s_1(X) = 0$ then also $s_2(X) = 0$. Is the converse true? Or for what spaces we have $s_2(X) =$ $= 0$? Also: is the inequality $s_2(X) \leq s_1(X)$ always true?

We do not know if $s_2(X) < 1$ implies X (UNS), but we can prove a partial result in this direction.

Lemma 3.9. *Let there exist in S a pair x, y such that $x \perp y$ and $\|x \pm y\| = 2$. Then $s_2(X) = 1$.*

Proof. Let be x, y as in the assumptions. Consider the convex functions of $t \in \mathbb{R}$: $g(t) = \|x + y + t(x - y)\|$ and $f(t) = \|x - y + t(x + y)\|$. We have $g(-1) = g(0) = g(1) = 2 = f(1) = f(0) = f(-1)$, so $g(t) \geq 2$ and $f(t) \geq 2$ for all $t \in \mathbb{R}$; moreover $f(t) = g(t) = 2$ for $-1 \leq t \leq 1$. Now take any $a \in (0, 1)$ and set $u = \frac{x-y+a(x+y)}{2}$, $v = \frac{x+y}{2}$. We have $g(0)/2 = \|v\| = 1 = \|u\| = f(a)/2$. Let $0 < t < 1$, so $\frac{t}{1+ta} < 1$. We obtain $2\|v+tu\| = \|x+y+t(x-y+a(x+y))\| = \|(1+ta)(x+y)+t(x-y)\| =$ $= (1 + ta)\|x + y + \frac{t}{1+ta}(x - y)\| = (1 + ta) \cdot g(\frac{t}{1+ta}) = 2(1 + ta)$. Therefore $\tau(v, u) = \lim_{t \to 0^+} \frac{\|v+tu\|-\|v\|}{t} = \lim_{t \to 0^+} \frac{1+ta-1}{t} = a$. Also, $u \perp v$: in fact, for all $t > 0$ and small enough ($a < a + t < 1$) we have $2\|u + tv\| = f(a + t) = 2 = 2\|u\|$, which implies $\tau(u, v) = 0$. Thus $s_2(X) \geq \tau(v, u) - \tau(u, v) = a$, and this implies the thesis. ◇

From the above lemma we obtain the following result.

Proposition 3.10. *If* $\dim(X) < \infty$ *and* $s_2(X) < 1$, *then* X *is* (UNS).

Proof. Assume X not (UNS). Then we have (see [2, Th. 6]) $\sup\{\|x + +y\| + \|x - y\|;\ x, y \in S; x \perp y\} = 4$. Now, by using the compactness of S and the fact that orthogonality is preserved when passing to the limit, we see that there exists in S a pair x, y with $x \perp y$ and such that $\|x + y\| + \|x - y\| = 4$. An application of Lemma 3.9 implies the thesis. ◇

We indicate another simple fact concerning $s_1(X)$.

Lemma 3.11. *Let X be smooth. Then*

$$s_1(X) \leq \sup\{\|x + y\|;\ x, y \in S;\ x \perp y\} - 1. \tag{3.3}$$

Proof. Let X be smooth, so $x \perp y$ is equivalent to $\tau(x, y) = 0$. Moreover, by using Prop. 3.6, for any $t > 0$ we have $s_1(X) = s_2(X) =$ $= \sup\{\tau(y, x);\ x, y \in S, x \perp y\} \leq \sup\{\frac{\|y+tx\|-1}{t};\ x, y \in S, x \perp y\}$. By setting $t = 1$ we obtain the thesis. ◇

We conclude this section with the following lemma.

Lemma 3.12. *For any space X and any $\lambda \in \mathbb{R}$ we have*

(3.4) $\lambda s(X) + 2 \leq \sup\{\|x + \lambda y\| + \|y - \lambda x\|;\ x, y \in S\}$,

(3.5) $\lambda s_1(X) + 2 \leq \sup\{\|x + \lambda y\| + \|y - \lambda x\|;\ x, y \in S, x \perp y\}$,

(3.6) $\lambda s_2(X) + 2 \leq \sup\{\|x + \lambda y\| + \|y - \lambda x\|;\ x, y \in S;\ y \perp x\}$.

