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D-CONHARMONIC CHANGE IN A SPECIAL PARA-SASAKIAN MANIFOLD

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Abstract: The notion of D-conformal change in a para-Sasakian and a special para-Sasakian manifold is introduced by G. Chūman [4]. The D-concircular change is a special kind of D-conformal change in a special para-Sasakian manifold. It is introduced and studied in [2].

In this paper, we introduce the D-conharmonic change, an another special kind of D-conformal change in a special para-Sasakian manifold. We obtain the tensor field invariant under this change and discuss the manifolds for which this tensor field vanishes.

1. A special para-Sasakian manifold and the D-conformal change.

Let us consider an n-dimensional differentiable manifold M with a positive definite Riemannian metric g_{ij}. We suppose that M admits a unit covariant vector field η_i satisfying

$$\nabla_j \eta_i = \bar{\varepsilon}(-g_{ij} + \eta_i \eta_j), \qquad \bar{\varepsilon} = \pm 1, \tag{1.1}$$

where ∇_j denotes the covariant differentiation with respect to g_{ij} and indices take the values $1, 2, \ldots, n$. If we put

$$\xi^i = g^{ik}\eta_k, \quad \psi_j^i = \nabla_j \xi^i, \quad \psi_{ij} = g_{ik}\psi_j^k,$$

we have

$$(1.2) \qquad \begin{cases} \eta_i \xi^i = 1, \quad \psi_j^i \psi_i^k = \delta_j^k - \eta_j \xi^k, \quad \psi_j^i \xi^j = 0, \quad \eta_i \psi_j^i = 0, \\ g_{ij}\psi_p^i \psi_q^j = g_{pq} - \eta_p \eta_q, \quad \psi_{ij} = \psi_{ji}, \quad \operatorname{rank}(\psi_j^i) = n-1. \end{cases}$$

The relations (1.2) show that M is an almost paracontact Riemannian manifold (ψ, ξ, η, g). Because of (1.1), it is a *special para-Sasakian manifold* [7].

There is in M an (n-1)-dimensional distribution D defined by a Pfaffian equation $\eta = 0$ and called the D-distribution. Assume in M two para-Sasakian structures (ψ, ξ, η, g) and $({}^*\psi, {}^*\xi, {}^*\eta, {}^*g)$ satisfy

$$(1.3) \qquad \begin{cases} {}^*g_{ij} = e^{2\alpha} g_{ij} + (e^{2\sigma} - e^{2\alpha})\eta_i \eta_j \\ {}^*\xi^i = \varepsilon e^{-\sigma}\xi^i, \quad {}^*\psi_j^i = \varepsilon \psi_j^i, \quad {}^*\eta_i = \varepsilon e^{\sigma}\eta_i, \qquad \varepsilon = \pm 1 \end{cases}$$

where α and σ are functions. Then (ψ, ξ, η, g) and $({}^*\psi, {}^*\xi, {}^*\eta, {}^*g)$ have the same D-distribution. The relation (1.3) is called by Chūman [4] a *D-conformal change* of (ψ, ξ, η, g). When the function α is constant, (1.3) is called *a D-homothetic change.* G. Chūman proved [4] that if a para-Sasakian manifold is not special (i.e. $\psi^2 \neq (n-1)^2$), then any D-conformal change is necessarily D-homothetic. That is why non D-homothetic D-conformal change occurs only in a special para-Sasakian manifold.

By the change (1.3), M is also transformed into an almost paracontact Riemannian manifold. Furthermore, if $\psi_j^i = \nabla_j \xi^i$ is invariant under the change (1.3), then a special para-Sasakian M is transformed into a special para-Sasakian manifold. Hereafter, we consider the D-conformal change (1.3) satisfying

$$\psi_j^i = \nabla_j \xi^i, \qquad {}^*\psi_j^i = {}^*\nabla_j^* \xi^i,$$

where ${}^*\nabla$ is covariant differentiation with respect to ${}^*g_{ij}$ in a special para-Sasakian manifold M. By [4] we have

$$(1.4) \qquad \sigma_i = \sigma_p \xi^p \eta_i, \quad \bar{\varepsilon}\alpha_p \xi^p = 1 - e^{\sigma}, \quad \sigma_i = \nabla_i \sigma, \quad \alpha_i = \nabla_i \alpha.$$

