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CONVERGENCE THEOREMS FOR A DANIELL-LOOMIS INTEGRAL

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Abstract: For a recent integral of Bobillo and Carrillo 1987, which subsumes the Daniell integral and the Dunford-Schwartz integration with respect to finitely additive measures, convergence theorems are obrtained, using local convergence in measure. Furthermore the relations between the Bobillo-Carrillo integral, the abstract Riemann integral and the Bourbaki integral are discussed.

For a semiring Ω of sets from an arbirtrary set X and $\mu : \Omega \to$ $\to [0, \infty)$ only finitely additive an analogue $R_1(\mu, \overline{\mathbb{R}})$ to the space of Lebesgue-μ-integrable functions $L^1(\mu, \overline{\mathbb{R}})$ and its Lebesgue integral has been introduced in Loomis [9], Dunford-Schwartz [5] and [8] and for which, following Loomis [9], we use the terms Riemann-μ-integrable and Riemann-μ-integral. The question, whether corresponding analogoues to the Daniell extension process, but without or weaker continuity assumptions on the elementary integral, exist, has been treated by Aumann [1], Loomis [9] and Gould [7]. Aumann's results are applicable only after the construction of a suitable integral seminorm; in Gould's paper [7] Stone's axiom is assumed, his results are therefore subsumed by the abstract Riemann integral (see for example [8], p.57, 268); Loomis [9] works without Stone's axiom, but still his three extension processes are only of Riemann power, for example if one starts with the Riemann integral on the continuous real-valued functions with compact support $C_0(\mathbb{R},\mathbb{R})$ or with the step functions $S(\Omega,\mathbb{R})$ corresponding to the ring Ω generated by the intervals $[a,b) \subset \mathbb{R}$ one gets $R_1(\mu_L|\Omega,\mathbb{R})$ and not $L^1(\mu_L,\mathbb{R})$.

The situation is different with the integral $\overline{I} : \overline{B} \to \mathbb{R}$ introduced recently by Bobillo and Carrillo [3]. It works for arbitrary function vector lattices B and non-negative linear $I : B \to \mathbb{R}$, and yields the usual L^1 in the two special cases above. In this note we prove convergence theorems for this integral, using an appropriate local "convergence in measure". In the case of Lebesgue's convergence theorem (§1 and 3), our results subsume the corresponding result for $R_1(\mu,\overline{\mathbb{R}})$ (§5); regarding the Monotone convergence theorem only a somewhat weaker version is true in $\overline{B}$ (§2). If $\mu : \Omega \to [0,\infty)$ is σ-additive, then the convergence used here is for sequences more general than pointwise convergence; if additionally Ω is a δ-ring, then $\overline{B} = R_1 = L^1$ modulo nulfunctions (§5 and 6) and one gets the usual Lebesgue convegence theorem. As an application we give in §5 a short proof of the recent result of Bobillo and Carrillo [4] that always $R_1 \subset \overline{B}$ modulo nulfunctions, in §6 we formulate a converse for the Lebesgue case, and discuss the possible relations between $\overline{B}$, R_1, L^1 and the Bourbaki extension.

Notations. $\overline{\mathbb{R}} := \{-\infty\} \cup$ reals $\mathbb{R} \cup \{\infty\} = [-\infty,\infty]$; we extend the usual $+$ in $\mathbb{R}$ to all of $\overline{\mathbb{R}} \times \overline{\mathbb{R}}$ by

$$(1) \quad \infty + (-\infty) = (-\infty) + \infty := 0, \infty \dot{+} (-\infty) = (-\infty) \dot{+} \infty := \infty, \\ \infty \underset{\cdot}{+} (-\infty) = (-\infty) \underset{\cdot}{+} \infty := -\infty,$$

$r - s := r + (-s)$, $r \dot{-} s := r \dot{+} (-s)$, $r \underset{\cdot}{-} s := r \underset{\cdot}{+} (-s)$ for $r, s \in \overline{\mathbb{R}}$. With $r \vee s := \max(r,s)$, $\wedge :=$ min and $r \cap t := (r \wedge t) \vee (-t)$ one has for $r, s \in \overline{\mathbb{R}}$, $0 \le t \in \overline{\mathbb{R}}$

$$(2) \qquad |r \cap t - s \cap t| \le 2(|r-s| \wedge t), |r \wedge t - s \wedge t| \le |r - s|, \\ |r \vee t - s \vee t| \le |r - s|;$$

for further properties used below see also Aumann's paper [1], p.442, ($^*a$ $-^*$ c). In all of the following we assume, with arbitrary set X

$$(3) \quad X \neq \emptyset,\ B \text{ function vector lattice} \subset \mathbb{R}^X,\ I : B \to \mathbb{R} \text{ linear},\ I \ge 0,$$

i.e. under the on X pointwise defined $+$, $r\cdot$, $=, \le$, $\wedge$, $\vee$, $|\ |$ B is real linear space of functions $f : X \to \mathbb{R}$ containing with f, g also $f \wedge g$, $f \vee g$, $|f|$, and $0 \le I(f)$ if $0 \le f \in B$, where $|f|(x) := |f(x)|$ for $x \in X$.

From Bobillo's and Carillo's paper [3] we need the following definitions and results: $B^+ := \{g \in (-\infty, \infty]^X$: to each $x \in X$ exist $h_n \in B$ with $h_n \le g$ and $h_n(x) \to g(x)\}$,

(4) $I^+(k) := \sup\{I(h) : h \in B \text{ and } h \le k\}$, with $\sup \emptyset := -\infty$, $k \in \overline{\mathbb{R}}^X$,

with $I^+(k) \dot{+} I^+(l) \le I^+(k \dot{+} l)$ for $k, l \in \overline{\mathbb{R}}^X$; $B^- : -B^+$, $I^-(k) := -I^+(-k)$, $B_+ := \{\varphi \in B^+ : I^+(\varphi + g) = I^+(\varphi) + I^+(g)$ for all $g \in B^+\}$, $B_- := -B_+$; B^+ and B_+ are $+$ and $\vee$ closed, B^+ is also $\wedge$ closed.

$$\text{(5)} \quad \begin{aligned} \overline{I}(k) &:= \inf\{I^+(\varphi) : k \le \varphi \in B_+\}, \text{ with } \inf \emptyset := \infty, k \in \overline{\mathbb{R}}^X; \\ \underline{I}(k) &:= -\overline{I}(-k), = \sup\{I^-(-\varphi) : \varphi \in B_+, -\varphi \le k\}. \end{aligned}$$

$$\text{(6)} \quad \overline{I}(k \dot{+} l) \le \overline{I}(k) \dot{+} \overline{I}(l) \text{ for } k, l \in \overline{\mathbb{R}}^X,$$

$$\text{(7)} \quad I^+(k) \le \underline{I}(k) \le \overline{I}(k) \le I^-(k) \quad \text{for any } k \in \overline{\mathbb{R}}^X,$$

$I^+, \underline{I}, \overline{I}$ are monotone (increasing) on $\overline{\mathbb{R}}^X$.

The elements of $\overline{B} := \{k \in \overline{\mathbb{R}}^X : \underline{I}(k) = \overline{I}(k) \in \mathbb{R}\}$, $= B_0$ in [3], are called *I-summable*; $\overline{B}$ is a lattice, containing with f, g also $|f|$, rf with $r \in \mathbb{R}$, and any $h : X \to \overline{\mathbb{R}}$ with $h(x) = f(x) + g(x)$ only for those x, for which $f(x) \in \mathbb{R}$ and $g(x) \in \mathbb{R}$ (a slight extension of Theorem 5.2 of [3]), then $I(rf) = rI(f)$, $I(h) = I(f) + I(g)$, where $I := \underline{I} = \overline{I}$ on $\overline{B}$; $\overline{B}$ is closed under $\pm$, $\dot{\pm}$, $\pm$. B is dense in $\overline{B}$ with respect to $||k||_I := \overline{I}(|k|)$. $B_{(+)} \cup B_{(-)} \subset \overline{B}$, where $B^{(+)} := \{g \in B^+ : I^+(g) < \infty\}$, $B_{(+)} := B_+ \cap B^{(+)}$, $B_{(-)} = -B_{(+)}$. $I|\overline{B}$ is the maximal extension of $I|B$ in the sense of Aumann [1] p.443 with respect to the integral(semi)norm $\overline{I}$.

1. Dominated convergence

To get convergence theorems also in the finitely additive case, a.e. or everywhere convergence is not sufficient; as in Dunford-Schwartz's work [5] (p. 101 – 104) one has to use a kind of convergence in measure, but localized (see §§4,5):

For any $T : [0,\infty]^X \to [0,\infty]$, arbitrary nets $(k_i)_{i\in J}$ with $k_i \in \overline{\mathbb{R}}^X$ for $i \in J =$ directed set, arbitrary $k \in \overline{\mathbb{R}}^X$ we need

Definition 1. $k_i \to k(T)$ means for each fixed $h \in B$ with $0 \le h$ one has $T(|k_i - k| \wedge h) \to 0$ (where e.g. $\infty - \infty = 0$ by (1)).