Proof. Since all these constants are non negative and $x \perp y$ implies $x \perp -y$, it is enough to reason for $\lambda \geq 0$. For $\lambda = 0$ there is nothing to prove. Now fix $\lambda > 0$; if $x, y \in S$, then we have $\|x + \lambda y\| + \|y - -\lambda x\| \geq \tau(x, x + \lambda y) + \tau(y, y - \lambda x) = 1 + \lambda\tau(x, y) + 1 + \lambda\tau(y, -x) \geq \geq 2 + \lambda(\tau(x, y) - \tau(y, x))$. This implies (3.4). A similar reasoning, applied to orthogonal pairs, implies (3.5) or (3.6). ◇

Remark 3.13. By the above lemma we reobtain easily that *if* X *is* (UNS), *then* $s_1(X) \leq s(X) < 2$.

4. Radial constant and skewness

We recalled in the introduction the definition of the radial constant $k(X)$ and its main properties. Now we shall indicate some relations between this constant and those dealing with skewness.

Proposition 4.1. *For any space* X *we have:*

$$k(X) \leq 1 + s_1(X). \tag{4.1}$$

Proof. Let $s_1 = s_1(X)$; let $x, y \in S, x \perp y$, thus $\tau(x, y) \geq 0$ and then we have $\tau(y, x) \geq \tau(x, y) - s_1 \geq -s_1$. But also $-x \perp y$ so $\tau(y, -x) \geq -s_1$. Let $\alpha = \|y + \lambda x\|$, so $\alpha \geq \tau(y, y + \lambda x)$. If $\lambda \geq 0$, then $\alpha \geq 1 + +\lambda\tau(y, x) \geq 1 - \lambda s_1$. Also, if $\lambda < 0$, then $\alpha \geq 1 + (-\lambda)\tau(y, -x) \geq 1 + +(-\lambda)(-s_1) = 1 + \lambda s_1$. Therefore $\alpha \geq 1 - |\lambda| s_1$; but also (from $x \perp y$) $\alpha \geq |\lambda|$. This implies $\alpha \geq \max\{|\lambda|, 1 - |\lambda| s_1\}$. Since $\min_{\lambda \in \mathbb{R}}(\max\{|\lambda|, 1 - |\lambda| s_1\}) = \frac{1}{1+s_1}$, we obtain $\alpha \geq \frac{1}{1+s_1}$. Therefore, by (1.6), we have $k(X) = \sup\{\frac{1}{\|y+\lambda x\|};\ x, y \in S;\ x \perp y;\ \lambda \in \mathbb{R}\} \leq 1 + s_1$, so (4.1). ◇

Proposition 4.1 has the following consequences, which contain Lemma 3.3:

Corollary 4.2. *If* $s_1(X) = 0$, *then orthogonality is symmetric. Also: if* $s_1(X) < 1$, *then* X *is* (UNS).

Proof. From $s_1(X) = 0$ we obtain $k(X) = 1$, so the first statement. Concerning the second statement, the contrapositive follows immediately: in fact, if X is not (UNS), then $k(X) = 2$, so (by (4.1)) $s_1(X) \geq 1$. ◇

Recall that, for any space X, we have

$$k(X) = k(X^*) \tag{4.2}$$

and

$$s(X) = s(X^*). \tag{4.3}$$

Prop. 3.4 shows that in general $s_1(X) \neq s_1(X^*)$, and also (see Prop. 3.6) $s_2(X) \neq s_2(X^*)$. But this cannot happen in "good" spaces. In fact we have the following result.

Proposition 4.3. *If both X and X^* are smooth, then*

$$s_1(X) = s_1(X^*). \tag{4.4}$$

Proof. If X is smooth but not reflexive, then it is not (UNS), so (by Cor. 4.2) $s_1(X) \geq 1$; therefore, by Prop. 3.6, $s_1(X) = 1$. For the same reasons, $s_1(X^*) = 1$, so (4.4) is proved in this case.

Now assume X reflexive, so our assumptions imply that it is also smooth and strictly convex. Under these assumptions J is a one-to-one isometry between X and X^*; moreover, $x \perp y$ if and only if $J(y) \perp J(x)$. Therefore, by setting $J(x) = f_x$ and $J(y) = f_y$, we obtain ($\hat{x}$, $\hat{y}$ denoting the point functionals in X^{**}) $s_1(X) = \sup\{\tau(x,y) - \tau(y,x);\ x, y \in$ $\in S;\ x \perp y\} = \sup\{f_x(y) - f_y(x);\ x, y \in S;\ x \perp y\} = \sup\{\hat{y}(f_x) -$ $-\hat{x}(f_y);\ f_x, f_y \in S^*;\ f_y \perp f_x\} = \sup\{\tau(f_y, f_x) - \tau(f_x, f_y);\ f_x, f_y \in$ $\in S^*;\ f_y \perp f_x\} = s_1(X^*)$, which concludes the proof of Proposition 4.3. ◊

The above proposition, together with Prop. 3.6, shows that in those spaces $s_2(X) = s_2(X^*)$. We can still ask whether, in any space X, we have $s_2(X) \leq s_2(X^*)$, or $s_2(X) \geq s_2(X^*)$.