From (1.3) we get

$$(1.5) \qquad {}^*g^{ji} = e^{-2\alpha} g^{ij} + (e^{-2\sigma} - e^{-2\alpha})\xi^i \xi^j \qquad ({}^*g^{ij\,*}g_{jk} = \delta_k^i).$$

Thus, in a special para-Sasakian manifold, we have the following relation between $^*\{{}^{k}_{ij}\}$ and $\{{}^{k}_{ij}\}$ (cf. [4]):

$$(1.6)\quad ^*\{{}^{h}_{ij}\} = \{{}^{h}_{ij}\} + \alpha_j(\delta_i^h - \eta_i\xi^h) + \alpha_i(\delta_j^h - \eta_j\xi^h) - \alpha^h(g_{ij} - \eta_i\eta_j) + \\ + \bar{\varepsilon}(e^{2\alpha-\sigma} - e^{\sigma})(g_{ij} - \eta_i\eta_j)\xi^h + \sigma_j\eta_i\xi^h.$$

Let $R_{kji}{}^h$, R_{ij} and R denote the Riemannian curvature tensor, the Ricci tensor and the scalar curvature of the manifold M respectively. Then the tensor field

$$(1.7)\quad B_{kji}{}^h = R_{kji}{}^h - \frac{R+2}{(n-2)(n-3)}(g_{ki}\delta_j^h - g_{ji}\delta_k^h) + \\ + \frac{1}{n-3}[R_{ki}(\delta_j^h - \eta_j\xi^h) - R_{ji}(\delta_k^h - \eta_k\xi^h) + (g_{ki} - \eta_k\eta_i)R_j^h - \\ -(g_{ji} - \eta_j\eta_i)R_k^h] + \frac{R+2(n-1)}{(n-2)(n-3)}(g_{ki}\eta_j\xi^h - g_{ji}\eta_k\xi^h + \eta_k\eta_i\delta_j^h - \eta_j\eta_i\delta_k^h)$$

is invariant under any D-conformal change in a special para-Sasakian manifold ($n > 4$) (cf. [4]).

In §4 we shall need the following

Theorem A. ([1]) *Let M be an n-dimensional special para-Sasakian manifold. Then M is transformed into a manifold of constant curvature -1 by a D-conformal change if and only if $B_{kji}{}^h = 0$ ($n > 4$).*

Also, we note that in a special para-Sasakian manifold M, we have the following equations

$$(1.8)\quad R_{kji}{}^h\eta_h = g_{ki}\eta_j - g_{ji}\eta_k, \qquad R_{ji}\xi^i = -(n-1)\eta_j.$$

2. The D-conharmonic change in a special para-Sasakian manifold.

In a special para-Sasakian manifold M, the Pfaffian equation $\eta =$ $= 0$ is completely integrable. The integral manifolds of $\eta = 0$ are called the level surfaces. In a local coordinate system of M, each level surface N is expressed by parametric equations

$$x^h = x^h(u^\lambda).$$

Here and in the sequel the Greek indeces have the range $(1, 2, \ldots, n-1)$. Putting $B_\lambda^h = \frac{\partial x^h}{\partial u^\lambda}$ we have

$$(2.1)\quad \eta_i B_\lambda^i = 0$$

while for the induced Riemannian metric $g_{\nu\mu}$ on N we have

$$g_{\nu\mu} = B^i_\nu B^j_\mu g_{ij}, \tag{2.2}$$

$$g^{ij} = g^{\mu\nu} B^i_\mu B^j_\nu + \xi^i \xi^j, \tag{2.3}$$

where $(g^{\mu\nu}) = (g_{\rho\omega})^{-1}$. Also

$$\nabla_\mu B^i_\nu = h_{\mu\nu}\xi^i, \tag{2.4}$$

where ∇_μ is the operator of the covariant differentiation with respect to $g_{\nu\mu}$ and $h_{\mu\nu}$ is the second fundamental tensor of N.