Lemma 1. *If k_i, $k \in \overline{\mathbb{R}}^X$, (k_i) net with $\overline{I}(|k_i - k|) \to 0$, then $\overline{I}(k_i) \to \to \overline{I}(k)$, $\underline{I}(k_i) \to \underline{I}(k)$, $k_i \to k(\overline{I})$.*

Proof. With (5) there are $z_i \in B_+$ with $k_i \dot{-} k \le |k_i - k| \le z_i$ and $I^+(z_i) \to 0$. Then $k_i \le k \dot{+} z_i$, $\overline{I}(k_i) \le \overline{I}(k \dot{+} z_i) \le \overline{I}(k) + \overline{I}(z_i)$ by (6), $\overline{\lim}\, \overline{I}(k_i) \le \overline{I}(k)$. Similarly $k - k_i \le z_i$, $k \le k_i \dot{+} z_i$, $\overline{I}(k) \le \le \overline{I}(k_i) + \overline{I}(z_i)$, $\overline{I}(k) \le \underline{\lim}\, \overline{I}(k_i)$, or $\overline{I}(k_i) \to \overline{I}(k)$, also if $\overline{I}(k) = \pm\infty$. Since $\overline{I}(|(-k_i) - (-k)|) = \overline{I}(|k_i - k|) \to 0$ and $\overline{I}(-l) = -\underline{I}(l)$, the $\underline{I}$-statement follows. ◇

Lemma 2. *If k_i,$k \in \overline{\mathbb{R}}^X$, $\varphi \in \overline{B}$, (k_i) net with $k_i \to k(\overline{I})$ and $\varphi \le k_i$ for $i \in J$, then $\overline{\lim}\, \underline{I}(k_i) \ge \underline{I}(k)$. If $k_i \le \varphi$, everything else unchanged, then $\underline{\lim}\, \overline{I}(k_i) \le \overline{I}(k)$.*

Proof. First with $\varphi = 0$: If $l \in B_+$, $0 \le -l \le k_+ := k \vee 0$, $l_i := k_i \wedge (-l) \ge 0$, by (2) one has $k_i = k_{i,+} \to k_+(\overline{I})$, $l_i \to k_+ \wedge (-l) = = -l(\overline{I})$, $0 \le (-l) - l_i \le -l \le$ some $h_0 \in B$ by definition of B^+, so $\overline{I}(|(-l) - l_i|) \to 0$. By Lemma 1 one has $\underline{I}(k_i) \ge \underline{I}(l_i) \to \underline{I}(-l)$ or $\underline{\lim}\, \underline{I}(k_i) \ge \underline{I}(-l) = -\overline{I}(l)$; since l was arbitrary $\ge -k_+$, $\underline{\lim}\, \underline{I}(k_i) \ge \ge \sup\{-I^+(l)\} = -\overline{I}(-k_+) = \underline{I}(k_+) \ge \underline{I}(k)$.

In the general case, one can assume $\underline{I}(k) > -\infty$ and $-\varphi = g \in \in B_{(+)}$. Then $0 \le l_i := k_i + g \to k + g(\overline{I})$, since $|(r+t) - (s+t)| \le |r - s|$ for $r, s, t \in \overline{\mathbb{R}}$. So by the above $\underline{\lim}\, \underline{I}(l_i) \ge \underline{I}(k + g)$. Now for $g \in B_+$, $k \in \overline{\mathbb{R}}^X$ with $\underline{I}(k) > -\infty$ one has

$$\underline{I}(k + g) = \underline{I}(k) + I^+(g): \tag{8}$$

$\ge$ follows from (6) since $\underline{I} = \overline{I} = I^+$ on B_+ and $\dot{+} = +$ on the right in (8). If $l \in B_+$ with $-l \le k + g$, then $-(l + g) \le k$ or $\underline{I}(k) = -\overline{I}(-k) \ge \ge -I^+(l + g) = -I^+(l) - I^+(g)$; this implies $I^-(-l) = -I^+(l) \le \underline{I}(k) + +I^+(g)$ or $\underline{I}(k + g) \le \underline{I}(k) + I^+(g)$. (8) applied to $\underline{\lim}\, \underline{I}(l_i) \ge \underline{I}(k + g)$ yields $\underline{\lim}\, \underline{I}(k_i) \ge \underline{I}(k)$ since $I^+(g) \in \mathbb{R}$.

The second statement of Lemma 2 follows from this since $\overline{B} \ni \ni -\varphi \le -k_i \to -k(\overline{I})$. ◇

Lemma 3. *If k_i, $k \in \overline{\mathbb{R}}^X$, $\varphi \in \overline{B}$, $k_i \to k(\overline{I})$ and $k_i - k \ge \varphi$ for $i \in J$, then $\underline{\lim}\, \underline{I}(k_i) \ge \underline{I}(k)$ and $\underline{\lim}\, \overline{I}(k_i) \ge \overline{I}(k)$; if instead $k_i - k \le \varphi$, then $\overline{\lim}\, \overline{I}(k_i) \le \overline{I}(k)$ and $\overline{\lim}\, \underline{I}(k_i) \le \underline{I}(k)$.*

Proof. If $\underline{I}(k) > -\infty$ there is $g \in B_{(+)}$ with $-g \leq k$; one can assume $-\varphi \in B_{(+)}$, then $k_i \geq k+\varphi \geq -(g+(-\varphi)) \in \overline{B}$. Lemma 2 yields $\underline{\lim}\underline{I}(k_i) \geq \underline{I}(k)$. Since $k_i - k \to 0(\overline{I})$, Lemma 2 gives $\underline{\lim}\underline{I}(k_i - k) \geq 0$; so to $\varepsilon > 0$ there are $i_\varepsilon \in J$ and $z_i \in B_{(+)}$ with $k_i - k \geq -z_i$ and $\overline{I}(z_i) = I^+(z_i) < \varepsilon$, $i \geq i_\varepsilon$; for such i then $\overline{I}(k) \leq \overline{I}(k_i + z_i) \leq \overline{I}(k_i) + \overline{I}(z_i) \leq \overline{I}(k_i) + \varepsilon$ by (6). The $\overline{\lim}$-statements follow as in Lemma 2. $\diamondsuit$

Lemma 3 applied to $l_i := |k_i - k|$, $l := 0$ (and Lemma 1) yield

Theorem 1. *If with $k_i,\ k \in \overline{\mathbb{R}}^X$, $\varphi \in \overline{B}$ one has $|k_i - k| \leq \varphi$ for $i \in J =$ directed set and $k_i \to k(\overline{I})$, then $\overline{I}(|k_i - k|) \to 0$, $\overline{I}(k_i) \to \overline{I}(k)$, $\underline{I}(k_i) \to \underline{I}(k)$.* $\diamondsuit$

Corollary 1. *If $k_i,\ k \in \overline{\mathbb{R}}^X$, $|k_i| \leq \varphi \in \overline{B}$, $k_i \to k(\overline{I})$, then $\overline{I}(k_i) \to \overline{I}(k \cap \varphi)$, $\underline{I}(k_i) \to \underline{I}(k \cap \varphi)$.*

Proof. $k_i \cap \varphi = k_i$, so $k_i \to k \cap \varphi(\overline{I})$ by (2), $|k_i - k \cap \varphi| \leq 2\varphi$. $\diamondsuit$

Corollary 2. *If with the assumptions of Theorem 1 additionally $k_i \in \overline{B}$, then $k \in \overline{B}$, $I(k_i) \to I(k)$. (Lebesgue's convergence theorem for $\overline{B}$).*

Corollary 3. *If $f_i \in \overline{B}$, $k \in \overline{\mathbb{R}}^X$, $f_i \to k(\overline{I})$, then $k \in \overline{B}$ if and only if $\overline{I}(|k|) < \infty$. Special case: $f_i \in \overline{B}$, $k \in \overline{\mathbb{R}}^X$, $\overline{I}(|f_i - k|) \to 0 \Rightarrow k \in \overline{B}$. ($\overline{B}$ is $\overline{I}$-closed).*

Proof. "If": $\overline{I}(|k|) < \infty$ is equivalent with $|k| \leq$ some $\varphi \in \overline{B}$, so $|f_i \cap \varphi - k| \leq 2\varphi$, $f_i \cap \varphi \to k \cap \varphi = k(\overline{I})$, $f_i \cap \varphi \in \overline{B}$ =lattice, Corollary 2. $\diamondsuit$

One has corresponding convergence theorems for B^+ (and B_+, = Corollary 7 in §3):

Corollary 4. *If in Lemma 2 additionally $k_i \leq k$ [resp. $k \leq k_i$], then $\underline{I}(k_i) \to \underline{I}(k)$ [resp. $\overline{I}(k_i) \to \overline{I}(k)$]. So if $k \in \overline{\mathbb{R}}^X$, $g_i \in B^+$, $\varphi \in \overline{B}$ with $\varphi \leq g_i \leq k$ and $g_i \to k(\overline{I})$, then $I^+(g_i) \to I^+(k) = \underline{I}(k)$.*

Proof. $\underline{I}(k_i) \leq \underline{I}(k)$, so $\overline{\lim}\underline{I}(k_i) \leq \underline{I}(k), \leq \underline{\lim}\underline{I}(k_i)$ by Lemma 2. Since $I^+ = \underline{I}$ on $B^+(-p \leq g \in B^+$ with $p \in B_+$ implies $0 \leq g+p$, $0 \leq I^+(g+p) = I^+(g) + I^+(p)$, $\underline{I}(g) \leq I^+(g))$, if $k_i = g_i \in B^+$ one gets $I^+(g_i) \to \underline{I}(k)$, but $I^+(g_i) \leq I^+(k) \leq \underline{I}(k)$. $\diamondsuit$

In the above statements usually all assumptions are essential (see however §3): Domination by $\varphi \in B^{(+)}$ is in Theorem 1/Corollary 2 not enough: $k_i \equiv 0$, $k = 1T$ of Example 2 below; similarly for Lemma 2 ($k_i \equiv 1T$, $k = 0$), Lemma 3, "[ ]" of Corollary 4.