We prove a result which will be used in the next section.

Lemma 4.4. *Let X be smooth; let $x, y \in S, x \perp y$. If we set $-\tau(y,x) = \beta$ and $\|x + \beta y\| = \alpha$, then $\alpha \leq 1 + s_1^2(X)$.*

Proof. Our assumptions imply $\tau(x,y) = 0$; $-s(x,y) = \tau(y,x)$, thus $\beta = s(x,y)$. Also $\tau(y, \frac{x+\beta y}{\alpha}) = 0, \|\frac{x+\beta y}{\alpha}\| = 1$, so $\tau(\frac{x+\beta y}{\alpha}, y) =$ $= s(\frac{x+\beta y}{\alpha}, y)$. So we obtain: $\alpha = \tau(x + \beta y, x + \beta y) = \tau(x + \beta y, x) +$ $+\beta\tau(x + \beta y, y) \leq 1 + \beta s(\frac{x+\beta y}{\alpha}, y) \leq 1 + |s(x,y)| \cdot |s(\frac{x+\beta y}{\alpha}, y)|$. Since $x \perp \pm y, y \perp \pm \frac{x+\beta y}{\alpha}$ and $\|x\| = \|y\| = \|\frac{x+\beta y}{\alpha}\| = 1$, we have $\max\{|s(x,y)|,$ $|s(\frac{x+\beta y}{\alpha}, y)|\} \leq s_1(X)$, so the thesis. ◊

Remark 4.5. In Prop. 4.1, in general we do not have equality (consider again the exagon); also, we do not known if Lemma 4.4 is true without smoothness. For a related result see (5.17).

We want to recall that in [4], Desbiens introduced the following constant:

$$\beta(X) = \sup\{\beta \in \mathbb{R};\ x + \beta y \perp y;\quad x, y \in S\}.$$

In fact, as the same author noticed later in [5] (and as it is not difficult to see by using (1.6)), $\beta(X) = k(X)$ for any X. Some properties of $\beta(X) = k(X)$ were indicated in [4]; we shall indicate them in the last section.

5. Projections

Let M be a linear subspace of X. Recall that M is said to be *proximinal* if for every $x \in X$ the set

$$\Pi_M(x) = \{x_0 \in M;\ \|x_0 - x\| \le \|m - x\| \text{ for every } m \in M\} =$$

$$= \{x_0 \in M;\ x - x_0 \perp M\}$$

is non-empty. Given a proximinal subspace M of X, set

(5.1) $$|||\,\Pi_M\,||| = \sup\{\|y\|;\ y \in \Pi_M(x);\ \|x\| = 1\}.$$

Also, set:

(5.2) $$MPB\ (X) = \sup\{|||\,\Pi_M|||;\ M \text{ is a proximinal subspace of } X\}$$

and

(5.3) $$\overline{MPB}(X) = \sup\{|||\Pi_M|||; M \text{ is a proximinal hyperplane of } X\}.$$

If $M = f^{-1}(0)$ for some norm-one functional $f \in X^*$, then f assumes its norm on S if and only if M is proximinal. Moreover, $f(y) = 1$ for $y \in S$ is equivalent to $y \perp M$. Also, there is exactly one $y \in S$ with that property if X is *strictly convex.*

A linear, continuous, idempotent operator $P : X \to M$ is called a *projection.* In case there exists some projection from X onto M, we set