It is easy to see that each level surface N is totally umbilical. In fact, differentiating (2.1) along N and using (1.1), (2.2) and (2.4), we find $h_{\mu\nu} = \bar{\varepsilon} g_{\mu\nu}$. Therefore (2.4) can be written in the form

$$\nabla_\mu B^i_\nu = \bar{\varepsilon} g_{\mu\nu}\xi^i. \tag{2.5}$$

If we put

$$B^\lambda_h = g^{\lambda\omega} g_{hk} B^k_\omega,$$

we have

$$B^\nu_i B^j_\nu = \delta^j_i - \eta_i \xi^j, \quad B^\nu_i B^i_\mu = \delta^\nu_\mu, \quad B^\lambda_h \xi^h = 0. \tag{2.6}$$

The D-conformal change (1.3) induces in N the conformal change

$$^*g_{\nu\mu} = e^{2\alpha} g_{\nu\mu}, \tag{2.7}$$

where $^*g_{\nu\mu} = {}^*g_{ij} B^i_\nu B^j_\mu$ and α is now considered as a function of u^λ in N. If this conformal change satisfies $\alpha_{\nu\mu} = \varphi g_{\nu\mu}$, where φ is a function of u^λ and

$$\alpha_{\nu\mu} = \nabla_\nu \alpha_\mu - \alpha_\nu \alpha_\mu + \tfrac{1}{2}\alpha_\omega \alpha^\omega g_{\nu\mu}, \quad \alpha_\nu = \nabla_\nu \alpha, \quad \alpha^\omega = g^{\omega\nu}\alpha_\nu$$

then (2.7) is the concircular transformation (cf. [8]). Using this T. Adati and G. Chūman in [2] defined and studied D-concircular transformations.

In this paper we suppose that the conformal change (2.7) is conharmonic one (cf. [6]), i.e. we suppose that the function α in (2.7) satisifies

$$\alpha_{\nu\mu} g^{\nu\mu} = 0. \tag{2.8}$$

Using this, we shall define the D-conharmonic change in M.

From $\alpha_\mu = B^i_\mu \alpha_i$ and (2.5), we have (cf. [1])

$$\nabla_\nu \alpha_\mu = B^j_\nu B^i_\mu (\nabla_j \alpha_i + \bar{\varepsilon} \alpha_p \xi^p g_{ij}).$$

On the other hand, using (2.3), we find

$$\alpha_\omega \alpha^\omega = g^{\nu\mu} \alpha_\nu \alpha_\mu = g^{\nu\mu} B^i_\mu B^j_\nu \alpha_i \alpha_j = (g^{ij} - \xi^i \xi^j) \alpha_i \alpha_j = \\ = \alpha_p \alpha^p - (\alpha_p \xi^p)^2$$

and taking (1.4) into account, we get

$$\alpha_{\nu\mu} = B^j_\nu B^i_\mu [\nabla_j \alpha_i - \alpha_j \alpha_i + \tfrac{1}{2}(\alpha_p \alpha^p - e^{2\sigma} + 1) g_{ji}].$$

Therefore

$$\alpha_{\nu\mu} g^{\nu\mu} = (g^{ij} - \xi^i \xi^j)[\nabla_j \alpha_i - \alpha_j \alpha_i + \tfrac{1}{2}(\alpha_p \alpha^p - e^{2\sigma} + 1) g_{ij}] = 0.$$

Thus, the D-conformal change (1.3) induces conharmonic changes on each level surface if and only if

$$(g^{ij} - \xi^i \xi^j)[\nabla_j \alpha_i - \alpha_j \alpha_i + \tfrac{1}{2}(\alpha_p \alpha^p - e^{2\sigma} + 1) g_{ij}] = 0. \tag{2.9}$$

Definition. The D-conformal change (1.3) satisfying (2.9) is called a *D-conharmonic change.*