In Corollary 1 one cannot substitute k for $k \cap \varphi$ ($k_i \equiv 0$, $k = 1T$), so Corollary 2 is false if only $|k_i| \leq \varphi$; in Corollary 3 the existence of a $g \in B^{(+)}$ with $|k| \le g$ cannot replace $\overline{I}(|k|) < \infty$, in Corollary 4 "$g_i \le k$" is essential: $g_i \equiv 0$, $k = -1T$; the "if" in Corollary 3 also becomes false for "(f_i) is $\| \|_I$-Cauchy, i.e. $I(|f_i - f_j|) \to 0$" instead of "$\overline{I}(|k|) < \infty$", contrary to the R_1-spaces (§5) the $\overline{B}$ is in this sense not closed ($k = 1T$). In Corollary 4, $\varphi \in \overline{B}$ cannot be weakened as in Corollary 7 to $\varphi \in B^{(-)}$, there exist counterexamples B_Ω, I_μ (see §5) with μ even σ-additive.

2. Monotone convergence

For $k, l, p \in \overline{\mathbb{R}}^X$ we define $k \le l(\overline{I})$ by $(k-l)_+ := (k-l) \vee 0 = 0(\overline{I})$ and $p = 0(\overline{I})$ by $p_n := p \to 0(\overline{I})$; by definition, $k \le l(\overline{I}) \Leftrightarrow 0 \le l - k(\overline{I})$.
Lemma 4. *If $k, l \in \overline{\mathbb{R}}^X$ with $k \le l(\overline{I})$ and $\overline{I}(k) < \infty$ [resp. $\underline{I}(l) > -\infty$], then $\overline{I}(k) \le \overline{I}(l)$ [resp. $\underline{I}(k) \le \underline{I}(l)$].*
Proof. If $\overline{I}(l) < \infty$ to $\varepsilon > 0$ there is $g \in B_{(+)}$ with $l < g$ and $I^+(g) < < \overline{I}(l) + \varepsilon$ resp. $< -1/\varepsilon$, with (1) one has $0 \le (k-g)_+ \le (k-l)_+ \to 0(\overline{I})$; there is $p \in B_{(+)}$ with $k \le p$ or $(k-g)_+ \le (p-g)_+ \in \overline{B}$, so Theorem 1 yields $\overline{I}((k-g)_+) = 0$. Now $k \le g \dot{+} (k-g)_+$, so $\overline{I}(k) \le \overline{I}(g) + \overline{I}((k-g)_+) = \overline{I}(g) = I^+(g) < \overline{I}(l) + \varepsilon$ resp. $-1/\varepsilon$. This applied to $(-l) \le (-k)(\overline{I})$ yields $\underline{I}(k) \le \underline{I}(l)$. ◊
Lemma 5. *If k_i, $k \in \overline{\mathbb{R}}^X$, (k_i) increasing net (i.e. $k_i \le k_j$ if $i \le j$) with $k_i \to k(\overline{I})$, then $k_i \le k(\overline{I})$ for $i \in J$; if additionally $\underline{I}(k) > -\infty$, then $\underline{I}(k_i) \le \underline{I}(k)$, $i \in J$; if furthermore $\underline{I}(k_{i_0}) > -\infty$ for some i_0 then $\underline{I}(k_i) \to \underline{I}(k)$.*
Proof. If $i \le j$, $0 \le (k_i - k)_+ \le (k_j - k)_+ \le |k_j - k|, \to 0(\overline{I})$, or $(k_i - k)_+ = 0(\overline{I})$; Lemma 4 yields the second statement. In the last there is g_0 with $\overline{B} \ni -g_0 \le k_{i_0} \le k_j$ if $i_0 \le j$, $\underline{I}(k) \le \underline{\lim}\, \underline{I}(k_j)$ by Lemma 2, so $\underline{I}(k_i) \to \underline{I}(k)$. ◊

Skipping a dualization of Lemma 5, we note, using Lemmas 3 – 5, the
Corollary 5. *If k_i, $k \in \overline{\mathbb{R}}^X$, (k_i) increasing net with $k_i \to k(\overline{I})$, $\overline{I}(|k|) < \infty$, $\underline{I}(k_{i_0}) > -\infty$ for some i_0 and all $\overline{I}(k_i) < \infty$, then $-\infty < < \underline{I}(k) = \lim \underline{I}(k_i) \le \lim \overline{I}(k_i) = \overline{I}(k) < \infty$.*
Corollary 6. *If $f_i \in \overline{B}$, $k \in \overline{\mathbb{R}}^X$, (f_i) increasing net with $f_i \to k(\overline{I})$, $\overline{I}(|k|) < \infty$, then $k \in \overline{B}$, $\overline{I}(|f_i - k|) \to 0$, $I(f_i) \to I(k)$. (Monotone convergence theorem for $\overline{B}$).*

Proof. Corollary 3 gives $k \in \overline{B}$; $f_i \leq k(\overline{I})$ or $0 \leq k - f_i(\overline{I})$ by Lemma 4; so $(|t| - t)_+ = 2(-t)_+$ implies $|k - f_i| \leq k - f_i(\overline{I})$, Lemma 4/5 then $\overline{I}(|k - f_i|) \leq \overline{I}(k - f_i) = I(k) - I(f_i) \to 0$. ◇

Again k_n, $k \in \{0, \pm 1T\}$ with T of Example 2 show that e.g. $\overline{I} < \infty$ resp. $\underline{I} > -\infty$ are essential in the above, also $\overline{I}(|k|) < \infty$ cannot be weakened to $\overline{I}(k) < \infty$ in Corollary 6, even if additionally $\sup I(|f_n|) < < \infty$; so the usual Monotone convergence theorem is false for $\overline{B}$ with $\to (\overline{I})$ (it becomes true for a suitable extension of $\overline{B}$ which will be treated elsewhere; see also §3). The "increasing" also cannot be omitted ($k_n = 1[n, n+1) \to 0$ (μ_L), §5).

3. Generalized dominated convergence

Lemma 6. *If $g \in B^+$ with $g \wedge |h| \in B_+$ for all $h \in B$, then $g \in B_+$.*
This is due to Bobillo and Carrillo [4], p. 261, Remark 2b. Here $g \in B^+$ can be weakened to: $g \in \overline{\mathbb{R}}^X$ such that to each $x \in X$ with $g(x) \neq 0$ there is $h \in B$ with $h(x) \neq 0$. For $g \in B^+$ the assumptions $g \wedge |B| \subset B_+$, $g \wedge B \subset B_+$, $g \wedge |B| \subset \overline{B}$ are equivalent; without $g \in B^+$ however $g \wedge B \subset \overline{B}$ does not imply $g \in \overline{B}$ even if $I^X(g) < \infty$ (see (11); $g = 1T$ of Ex. 2).
Theorem 2. *If $f_i \in \overline{B}$, $g \in B^+$, $f_i \to g(\overline{I})$, then $g \in B_+$; if additonally $\underline{\lim}\, I(f_{i,+}) < \infty$, then $g \in \overline{B}$.*
Proof. With h_0, $h \in B$ with $h_0 \leq g$ and (2) one gets $p_i := (f_i \vee h_0) \wedge \wedge h \to (g \vee h_0) \wedge h = g \wedge h(\overline{I})$, $p_i \in \overline{B}$, $|p_i| \leq |h_0| \vee |h|$, Corollary 2 yields $g \wedge h \in \overline{B}$, $I(p_i) \to I(g \wedge h)$. Since $B^+ \wedge B \subset B^+$ and $B^+ \cap \overline{B} = = B_{(+)}$ by Guerrero-Carrillo ([4], p. 261, Rem. 2a), Lemma 6 shows $g \in B_+$. Furthermore $p_i \leq f_{i,+} + |h_0|$, so $I(g \wedge h) = \lim I(p_i) \leq \leq \underline{\lim}\, I(f_{i,+}) + I(|h_0|) =: c_0 < \infty$ independent of $h \in B$, so $I^+(g) < \infty$, $g \in B_{(+)} \subset \overline{B}$. ◇
Corollary 7. *If $g_i \in B_+$, g and $q \in B^+$ with $I^+(q) < \infty$, $-q \leq g_i \leq g$, $g_i \to g(\overline{I})$, then $g \in B_+$ and $I^+(g_i) \to I^+(g)$. Special case: $g_i \geq \varphi \in \overline{B}$.*
Proof. If $I^+(g) = \infty$, $g \in B_+$; else $g_i \in \overline{B}$, then $g \in B_+$ by Theorem 2. The assumptions imply $0 \leq g_i + q \leq g + q$ and $g_i + q \to g + q(\overline{I})$ (see before (8)), Corollary 4 yields $I^+(g_i) + I^+(q) = I^+(g_i + q) \to I^+(g + q) = = I^+(g_i) + I^+(q)$. ◇
Corollary 8. *If $f_i \in \overline{B}$, $g \in B^+$, $f_i \to g(\overline{I})$, $g_0 \in B^+$ with $|f_i| \leq g_0$ for $i \in J$ and $I^+(g_0) < \infty$, then $g \in \overline{B} \cap B_+$ and $I(|f_i - g|) \to 0$,*

$I(f_i) \to I(g)$.