(5.4) $$\lambda(M, X) = \inf\{\|P\|;\quad P \text{ is a projection onto } M\}.$$

Moreover, we set

(5.5) $$F(X) = \sup\{\lambda(M, X);\ M \text{ is a hyperplane of } X\}.$$

For any space X we have (see [2])

$$k(X) = MPB(X) \tag{5.6}$$

and (see [10])

$$F(X) \leq \overline{MPB}(X). \tag{5.7}$$

Moreover, if $\dim(X) \geq 3$, then (see [1]) $F(X) = 1$ if and only if X is (H).
But we can also prove the following result:
Proposition 5.1. *For any space X*

$$\overline{MPB}(X) = MPB(X) = k(X). \tag{5.8}$$

Proof. Given $\varepsilon > 0$, there exist $x, y \in X, y \neq 0$, such that $x \perp y$ and $\frac{\|y\|}{\|x-y\|} > k(X) - \varepsilon$. Take a functional $f_x \in J(x)$ such that $f_x(y) = 0$; let M be the kernel of f_x. Note that M is a proximinal hyperplane and that $x \perp M$, therefore $-y \in \Pi_M(x - y)$. Thus $\overline{MPB}(X) \geq |||\Pi_M||| \geq \geq \frac{\|y\|}{\|x-y\|} > k(X) - \varepsilon$, which shows that (use (5.6)): $\overline{MPB}(X) \geq k(X) = = MPB(X) \geq \overline{MPB}(X)$, and then all these are equalities. ◊

Recall that given a hyperplane $M = f^{-1}(0)$, a projection $P : X \to \to M$ has a specified form, namely:

$$P(x) = P_{y,M}(x) = x - f(x)y, \text{ where } f(y) = 1. \tag{5.9}$$

If in (5.9) y is chosen so that $\|y\| = 1$, then we say that $P_{y,M}$ is a *polar projection* over M. In this case $\|I - P_{y,M}\| = 1$ (since $y \perp M$) and $x - f(x)y \in \Pi_M(x)$. If in addition there is a unique y as above, then we set

$$P'_M = P_{y,M}, \text{ i.e., } P'_M(x) = x - f(x)y \quad (f(y) = 1). \tag{5.10}$$

In this case, if X is also reflexive, then $\Pi_M(x) = \{x - f(x)y\}$, thus $|||\Pi_M||| = \|P'_M\|$. Polar projections have been used in [11], where it was shown that in some classical Banach spaces they coincide with the projections of minimal norm.

We can state the following result, which slightly improves Lemma 8 in [11].
Proposition 5.2. *For any space X*

$$k(X) = \sup\{\|P_{y,M}\|;\ M \textit{ is a proximinal hyperplane of } X;\ y \perp M\}. \tag{5.11}$$

Moreover, if X is reflexive and strictly convex, then

$$k(X) = \sup\{\|P'_M\|;\ M \text{ is a hyperplane of } X\}. \tag{5.12}$$

Proof. By using Proposition 5.1 we have (see the discussion above): $k(X) = \overline{MPB}(X) = \sup\{|||\Pi_M|||;\ M$ is a proximinal hyperplane of X$\} = \sup\{\|P_{y,M}\|;\ M$ is a proximinal hyperplane of $X; y\perp M\}$, so (5.11). Moreover, if X is reflexive and strictly convex, then every hyperplane is proximinal, so we can write $P_{y,M} = P'_M$, and then we obtain (5.12). $\diamondsuit$

Now let X be smooth. If $M = f^{-1}(0), \|f\| = 1$, and $y\perp M$, then $\tau(y,x) = f(x)$ for every $x \in X$ and we can write:

$$\|P_{y,M}\| = \sup\{\|x - \tau(y,x)y\|;\ \|x\| = 1\}. \tag{5.13}$$

Also, by smoothness we have $\|-x-\tau(y,-x)y\| = \|x-\tau(y,x)y\|$, while $\tau(y,-x) = -\tau(y,x) > 0$ if $\tau(y,x) < 0$; thus

$$\|P_{y,M}\| = \sup\{\|x - \tau(y,x)y\|;\ \|x\| = 1, \tau(y,x) \geq 0\} =$$
$$= \sup\{\|x - \tau(y,x)y\|;\ \|x\| = 1, \tau(y,x) \leq 0\}.$$

Now we recall the following result from [11, Lemma 7]:

Lemma 5.3. *If $M = f^{-1}(0)$ for some $f \in X^*, \|f\| = 1$, then*

$$\|P_{y,M}\| = \sup\{\|P(x)\|, x \in S, x\perp y\}. \tag{5.14}$$

Proposition 5.4. *Let X be smooth. Then we have*

$$k(X) \leq 1 + s_1^2(X). \tag{5.15}$$

Proof. If X is smooth but not reflexive, so not (UNS), then we have (see Corollary 4.2) $s_1(X) = 1$, thus (5.15) is trivial. If X is smooth and reflexive, then every hyperplane is proximinal; moreover, (5.14), (5.13) and Lemma 4.4 together imply ($y \in S$):

$$\begin{aligned}\|P_{y,M}\| = \sup\{\|P(x)\|, x \in S, x\perp y\} = \sup\{\|x - \tau(y,x)y\|,\\ x \in S, x\perp y\} \leq 1 + s_1^2(X).\end{aligned} \tag{5.16}$$