The condition (2.9) can be written in the form

$$g^{ij}(\nabla_j \alpha_i - \alpha_i \alpha_j) - \nabla_j \alpha_i \xi^i \xi^j + (\alpha_p \xi^p)^2 + \tfrac{1}{2}(n-1)(\alpha_p \alpha^p - e^{2\sigma} + 1) = 0,$$

or, using (1.4), in the form

$$g^{ij}(\nabla_j \alpha_i - \alpha_i \alpha_j) - \nabla_j \alpha_i \xi^i \xi^j + (1 - e^\sigma)^2 + \\ + \tfrac{1}{2}(n-1)(\alpha_p \alpha^p - e^{2\sigma} + 1) = 0. \tag{2.10}$$

On the other hand, differentiating the second equation (1.4) and using (1.1), we get

$$(\nabla_j \alpha_i)\xi^i - \bar{\varepsilon} \alpha_j + \bar{\varepsilon} \alpha_i \xi^i \eta_j = -\bar{\varepsilon} \sigma_j e^\sigma.$$

Therefore

$$(\nabla_j \alpha_i)\xi^i \xi^j = -\bar{\varepsilon} \sigma_j \xi^j e^\sigma.$$

Substituting this into (2.10), we obtain

$$g^{ij}(\nabla_j \alpha_i - \alpha_i \alpha_j) + \bar{\varepsilon} \sigma_p \xi^p e^\sigma + (1 - e^\sigma)^2 + \\ + \tfrac{1}{2}(n-1)(\alpha_p \alpha^p - e^{2\sigma} + 1) = 0. \tag{2.11}$$

Also, (2.9) can be expressed as follows

$$\nabla_j \alpha_i (g^{ij} - \xi^i \xi^j) + \tfrac{n-3}{2} \alpha_t \alpha^t + (\alpha_t \xi^t)^2 + \tfrac{1}{2}(n-1)(1 - e^{2\sigma}) = 0,$$

from which, taking into account (1.4), we get

(2.12) $$\nabla_j\alpha_i(g^{ij}-\xi^i\xi^j)+\tfrac{n-3}{2}\alpha_t\alpha^t+\tfrac{n+1}{2}-2e^{\sigma}-\tfrac{n-3}{2}e^{2\sigma}=0.$$

3. The second access to the notion of the D-conharmonic change.

Let us consider a function A in M. It is, in the level surface N, a function of the coordinates u^λ. If this last function satisfies

$$g^{\nu\mu}\nabla_\nu A_\mu=0,\qquad A_\mu=\nabla_\mu A,$$

it is said to be a *harmonic function* in N (cf. [6]). Let us search for the corresponding condition in M. Since

$$A_\mu=B^i_\mu A_i,\qquad A_i=\nabla_i A$$

and (2.5), we have

$$\nabla_\nu A_\mu=B^i_\mu B^j_\nu\,\nabla_j A_i+\bar\varepsilon g_{\nu\mu}A_t\xi^t,$$

from which, in view of (2.3), we have

$$g^{\nu\mu}\nabla_\nu A_\mu=(g^{ij}-\xi^i\xi^j)\,\nabla_j A_i+\bar\varepsilon(n-1)A_t\xi^t.$$

Thus, the function A is the harmonic function on each level surface if and only if

(3.1) $$(g^{ji}-\xi^j\xi^i)\,\nabla_j A_i+\bar\varepsilon(n-1)A_t\xi^t=0.$$

Now, let us consider in M the function

$$^*A=e^{2p\alpha}A,$$

where p is a suitable constant and A is a function satisfying (3.1). Let us look for the condition upon α ensuring that

(3.2) $$({}^*g^{ji}-{}^*\xi^j\,{}^*\xi^i){}^*\nabla_j\,{}^*A_i+\bar\varepsilon(n-1){}^*A_t{}^*\xi^t=0.$$

We have

(3.3) $$^*A_i=e^{2p\alpha}(2p\alpha_i A+A_i),$$

(3.4) $${}^*\nabla_j{}^*A_i=e^{2p\alpha}[(4p^2\alpha_i\alpha_j+2p\,{}^*\nabla_j\alpha_i)A+2p(\alpha_jA_i+\alpha_iA_j)+{}^*\nabla_jA_i].$$