Proof. (See Addendum and Lemma 7′ too). If $k \le g \in B^+$, $p \in B_+$, $-p \le k$, then $0 \le g + p$, $0 \le I^+(g+p) = I^+(g) + I^+(p)$, $I^-(-p) = -I^+(p) \le I^+(g)$, or

$$(9) \qquad k \in \overline{\mathbb{R}}^X,\ k \le g \in B^+ \text{ imply } \underline{I}(k) \le I^+(g) = \underline{I}(g).$$

Now $f_{i,+} \le |f_i| \le g_0$, so $I(f_{i,+}) \le I^+(g_0)$ and therefore $g \in \overline{B}$ by Theorem 2. Then $k_i := |f_i - g| \le g_0 + |g| \le g_0 + g + 2h_0 \in B^+$ with $h_0 \in B$, $h_0 \le g$, $I^+(g_0 + g + 2|h|) = I^+(g_0) + I^+(g) + 2I(|h|)$ since $g \in B_+ (= \overline{B} \cap B^+)$, $< \infty$; one can apply Lemma 7′, $I^X(k_i) \to 0$; but $k_i \in \overline{B}$, and $I^X = I$ on $\overline{B}$ by (12). ◊

Similar examples as above show that $\overline{\lim} I(f_i) < \infty$ does not suffice in Theorem 2, $f_{i,+} \le g_0$ or $I^+(g_0) = \infty$ do not imply $I(f_i) \to I^+(g)$ in Corollary 8; $g \in B^+$ is essential in Theorem 2 and Corollary 7/8, an analogue to Corollary 3 with $I^X(|k|) < \infty$ is false; replacing $g_i \to g(\overline{I})$ by $\to (I^+)$ or by $I^+(g_i) \to I^+(g)$ does not imply $g \in B_+$ in Corollary 7 ($g_i \equiv 0$, $g = g_0$ of Example 2), also $I^+(q) < \infty$ is essential.

Lemma 7. *If $k_i \in \overline{\mathbb{R}}^X$, $g_0 \in B^{(+)}$ with $k_i \le g_0$ for $i \in J$ and $k_i \to 0(I^+)$, then $\overline{\lim}\, I^+(k_i) \le 0$; if additionally $k_i \ge 0$ for $i \in J$, then $I^+(k_i) \to 0$.*

Proof. For $k \in \overline{\mathbb{R}}^X$, $h \in B$ one has

$$(10) \qquad I^+(k) = I^+(k \wedge h) \dot{+} I^+(k - k \wedge h):$$

by definition of I^+ one has "$\ge$" with "$=$" if $I^+(k) = -\infty$; if $p \in B$, $p \le k$, then $p \wedge h, p - p \wedge h \in B$ with $p \wedge h \le k \wedge h$, $p - p \wedge h \le k - k \wedge h$, $I(p) = I(p \wedge h) + I(p - p \wedge h) \le I^+(k \wedge h) + I^+(k - k \wedge h)$; p being arbitrary, "$\le$" follows. Choosing $h \in B$ with $h \le g_0$ one gets $k_i - k_i \wedge h \le g_0 - h$, (10) yields $I^+(k_i) \le I^+(|k_i| \wedge h) + I^+(g_0 - h)$ with $I^+(g_0 - h) = I^+(g_0) - I(h) < \varepsilon$ for suitable h, all i. Definition 1 yields Lemma 7. ◊

Addendum. For the proof of Corollary 8 we define, for any $k \in \overline{\mathbb{R}}^X$, with $\inf \emptyset = \infty$, $\sup \emptyset = -\infty$

$$(11) \qquad \begin{aligned} I^X(k) &:= \inf\{I^+(g) : k \le g \in B^+\}, \\ I_X(k) &: \sup\{I^-(p) : k \ge p \in B^-\}. \end{aligned}$$

With the definition of $\overline{I}$, $\underline{I}$, B_+, B_- and (7) one gets

$$(12) \qquad I^+ \le \underline{I} \le \min(I_X, I^X) \le \max(I_X, I^X) \le \overline{I} \le I^- \text{ on } \overline{\mathbb{R}}^X.$$

Lemma 7′. *If* $0 \leq k_i \in \overline{\mathbb{R}}^X$, $k_i \leq g_0 \in B^{(+)}$, $k_i \to 0(\overline{I})$, *then* $I^X(k_i) \to 0$.

Proof. To $\varepsilon > 0$ there exist $g_{i,\varepsilon} \in B^+$ with $k_i \leq g_{i,\varepsilon} \leq g_0$ and $I^+(g_{i,\varepsilon}) \leq I^X(k_i) + \varepsilon$. So $k_i \leq k_i \wedge h + (g_{i,\varepsilon} - g_{i,\varepsilon} \wedge h)$ if $0 \leq h \in B$; there exist $z_i \in B_+$ with $k_i \wedge h \leq z_i$ and $I^+(z_i) < \varepsilon$ if $i \geq i_\varepsilon$, then $I^+(g_{i,\varepsilon}) - \varepsilon \leq I^+(k_i) \leq I^+(z_i + (g_{i,\varepsilon} - g_{i,\varepsilon} \wedge h)) = I^+(z_i) + I^+(g_{i,\varepsilon} - g_{i,\varepsilon} \wedge h) = I^+(z_i) + I^+(g_{i,\varepsilon}) - I^+(g_{i,\varepsilon} \wedge h)$ by (10) and using

(13) if $g \in B^+$, $h \in B$, then $g - g \wedge h \in B^+$.

So $g_{i,\varepsilon} \to 0(I^+)$ with directed set $J \times (0, \infty)$, Lemma 7 shows $I^+(g_{i,\varepsilon}) \to 0$, $I^+(g_{i,\varepsilon}) \to 0$, thus $I^X(k_i) \to 0$. ◇

There are analogues to Lemma 7/7′ for certain certain other combinations from I^+, $\underline{I}$, I^X, $\overline{I}$, I^-, for example: Lemma 7′ still holds if only $k_i \to 0\ (I^X)$, provided $\overline{B}$ satisfies Stone's axiom and $I(h \wedge \frac{1}{n}) \to 0$, $I(h - h \wedge n) \to 0$ as $n \to \infty$, $0 \leq h \in B$.

4. Improper integrals

Under some additional assumptions, $\overline{B}$ is closed with respect to improper integrals, just as in the case of Riemann- or Lebesgue-integration ([8], p. 259/261):

(14) $= S_+(B)$ means: (3), $0 \leq h \in B \Rightarrow h \wedge 1$ and $h - h \wedge 1 \in B_+$

(15) $= C_\infty = C_\infty(B, I)$: (14), $0 \leq h \in B \Rightarrow I(h \wedge n) \to I(h)$ as $n \to \infty$.

(14) implies $h \wedge t$ and $h - h \wedge t \in B_{(+)}$ if $0 \leq h \in B$, $0 \leq t \in \mathbb{R}$. Stone's condition "$0 \leq h \in B \Rightarrow h \wedge 1 \in B$" implies (14).

Lemma 8. $S_+(B)$, $0 \leq g \in B_{(+)}$, $0 \leq t \in \mathbb{R}$ *imply* $g \wedge t$ *and* $g - g \wedge t \in B_{(+)}$; *conversely,* $C_\infty(B, I)$, $0 \leq k \in \overline{\mathbb{R}}^X$ *with* $k \wedge n \in B_+$ *for* $n = 1, 2, \ldots$ *imply* $k \in B_+$ *with* $I^+(k \wedge n) \to I^+(k)$ *as* $n \to \infty$.