An application of (5.11) implies the thesis. $\diamondsuit$

Remark 5.5. By using (5.7), (5.8) and (5.15), *we also have (in any smooth space X)*

$$F(X) \leq 1 + s_1^2(X). \tag{5.17}$$

A direct proof of (5.17) can be achieved in this way. If $M = f^{-1}(0)$ is proximinal ($f \in X^*$; $\|f\| = 1$) and there exists $y \in S$ such that $f(y) = 1$, then we have (see (5.16))

$$\lambda(M,X) \leq \|P_{y,M}\| \leq 1 + s_1^2(X). \tag{5.18}$$

Moreover, it is not difficult to see that a constant $k \in \mathbb{R}$ exists such that if $M_1 = f^{-1}(0), M_2 = g^{-1}(0)$ and $\|f - g\| < \varepsilon$ ($f, g \in X^*$; $\|f\| = \|g\| = 1$), then $|\lambda(M_1,X) - \lambda(M_2,X)| < k\varepsilon$. By combining this fact with the Bishop-Phelps theorem, we see that, in a smooth space, (5.18) is true for every M, so we obtain again (5.17).

6. Uniformly convex and uniformly smooth spaces

Recall that X is said to be *uniformly convex* when the function of $\varepsilon \in [0,2], \delta(\varepsilon) = \inf\{1 - \frac{\|x+y\|}{2};\ x,y \in S;\ \|x-y\| \geq \varepsilon\}$ is positive for all $\varepsilon > 0$.

By using the function δ-called the *modulus of convexity* of X - we can give some rough estimates concerning some of the constants considered in the paper.

Proposition 6.1. *We always have*

$$s_1(X) \leq 2 - 2\delta(1) \tag{6.1}$$

Moreover, if X is smooth, then

$$s_1(X) \leq 1 - 2\delta(1). \tag{6.2}$$

Proof. Let $x, y \in S$; $x \perp y$, so $\|x - y\| \geq 1$. This implies $\|x + y\| \leq 2 - 2\delta(1)$. By using (3.5) with $\lambda = 1$, we obtain: $s_1(X) + 2 \leq 2 - 2\delta(1) + 2$, so we have (6.1). If X is smooth, then we obtain (6.2) in a similar way, by using (3.3). ◇

We have immediately the following

Corollary 6.2. *The condition $\delta(1) > 0$ implies $s_1(X) < 2$, and also $s_1(X) < 1$ if X is smooth.*

Of course, if we assume X^* to be uniformly convex, then Prop. 4.3 can be used again to obtain estimates for $s_1(X)$; note that in this case X is smooth. But we can also indicate some direct estimates for uniformly smooth spaces. Recall that the *modulus of smoothness* is defined, for $\lambda \in \mathbb{R}$, in this way:

$$\rho(\lambda) = \sup\{\frac{\|x+\lambda y\| + \|x-\lambda y\|}{2} - 1;\ x, y \in S\}.$$

The space is *uniformly smooth* if $\lim_{t\to 0} \frac{\rho(t)}{t} = 0$; this happens if and only if X^* is uniformly convex. Also, if we denote by ρ^* and δ^* respectively the moduli of smoothness and rotundity of X^*, then we have (see [6,pp.63 – 64]):

$$2\rho^*(1) = \sup\{\varepsilon - 2\delta(\varepsilon);\ 0 \le \varepsilon \le 2\} \tag{6.3}$$

and

$$2\rho(1) = \sup\{\varepsilon - 2\delta^*(\varepsilon);\ 0 \le \varepsilon \le 2\}. \tag{6.4}$$

We can also define (see [7,p.129])

$$\rho_1(\lambda) = \sup\{\frac{\|x+\lambda y\| + \|x-\lambda y\|}{2} - 1;\ x, y \in S, x \perp y\}.$$