Proof. To g exist $0 \leq h_n \in B$ with $I(h_n) \to I^+(g)$, so $\overline{I}(|g - h_n|) \to 0$; $\overline{I}(|g \wedge t - h_n \wedge t|) \leq \overline{I}(|g - h_n|) \to 0$, implying $g \wedge t \in \overline{B}$, by definition of B^+ and with $B \wedge t \subset B^+$ the $g \wedge t$ are in B^+, so $g_1 t \in B^* \cap \overline{B} = B_{(+)}$; similarly $g - g \wedge t \in B_{(+)}$. If all $k \wedge n \in B_+$, $k \in B^+$ by the remark after Lemma 6; if $0 \leq h \in B$, $k \wedge h$ and $k \wedge n \wedge h \in B^+$ with $0 \leq k \wedge h - k \wedge h \wedge n \leq h - h \wedge n$, Corollary 4 shows $I^+(k \wedge n) \geq I^+(k \wedge h \wedge n) \to I^+(k \wedge h)$,

$\approx I^+(k)$ for suitable h, so $I^+(k \wedge n) \to I^+(k)$. If $I^+(k) = \infty$, $k \in B_+$; else $(k \wedge n) \wedge h \in B_{(+)}$, which is min-closed by [3], p.248, Corollary 7 shows $k \wedge h \in B_+$. So $k \in B_+$ by Lemma 6. $\diamondsuit$

The first part of Lemma 8 becomes false if only $g \in B_+$.

Theorem 3. *If $C_\infty(B,I) = (15)$ holds and $k \in \overline{\mathbb{R}}^X$, then $k \in \overline{B}$ if and only if $k \cap n \in \overline{B}$ for $n = 1,2,\dots$ and $\sup\{I(|k \cap n|) : n \in \mathbb{N}\} < \infty$; then $I(|k - k \cap n|) \to 0$, $I(k \cap n) \to I(k)$ (with $k \cap n = (k \wedge n) \vee (-n)$).*

Theorem 3 becomes false with $k \cap h$ instead of $k \cap n$, even if $0 \leq k \leq 1$, $0 \leq h \in B = B_\Omega$, $I = I_\mu$ as in (17) below, $I(k \wedge h) \equiv 0$: $k = 1T$ of Example 2, §5; so "improper" here is meant only with respect to unbounded functions, not with respect to "unbounded support". Theorem 3 also becomes false without C_∞ by

Example 1. $X = \mathbb{N}$, $B = \{(x_n)_{n \in \mathbb{N}} : \lim(x_n/n) \text{ exists } \in \mathbb{R}\}$, $I =$ this lim, $k = (n^2)$; $k \notin \overline{B}$, though X is a I-nulset, $I(1X) = 0$; even $h \wedge 1 \in B$ if $h \in B$.

Proof of Theorem 3. "If": Since $k_\pm \wedge n = (k \cap n)_\pm$, $k = k_+ - k_-$, $|k - k \cap n| = |k| - |k| \wedge n$, one can assume $k \geq 0$. To $k_n := k \wedge n - -k \wedge (n-1) \in \overline{B}$ and $\varepsilon > 0$ there are $g_n \in B_{(+)}$ with $k_n \leq g_n$ and $I^+(g_n) \leq I(k_n) + \varepsilon 2^{-n}$, $n \in \mathbb{N}$. By recursive definition there is a unique sequence (z_n) with $z_n \in B_{(+)}$,

$$k_{n+1} \leq z_{n+1} = g_{n+1} \wedge (2(z_n - z_n \wedge \tfrac{1}{2})) \wedge 1 \text{ and}$$
$$I(z_n) \leq I(g_n),\ n \in \mathbb{N},\ z_1 = g_1:$$

$z_{n+1} \in B_{(+)}$ by Lemma 8 and the $\wedge$-closedness of $B_{(+)}$; if $k_{n+1}(x) > 0$, there $k > n$, $1 = k_n \leq z_n$, $z_{n+1} = 1 \geq k_{n+1}$. If $w_m := \Sigma_{j=m}^\infty z_j$, $w_m \in B^+$ by [3], p. 246. One has $w_m \wedge n = (\Sigma_{j=m}^{m+2n} z_j) \wedge n$: If $z_q(x) > 0$ for some $q > m + 2n$, $z_{q-1}(x) > \frac{1}{2}$ and thus $z_j(x) > 1/2$ if $1 \leq j \leq q$, implying " = ". Thus $w_m \wedge n \in B_{(+)}$ for $n \in \mathbb{N}$ by Lemma 8, again by Lemma 8 the $w_m \in B_+$ with $I^+(w_m) = \lim_{n\to\infty} I(w_m \wedge n) \leq$ $\leq \lim I(\Sigma_m^{m+2n} z_j) \leq \Sigma_m^\infty I(g_j) \leq \Sigma_m^\infty I(k_j) + \varepsilon$. Since $\Sigma_1^n k_j = k \wedge n$, $\Sigma_1^n I(k_j) \leq \sup_n I(k \wedge n) < \infty$, so $I^+(w_m) < 2\varepsilon$ if $m > m_\varepsilon$. Then $0 \leq k - k \wedge n = \Sigma_{n+1}^\infty k_j \leq w_{n+1}$ implies $\overline{I}(|k - k \wedge n|) \to 0$, $k \in \overline{B}$. $\diamondsuit$

Corollary 9. *If $C_\infty = (15)$ holds, $P \subset X$, $1P \in \overline{B}$, $I(1P) = 0$, then $\infty P \in \overline{B}$ with $I(\infty P) = 0$.*

Proof. $(\infty P) \cap n = n \cdot 1P \in \overline{B}$, $I(|\infty P \cap n|) = 0$, Theorem 3. $\diamondsuit$

Corollary 9 is false without C_∞ : $P = X$ in Example 1, even $1X \in B$.

5. Riemann-integrals

We consider now B, I arising from finitely additive set functions μ, with arbitrary set $X \neq \emptyset$:

(16) $= \mu|\Omega$ means: Ω is a semiring from X, $\mu : \Omega \to [0, \infty)$ is additive on Ω

(17) $B_\Omega :=$ step functions $S(\Omega,\mathbb{R})$, $I_\mu(h) := \int h d\mu$, $h \in B_\Omega$,

where $S(\Omega,\mathbb{R})$ contains all $h = \Sigma_1^n a_m A_m$ with $n \in \mathbb{N}$, $a_m \in \mathbb{R}$, $A_m \in \Omega$, $aA := a$ on A, $:= 0$ on $X - A$, $\int h d\mu = a_1\mu(A_1) + \cdots + a_n\mu(A_n)$ (see [8], p. 17); with $\mu|\Omega$ one has (3) for B_Ω and I_μ, B_Ω satisfies Stone's axiom and C_∞. In this situation one can define μ-local convergence, $k_i \to k(\mu)$, [8] p. 69, which localizes the convergence in μ-measure of Dunford-Schwartz ([5], p. 104). By the Lemma in [8], p. 70, A 2.72, for nets one gets with Definition 1 and $I_\mu^-(k) = \inf\{I_\mu(h) : k \leq h \in B_\Omega\}$

Lemma 9. *If $\mu|\Omega$ holds and k_i, $k \in \overline{\mathbb{R}}^X$, then $k_i \to k(\mu)$ if and only if $k_i \to k(I_\mu^-)$.*

By Lemma 9 and (7), $k_i \to k(\mu)$ always implies $k_i \to k(\overline{I}_\mu)$; the converse is in general false: $X = [0,1)$, $\Omega = \{[a,b) : 0 \leq a \leq b \leq 1\}$, $\mu =$ Lebesgue measure on Ω, $\mathbb{Q} =$ rationals $\subset X$; then $k_n := 0 \to$ $\to 1\mathbb{Q}(\overline{\mathrm{I}}_\mu)$ by §6, (38), but not $\to (\mu)$. This is different for "Riemann-μ-integrable" functions: The space $L(\mu,\mathbb{R}) = \mathrm{L}(\mathrm{X}, \Omega, \mu,\mathbb{R})$ of μ-integrable functions of Dunford-Schwartz ([5], III.2.17, p.112) has been generalized to $R_1(\mu,\mathbb{R})$ resp. $R_1(\mu,\overline{\mathbb{R}})$ in [8], p.70, 199; if $X \in \Omega$, then $L(\mu,\mathbb{R}) =$ $= R_1(\mu,\mathbb{R})$, but even for $X = \mathbb{R}$, $\Omega = \{[a,b)\}$ and $\mu =$ Lebesgue-measure μ_L on Ω the $L(\mu_L,\mathbb{R})$ strictly $\subset R_1(\mu_L,\mathbb{R})$, there are $f \in$ $\in R_1(\mu,\mathbb{R})$ which are not equivalent.

Lemma 10. *For $\mu|\Omega$ and f_i, $f \in R_1(\mu,\overline{\mathbb{R}})$ the convergences $f_i \to$ $\to f(\mu)$ and $f_i \to f(\overline{I}_\mu)$ are equivalent.*

Proof. If $f_i \to 0(\overline{I}_\mu)$, $0 \leq h \in B := B_\Omega$, $\varepsilon > 0$, there are $i_\varepsilon = i_{\varepsilon,h}$, $z_i = z_{i,h} \in B_+$ with $g_i := |f_i| \wedge h \leq z_i$ and $I^+(z_i) < \varepsilon$, $i > i_\varepsilon$. Now $g_i \in$ $\in R_1(\mu,\mathbb{R})$, the g_i are bounded with Ω-bounded support, so by [8], A 7.114, p.257 the g_i are "proper Riemann-μ-integrable", i.e. $\in R_e^1(\mu,\mathbb{R}) = \mathrm{I}_\mu^-$-closure of B in $\mathbb{R}^{\mathrm{X}}$ in the sense of Aumann [1]. For $g \in R_e^1(\mu,\mathbb{R})$ one has almost by definition Riemann-μ-integral $\int g d\mu =$ $= I_\mu^+(g) = I_\mu^-(g)$, so $I_\mu^-(g_i) = I_\mu^+(g_i) \leq I_\mu^+(z_i) < \varepsilon$, $i > i_\varepsilon$; by Lemma 9, $f_i \to 0(\mu)$. ◊

By Bobillo and Carrillo [4], $R_1(\mu,\mathbb{R}) \subset \overline{\mathrm{B}}_\Omega$ mod μ-nulfunctions; a slight generalization of this follows easily with the results of §1:

Corollary 10. *If $\mu|\Omega$ holds, then $R_1(\mu,\overline{\mathbb{R}}) \subset \overline{B}_\Omega + \{k \in R_1(\mu,\overline{\mathbb{R}}) : \int |k| d\mu = 0\}$.*

Proof. If $0 \leq f \in R_1 := R_1(\mu,\overline{\mathbb{R}})$, by [8]. A. 7.124 c, p. 259, there are $h_n \in B := B_\Omega$ with $0 \leq h_n \leq h_{n+1} \leq f$, $h_n \to f(\mu)$, $I_\mu(h_n) \to \int f d\mu$. Then $g := \lim h_n \in B^+$, $\leq f$, and $h_n \to g(\mu)$; Lemma 9, (7) and Corollary 8 give $g \in B_{(+)} \subset \overline{B}$, $I(h_n) \to I(g) = \int f d\mu$. Since (h_n) is Cauchy with respect to $\|\cdot\|_\mu := \int |\cdot| d\mu$, by definition $g \in R_1(\mu,\overline{\mathbb{R}})$, $\int g d\mu = \lim \int h_n d\mu = \int f d\mu$; if $0 \leq k := f - g$, $0 = h_n - h_n \to k(\mu)$, $k \in R_1(\mu,\overline{\mathbb{R}})$, $\int |k| d\mu = 0$. With $f = f_+ - f_-$ and the linear and lattice properties of R_1 and $\overline{B}$, $\int \cdot \cdot d\mu$ and $I = I_\mu$ one gets (where $f_- \neq 0$, $f_+ = 0$, there the g, k for f_+ vanish too; one can arrange even $g(x) \neq \infty$ for $x \in X$): If $f \in R_1(\mu,\overline{\mathbb{R}})$, then

$$\text{(18)} \qquad \begin{aligned} &f = g + k, g \in \overline{B}_\Omega \cap R_1(\mu,\mathbb{R}),\ k \in R_1(\mu,\overline{\mathbb{R}}) \text{ with} \\ &\int |k| d\mu = 0,\ k = 0(\mu),\ \int f d\mu = I(g).\ \Diamond \end{aligned}$$

$R_1 \subset \overline{B}$ mod μ-nulfunctions of [4] is the only relation one has in general between R_1 and $\overline{B}_\Omega$: There exists X, a semiring Ω of sets from X and an even σ-additive $\mu : \Omega \to \{0,1\} \subset \mathbb{R}$ such that simultaneously $R_1(\mu,\mathbb{R}) - \overline{\mathrm{B}}_\Omega \neq \emptyset$, $\overline{B}_\Omega - (L_1(\mu,\overline{\mathbb{R}}) \cup R_1(\mu,\overline{\mathbb{R}})) \neq \emptyset$, $L_1(\mu,\mathbb{R}) - (\overline{\mathrm{B}}_\Omega \cup \cup R_1(\mu,\overline{\mathbb{R}})) \neq \emptyset$, $(\overline{B}_\Omega \cap L_1(\mu,\mathbb{R})) - \mathrm{R}_1(\mu,\overline{\mathbb{R}}) \neq \emptyset$ and $X = \cup\Omega$. We give only

Example 2. There is a semiring Ω, a σ-additive $\mu : \Omega \to \{0,1\}$, a set $T \subset X$ and a $g_0 \in B^{(+)}$ with $1T = 0(\mu)$, so $1T \in R_1(\mu,\mathbb{R})$, but $1T \notin \overline{B}_\Omega$, though even $1T \leq g_0$ and $I_\mu^+(g_0) = 0$: $X := \mathbb{N}_0 \times \mathrm{J}$, $\mathbb{N}_0 = \{0,1,2,\ldots\}$, $J = [0,1) \subset \mathbb{R}$, Ω contains all M of the form $\{n\} \times E$, $\{n\} \times (J - E)$, $F \times \{y\}$ or $(N_0 - F) \times \{y\}$ with $0 \neq n \in \mathbb{N}_0$, E finite $\subset J$, $0 \notin F$ finite $\subset \mathbb{N}_0$, $y \in J$; $\mu : \Omega \to \{0,1\}$ is defined by $\mu(\{n\} \times (J - E)) = 1$, $\mu(M) = 0$ for all other $M \in \Omega$; $T := := \{0\} \times J$, $g_0 := 1W$ with $W = \cup_{y \in J} A_y$ with $A_y := \mathbb{N}_0 \times \{y\}$ if $y \neq$ all $y_{n,k}$, $A_{y_{n,k}} := \{0, k, k+1, \ldots\} \times \{y_{n,k}\}$, where the $y_{n,k}$ are choosen as follows: for $n \in \mathbb{N}$, $I_n := (1 - \frac{1}{n}, 1 - \frac{1}{n+1})$, $y_{n,k} \in I_n$ for $k = 1, 2, \ldots$ with $y_{n,k} \neq y_{n,l}$ if $k \neq l$. Furthermore $\overline{B}_\Omega \subset R_1(\mu,\overline{\mathbb{R}})$. By definition $h_n := 0 \to 1T(\mu)$, but $1T \notin \overline{B}$, $B := B_\Omega$: Else $I(1T) = \int 1T d\mu = 0$ by Corollary 2, there is $g \in B_+$ with $1T \leq g$, $I^+(g) < 1/2$, by Lemma 1 of [2] and since $B^+ \cap \overline{B} = B_{(+)}$ one can assume $g = 1M$. If $J_n := \{y \in J : \{0, n, n+1, \cdots\} \times \{y\} \subset M$, $J = \cup_1^\infty J_n$, so one J_{n_0}

is infinite; then $l := 1(\{n_0\} \times (J - J_{n_0})) \in B^+$ with $I^+(l) = 0$; since $1(\{n_0\} \times J) \leq 1M + l$, one has $1 \leq I^+(1M+l) = I^+(1M) + I^+(l) < \frac{1}{2} + 0$, a contradiction. ($\overline{I}(1T) = \infty$ by Corollary 11.)

Finally $\overline{B} \subset R_1$ by the following criteria, since one can easily verify (c), for $\mu(A) > 0$ with (20), then $-g \in B^+$.

If $\mu|\Omega$ holds, the following conditions are equivalent for $B := B_\Omega$, I_μ:

(19) (a) $\overline{B} \subset R_1 := R_1(\mu, \overline{\mathbb{R}})$ (b) $B_{(+)} \subset R_1$
(c) if $0 \leq g \leq 1A$ with $A \in \Omega$, $g \in B_+$, then $g \in R_1$.

(20) $\Rightarrow \mu|\Omega,\ 0 \leq g \leq 1A,\ A \in \Omega,\ g \in B_{\Omega,+}\ \Rightarrow,$
$\Rightarrow [g \in R_1(\mu, \overline{\mathbb{R}}) \Leftrightarrow \mathrm{I}^-(\mathrm{g}) = \mathrm{I}^+(\mathrm{g}) \Leftrightarrow \mathrm{I}^+(\mathrm{g}) + \mathrm{I}^+(-\mathrm{g}) = 0]$.

(19) follows from the closure properties of R_1 ([8], A 7.124 (f) $\Rightarrow$ (a), A 7.121, A 3.56) with Lemma 10 for $p_n + g_n$, $B_- \ni -p_n \leq f \leq g_n \in B_+$; [8], A 7.114 gives (20).

So generally $R_1 \not\subset \overline{B}_\Omega$; we can however characterize the sets on which $R_1 \subset \overline{B}_\Omega$. For this we need

Corollary 11. *If $\mu|\Omega$ holds and $f \in R_1(\mu, \overline{\mathbb{R}})$, then $f \in \overline{B}_\Omega$ if and only if there is $g \in \overline{B}_\Omega$ with $|f| \leq g$; then $\int f d\mu = I_\mu(f)$.*

Proof. Only "if": $g := |f| \in \overline{B}_\Omega$. For the "if", one can assume $f \geq 0$ as in the proof of Corollary 10; with h_n as there one has $\int h_n d\mu \to \int f d\mu$, $0 \leq f - h_n \leq g$, $h_n \to f(\overline{I}_\mu)$, so $f \in \overline{B}_\Omega$ and $I_\mu(f) = \lim I_\mu(h_n) = \int f d\mu$ by Corollary 2. ◇

Definition 2 with $\mu|\Omega$: $\mathcal{R}(\mu) := \{M \subset X : f \in R_1(\mu, \overline{\mathbb{R}}),\ f = 0$ on $X - M \Rightarrow f \in \overline{B}_\Omega\}$.