Proposition 6.3. *For any space X*

$$s(X) \le 2\rho(1);\ s_1(X) \le 2\rho_1(1). \tag{6.5}$$

Proof. Formulas (6.5) are trivial (see (3.4) and (3.5)). Moreover, $s(X) = s(X^*) \le 2\rho^*(1) = \sup\{\varepsilon - 2\delta(\varepsilon);\ 0 \le \varepsilon \le 2\}$. ◇

Concerning the radial constant, (6.5) and (4.1) together imply

$$k(X) \le 1 + 2\rho_1(1). \tag{6.6}$$

By using the modulus of convexity, we have

Proposition 6.4. *Let X be* (UNS). *Then*

$$k(X) + \delta(k(X)) \le 2. \tag{6.7}$$

Also:

$$k(X) \le 2 - \frac{\delta(\varepsilon_0)}{2},\ \textit{where}\ \varepsilon_0 = \sup\{\varepsilon > 0;\ \varepsilon + \delta(\varepsilon) \le 2\}. \tag{6.8}$$

Proof. Our assumptions imply $k(X) \in [1, 2)$. If $k(X) = 1$ there is nothing to prove. Now let $k(X) \in (1, 2)$; set, for $\varepsilon \in (0, k(X) - 1), k_\varepsilon = = k(X) - \varepsilon$. By using (1.6), we can find $x, y \in S, x \perp y$ and $t_\varepsilon \in \mathbb{R}$, so that $\|t_\varepsilon x - y\| \le \frac{1}{k_\varepsilon}$; from $|k_\varepsilon| - |k_\varepsilon t_\varepsilon| \le \|k_\varepsilon y - k_\varepsilon t_\varepsilon x\| \le 1$ we obtain $2(k_\varepsilon - 1) \le 2k_\varepsilon|t_\varepsilon| = \|2k_\varepsilon t_\varepsilon x\| \le \|k_\varepsilon t_\varepsilon x + k_\varepsilon t_\varepsilon x - k_\varepsilon y\|$. Now we observe that $\|k_\varepsilon t_\varepsilon x\| \le \|k_\varepsilon y - k_\varepsilon t_\varepsilon x\| \le 1$ by construction, while $\|k_\varepsilon t_\varepsilon x - (k_\varepsilon t_\varepsilon x - k_\varepsilon y)\| = k_\varepsilon$; this implies, by definition of δ, $\|k_\varepsilon t_\varepsilon x + $ $+ k_\varepsilon t_\varepsilon x - k_\varepsilon y\| \le 2(1-\delta(k_\varepsilon))$, thus $2(k_\varepsilon - 1) \le 2(1-\delta(k_\varepsilon))$, and then $k(X) + \delta(k(X)) \le 2$.

For the second part of the thesis, recall that the following parameter was used in [12]:

$$\mu(X) = \sup\{\frac{\|x\| + \|ty\|}{\|x+ty\|};\ x \perp y;\ t \in \mathbb{R}\}.$$

It was proved there that $\mu(X) \le 3 - \delta(\varepsilon_0)$ (ε_0 defined above). Then it was proved in [2] that, in any space:

$$2k(X) - 1 \le \mu(X) \le k(X) + 1. \tag{6.9}$$

This implies $k(X) \le \frac{1+\mu(X)}{2} \le 2 - \frac{\delta(\varepsilon_0)}{2}$. $\diamondsuit$

Better (but more complicated) relations similar to (6.9) were given in [5], where also the estimate (6.7) was proved, in the form $k(X) \le \varepsilon_0$. By using those results, the second part of Proposition 6.4 could be slightly improved.

Remark 6.5. The function δ is non decreasing and continuous for $\varepsilon < 2$; therefore we have $\varepsilon_0 < 2$ when X is (UNS) (and also the converse is true). Moreover, for any space X (see e.g. [6, p.60]) $\delta(\varepsilon) \le 1 - (1 - $ $-\frac{\varepsilon^2}{4})^{\frac{1}{2}}$, so $k(X) + \delta(k(X)) \le k(X) + 1 - \sqrt{1 - \frac{k(X)^2}{4}}$; therefore, $\varepsilon_0 \ge \frac{8}{5}$. Thus, the estimate given by (6.7), at most, can say that $k(X) \le \alpha$ for some $\alpha \ge \frac{8}{5}$. Concerning the estimate (6.8), note that we always have $2 - \frac{\delta(\varepsilon_0)}{2} \ge \frac{9}{5}$. Also, note that the second part of (6.9), together with (4.1), implies $\mu(X) \le 2 + s_1(X)$. Again, slightly better estimates can be given by using the results of [5].

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