$\mathcal{R}(\mu)$ is complete, i.e. if $P \subset M \in \mathcal{R}(\mu)$, then $P \in \mathcal{R}(\mu)$.

Theorem 4. *If $\mu|\Omega$ holds and $M \subset X$, then $M \in \mathcal{R}(\mu)$ if and only if $1P \in \overline{B}_\Omega$ for each strong μ-nulset $P \subset M$.*

Here P is called a *strong μ-nulset* iff $1P \in R_1(\mu, \overline{\mathbb{R}})$ and $\int 1P d\mu = 0$, or equivalently iff $1P \to 0(I_\mu^-)$ ([8], p. 69).

Proof of "if": By corollary X and as there we can assume $0 \leq f \in R_1(\mu, \overline{\mathbb{R}})$ with $\int f d\mu = 0$, $f = 0$ outside M; with Theorem 3 it is enough to show $f \wedge 1 \in \overline{B}$, $B = B_\Omega$, $C_\infty(B, I_\mu)$ holds since step functions are bounded, so we assume $f \leq 1$. The $P_n := \{x \in X : f(x) > 1/n\}$ are strong μ-nulsets $\subset M$, since $\frac{1}{n} P_n \leq f = 0(\mu)$, so $1P_n \in \overline{B}$, $I(1P_n) = 0$ by Corollary 11, $n \in \mathbb{N}$. $0 \leq f_n := f \wedge 2^{-n} - f \wedge 2^{-n-1} \leq 2^{-n} P_{n+1}$ with $f = \Sigma_1^\infty f_n$; to $\varepsilon > 0$ there are $g_n \in B_+$ with $2^{-n} P_{n+1} \leq g_n \leq 2^{-n}$, $I(g_n) < \varepsilon \cdot 2^{-n}$, using Lemma 8. So $g := \Sigma_1^\infty g_n \in B^+$ and $l_n := \Sigma_1^n g_j \to$

$\to g$ uniformly on X; this implies $l_n \to g(\mu)$, with $I(l_n) < \varepsilon$ for $n \in \mathbb{N}$. Then $g \in B_+$ with $I^+(g) \leq \varepsilon$, $g \in \overline{B}$, by Corollary 7. Obviously $f \leq g$, Corollary 11 yields $f \in \overline{B}$. $\diamondsuit$

Corollary 12. *If $\mu|\Omega$ holds, $\mathcal{R}(\mu)$ of Definition 2 is a ring containing all $M \subset X$ to which there is $g \in \overline{B_\Omega}$ with $1M \leq g$, especially*

$$\Omega \subset \{P \subset M : 1M \in R_1(\mu,\mathbb{R}) \cap B_\Omega^+\} \subset \{P \subset M : 1M \in \overline{B_\Omega}\} \subset \mathcal{R}(\mu).$$

Proof. Theorem 4, Corollary 11 and the linearity of $\to (I_\mu^-)$ give the "ring" and $M \in \mathcal{R}(\mu)$ if $1M \leq g \in \overline{B}$, $B = B_\Omega$. One even has

$$R_1(\mu,\overline{\mathbb{R}}) \cap B^+ \subset B_{(+)} \subset \overline{B}: \tag{21}$$

If $g \in R_1 \cap B^+$, then $g \geq$ some $h \in B$, so one can assume $g \geq 0$; the proof of Corollary 10 yields $g \in B_{(+)}$. $\diamondsuit$

One can also show that $\mathcal{R}(\mu)$ is closed with respect to certain countable unions: If $M = \cup_1^\infty M_m$ with $1M_m \in \overline{B_\Omega}$, $1M_n \to 1M(\overline{I}_\mu)$, $1M \in B_\Omega^+$, then $M \in \mathcal{R}(\mu)$. Special case: $M = \cup_1^\infty A_m \in \mathcal{R}(\mu)$ if $A_m \in \Omega$ and $1 \cup_1^n A_m \to 1M(\mu)$ (directly: Theorem 4, Corollary 7, 11). $M = X$ gives $R_1(\mu,\mathbb{R}) \subset \overline{B_\Omega}$ if one of the following conditions is true:

(22) $1P \in \overline{B_\Omega}$ if P strong μ-nulset

(23) there are $A_m \in \Omega$ with $X = \cup_1^\infty A_m$ and $1(\cup_1^n A_m) \to 1X(\overline{I}_\mu)$

(24) there is a locally finite countable Ω-partition of X

(25) there are $A_m \in \Omega$ with $X = \cup_1^\infty A_m$ and μ is σ-additive on Ω

(26) $I^+(1X) < \infty$ (equivalently: μ is bounded on the ring generated by Ω, or $1X \in \overline{B_\Omega}$, or $1X \in R_1(\mu,\mathbb{R})$)

(27) all $\{x\} \in \Omega$, $x \in X$ (equivalently: $[0,\infty]^X \subset B_\Omega^+$), then even $R_1(\mu,\overline{\mathbb{R}}) = \overline{B}_\Omega$ by (19), (20), (21).

Example for (24) or (25): $X = \mathbb{R}^n$, $\mu =$ Lebesgue measure μ_L^n,

$$\Omega = \Omega_n := \{\textstyle\prod_1^n [a_j, b_j) : a_j \leq b_j,\ a_j,\ b_j \in \mathbb{R}\}. \tag{28}$$

By (25) an "example 2" with $X =$ countable union of $A_m \in \Omega$ does not exist.

Finally, one can always force $R_1(\mu,\overline{\mathbb{R}}) \subset$ some $\overline{B}$ with $\int \cdot\cdot\, d\mu = I$; $\Sigma := \{D = (A - M) \cup P : A \in \Omega,\ M$ and P strong μ-nulsets$\}$, $\nu(D) := \mu(A)$, $B = B_\Sigma$, $I = I_\nu$; then $R_1(\nu,\overline{\mathbb{R}}) = R_1(\mu,\overline{\mathbb{R}})$, integrals and strong nulsets coincide ([8], p.199, A 6.148).

Also always $0 \leq f \in R_1(\mu,\overline{\mathbb{R}}) \Rightarrow f \in (B_\Omega)^*_+$ of [3], p.235, though $\int f d\mu < \overline{I}(f)$ if $\overline{I}(f) = \infty$ ([4], p.263, Proposition 1).

6. Lebesgue-, Daniell- and Bourbaki-integrals

In this section we additonally assume Daniell's continuity condition, = σ-stetig in Floret's work [6], p.43:

(29) (3) and $I(h_n) \to 0$ if $0 \leq h_{n+1} \leq h_n \in B$, $n \in \mathbb{N}$, with $h_n(x) \to 0$ for each $x \in X$;

then the space $L^1 := L^1(B,I) := L^1(B,I,\overline{\mathbb{R}}) \subset \overline{\mathbb{R}}^X$ of Daniell-I-integrable functions with integral extension $J : L^1 \to \mathbb{R}$ is well defined (e.g. [6], p. 77; Daniell- "summable" in Pfeffer's work [10] (p. 60); = closure of B in $\overline{\mathbb{R}}^X$ with respect to a suitable integral-seminorm in Aumann's paper [1] (p. 448 – 450)). Here one has an analogue to the statement of Corollary 10:

Theorem 5. *In the Daniell situation* (29), $\overline{B} \subset L^1(B,I) \cap \overline{B} \cap \mathbb{R}^X + \overline{B}_n$, *with* $\overline{B}_n := \{f \in \overline{B} : \overline{I}(|f| = 0\}$, *and* $I = J$ *on* $L^1 \cap \overline{B}$.

Our proof is fundamentally similar to that of Corollary 10, but more involved and somewhat lengthy, so we omit it here.

Corresponding to the remarks before Example 2, $\overline{B} \subset L^1 + \overline{B}_n$ is the only generally true relation of this type:

Example 3. There is a σ-algebra Ω and a σ-additive $\mu : \Omega \to \{0,1\}$ such that

$$(30) \qquad R_1(\mu,\overline{\mathbb{R}}) = L^1(\mu,\overline{\mathbb{R}}) = L^1(B_\Omega, I_\mu) = L_1(\mu,\overline{\mathbb{R}}) \subsetneq \overline{B}_\Omega :$$

X uncountable, $\Omega = \{M \subset X : M$ or $X - M$ countable and $\not\ni x_0\}$, $\mu =$ Dirac measure δ_{x_0} in $x_0 \in X$, even τ-continuous (see after (35)). There are also algebras Ω for which with $\mu = \delta_{x_0}$,

$$(31) \qquad R_1 \subsetneq L^1 = L_1 := L^1 + \{k \in \overline{\mathbb{R}}^X : k \cdot 1A = 0 \ \mu - a.e. \text{ for each } A \in \Omega\} \subsetneq \overline{B}_\Omega .$$

Example 4. There is an algebra K and a σ-additive $\mu : K \to [0,1]$ such that

$$(32) \ R_1(\mu,\overline{\mathbb{R}}) = \overline{B}_K \subsetneq \overline{B}_K + \{f \in \overline{\mathbb{R}}^X : f = 0 \mu - a.e.\} \subsetneq L^1(\mu|K,\overline{\mathbb{R}}) = \overline{B}_{\Omega_1},$$

i.e. L^1 differs from $\overline{B}$ by more than just L^1-nulfunctions (see (38)):

$X = [0,1]$, Ω =ring K generated by all intervals $\subset X$ (thus $\{t\} \in K$, $t \in X$), μ = Lebesgue measure $\mu_L^1|K$, $\Omega_1 = \{[a,b)\}$ of (28); here $L^1 = L^1(B_K, I_\mu)$ = usual {Lebesgue integrable $f : X \to \overline{\mathbb{R}}$}, if $G :=$ $:= \cup_1^\infty (r_m, r_m + 3^{-m})$, $P := \{r_m : m \in \mathbb{N}\}$:= rationals $\subset X$, $B := B_K$, then $1P \notin \overline{B}$, $1G \notin \overline{B} + \{f = 0\ \mu - a.e.\}$: In $f = 1G + p \in \overline{B}$ with $p = 0$ a.e. one can assume $0 \leq f \leq 1$ since $\overline{B}$ is a lattice ([3], p. 252), thus f and $1 - f \in [0,\infty)^X \subset B^+$, $f \in B_{(+)}$, $1 = I^+(f) + I^+(1-f)$, $I^+(f) \leq \int f dx = \int 1G dx \leq \frac{1}{2}$; by definition of G to each $x \in \dot{X}$ and $\varepsilon > 0$ there is $y \in (G - \{p \neq 0\}) \cap (x - \varepsilon, x + \varepsilon)$, $f(y) = 1$, thus $I^+(1-f) = 0$, a contradiction. $\overline{B} = R_1(\mu|K,\overline{\mathbb{R}}) = R_1(\mu|\Omega_1,\overline{\mathbb{R}}) \subset L^1 =$ $= \overline{B}_{\Omega_1}$ by (27) and (38).

By simple disjoint union one can combine Examples 2 – 4 into one X, Ω, μ. Example 4 shows that a converse of Theorem 5 is false; it also shows that the extension process $B \to \overline{B}$ is in general not monotone in B – contrary to the Lebesgue, Daniell and Bourbaki extensions.

Furthermore the convergence $\to (\overline{I})$ used here is in general not comparable with that of L^1, i.e. pointwise (almost everywhere) convergence, not even in the situation $\mu|\Omega$ with σ-additive μ.

Only under additional assumptions can one say more:

If $I|B$ is monotone-net-continuous, = Bourbaki-integral (Pfeffer [10], p. 44), = τ–stetig (Floret [6], p.336), then the space $L^\tau :=$ $:= L^\tau(B,I) = L^\tau(B,I,\overline{\mathbb{R}})$ of Bourbaki-I-integrable functions ($\mathcal{L}^\#$ in [10], ${}^\tau\mathcal{L}_1$ in [6], p.338) and the corresponding integral extension $I^\tau :$ $: L^\tau \to$ R are well defined with $L^1(B,I,\overline{\mathbb{R}}) \subset L^\tau(B,I,\overline{\mathbb{R}})$, $J = I^\tau|L^1$; then I^+ is additive on B^+, i.e. $B_+ = B^+$ and $\overline{I}$ = upper Bourbaki integral (Bobillo-Carrillo [3], p. 247), thus

$$(33) \qquad (3),\ I\ \tau\text{-continuous on } B \Rightarrow \overline{B} = \text{Bourbaki extension } L^\tau(B,I,\overline{\mathbb{R}}),\ I = I^\tau,$$

Daniell-$L^1(B,I) =: L^1 \subset \overline{B} = L^\tau = L^1 + L_n = L^1 + \overline{B}_n$ by [6] (p. 340) or our Theorem 5, n := nulfunctions, with $\subset$ generally strict by Example 3.

If $B = C_0(X,\mathbb{R})$ with arbitrary Hausdorff space X, then any nonnegative linear $I : B \to \mathbb{R}$ is τ-continuous ([6], p.337), L^τ is defined and (33) holds. With Pfeffer ([10], p.37) one gets for $B = C_0 :=$ $:= C_0(X,\mathbb{R})$ and any nonnegative linear $I : C_0 \to \mathbb{R}$, automatically τ-continuous

$$(34) \qquad X \text{ locally compact, all open } G \subset X \text{ are } \sigma\text{-compact} \Rightarrow$$

$$\Rightarrow L^1(C_0,I) = L^\tau(C_0,I) = \overline{C}_0;$$

example: $X = \mathbb{R}^n$, I =Riemann/Lebesgue integral (see (38)).

In the situation $\mu : \Omega \to [0,\infty)$ with σ-additive μ on the semiring Ω, $B = B_\Omega$,

(35) $I_\mu = \int \cdot\cdot\, d\mu$ is τ-continuous on B_Ω iff μ is τ-additive on Ω (see (17))

(there are such μ, Ω = algebra, but μ not τ-continuous – the converse holds for rings). As a sufficient criterium on has:

If $\mu|\Omega$ = (16) holds, μ is σ-additive on Ω and if for any index set S

(36) to $A, A_s \in \Omega$, $s \in S$, with $A_s \supset A$, exist countable $S_0 \subset S$, μ-nulset P with $\bigcup_{s\in S} A_s = P \cup \bigcup_{s\in S_0} A_s$

is true, then for $B = B_\Omega, I = I_\mu$ of (17) one has with $\overline{B}_n$, L_1 of Theorem 4, (33)

(37) $I|B$ is τ-continuous,

$$L^1(B,I) = L^1(\mu,\overline{\mathbb{R}}) \subset \overline{B} = L^\tau(B,I) = L^1 + \overline{B}_n \subset L_1(\mu,\overline{\mathbb{R}}):$$

By (36) μ is τ-additive on Ω, by (35) I τ-continuous on B; with (33) and Theorem 5 it is enough to show $\overline{B}_n \subset L_{1,n}$. This being a local property, it suffices to show that $g \in L^1$ if $0 \le g \le 1A$, $g \in B^+$, $A \in \Omega$, since then $I^+(g) = \int g d\mu$ by (33); $g \in L^1$ follows if g is μ-measurable, and the latter is an immediate consequence of (36) and the definition of B^+.

Example 5. There is a ring Ω and a τ-additive $\mu : \Omega \to [0,\infty)$ with (36) and

$$L^1 \subsetneqq \overline{B} = L^\tau \subsetneqq L_1:$$

$X :=$ disjoint $\cup_{i\in S} S_i$ with $S_i := S := [0,1)$, Ω_0, μ_0 as in Example 3, $\Omega_i := \Omega_1$ of (28) in S_i and $\mu_i :=$ Lebesgue measure μ_L, $0 < i < 1$, $\Omega :=$ ring generated by $\cup_{i\in S}\Omega_i$, $\mu = \mu_i$ on Ω_i; (36) for $\mu_L|\Omega_1$ holds by the remarks following: if $f := 1$ on all $0_i \in S_i$ with $i > 0$, else $f := 0$, then $f \in L_{1,n}$ but $\overline{I}(f) = \infty$.

The condition (36) follows from an abstract Vitali covering condition for $\mu|\Omega$; the latter is true for example for $X = \mathbb{R}^n$, μ = Lebesgue measure μ_L^n, Ω = intervals Ω_n of (28); since here additionally $X = \cup_1^\infty A_m$ with $A_m \in \Omega$, one has $L^1 = L_1$.

We collect the above results with (27) and $R_1 = L_1$ for δ-rings

([8], p. 265), $L^1(C_0, I_\mu, \overline{\mathbb{R}}) = L^1(\mu|M_n, \overline{\mathbb{R}})$:
If $\Omega_n \subset \Sigma$ semiring $\subset M_n :=$ {Lebesgue-measurable sets $\subset \mathbb{R}^n$ with finite L-measure}, $\Omega \in \{\Omega_n, \Sigma, M_n\}$, $\mu = \mu_L^n$, $I = I_\mu$, $B = B_\Omega$ or $C_0 := C_0(\mathbb{R}^n, \mathbb{R})$ (see (17), (28)), $X = \mathbb{R}^n$ or more generally open $\subset \mathbb{R}^n$ with corresponding $\Omega, \cdots$, then the following L-spaces, and their integrals, all coincide

$$(38) \qquad \begin{aligned} L^1(\mu|\Omega,\overline{\mathbb{R}}) &= L_1(\mu|\Omega,\overline{\mathbb{R}}) = L^1(B,I) = L^\tau(B_{\Omega_n}, I) = \\ &= L^\tau(C_0, I) = \overline{C}_0 = \overline{B}_{\Omega_n} = \overline{B}_{M_n} = R_1(\mu|M_n,\overline{\mathbb{R}}). \end{aligned}$$

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