Using (1.6), we compute

$${}^*\nabla_j\,\alpha_i=\nabla_j\alpha_i-2\alpha_j\alpha_i+(\alpha_j\eta_i+\alpha_i\eta_j)\xi^t\alpha_t+(g_{ij}-\eta_i\eta_j)\alpha_t\alpha^t-$$

$$-\bar{\varepsilon}(e^{2\alpha-\sigma} - e^{\sigma})(g_{ij} - \eta_i\eta_j)\xi^t\alpha_t - \sigma_j\eta_i\xi^t\alpha_t$$

and

$$\begin{gathered}{}^*\nabla_j A_i = \nabla_j A_i - (\alpha_j A_i + \alpha_i A_j) + (\alpha_j\eta_i + \alpha_i\eta_j)\xi^t A_t + \\ +(g_{ij} - \eta_i\eta_j)\alpha^t A_t - \bar{\varepsilon}(e^{2\alpha-\sigma} - e^{\sigma})(g_{ij} - \eta_i\eta_j)\xi^t A_t - \sigma_j\eta_i\xi^t A_t.\end{gathered}$$

Substituting this into (3.4), we find

$$\begin{gathered}{}^*\nabla_j{}^*A_i = e^{2p\alpha}\{2pA[\nabla_j\alpha_i + 2(p-1)\alpha_j\alpha_i + (\alpha_j\eta_i + \alpha_i\eta_j)\xi^t\alpha_t + \\ +(g_{ij} - \eta_i\eta_j)\alpha_t\alpha^t - \bar{\varepsilon}(e^{2\alpha-\sigma} - e^{\sigma})(g_{ij} - \eta_i\eta_j)\xi^t\alpha_t - \sigma_j\eta_i\xi^t\alpha_t] + \\ + \nabla_j A_i + (2p-1)(\alpha_j A_i + \alpha_i A_j) + (\alpha_j\eta_i + \alpha_i\eta_j)\xi^t A_t + \\ +(g_{ij} - \eta_i\eta_j)A_t\alpha^t - \bar{\varepsilon}(e^{2\alpha-\sigma} - e^{\sigma})(g_{ij} - \eta_i\eta_j)\xi^t A_t - \sigma_j\eta_i A_t\xi^t\}.\end{gathered}$$

Using (1.3) and (1.5), we have

$${}^*g^{ij} - {}^*\xi^i\,{}^*\xi^j = e^{-2\alpha}(g^{ij} - \xi^i\xi^j).$$

Therefore,

$$\begin{gathered}({}^*g^{ji} - {}^*\xi^j{}^*\xi^i)^*\nabla_j{}^*A_i = \\ = e^{2(p-1)\alpha}\{2pA[\nabla_j\alpha_i(g^{ij} - \xi^i\xi^j) + (2p+n-3)\alpha_t\alpha^t - 2(p-1)(\xi^t\alpha_t)^2 - \\ -\bar{\varepsilon}(n-1)(e^{2\alpha-\sigma} - e^{\sigma})\xi^t\alpha_t] + \\ + \nabla_j A_i(g^{ij} - \xi^i\xi^j) + (4p+n-3)\alpha^t A_t - 2(2p-1)\xi^t\alpha_t\xi^p A_p - \\ -\bar{\varepsilon}(n-1)(e^{2\alpha-\sigma} - e^{\sigma})\xi^t A_t\}.\end{gathered}$$

Now, if we use (3.3) and take $\varepsilon = +1$ in (1.3), we find

$$\bar{\varepsilon}(n-1){}^*A_t{}^*\xi^t = \bar{\varepsilon}(n-1)e^{2p\alpha-\sigma}(2p\alpha_t\xi^t A + A_t\xi^t).$$

Therefore

$$\begin{gathered}({}^*g^{ji} - {}^*\xi^j{}^*\xi^i)^*\nabla_j{}^*A_i + \bar{\varepsilon}(n-1){}^*A_t{}^*\xi^t = \\ = 2pA\{e^{2(p-1)\alpha}[\nabla_j\alpha_i(g^{ij} - \xi^i\xi^j) + (2p+n-3)\alpha_t\alpha^t - 2(p-1)(\xi^t\alpha_t)^2 - \\ -\bar{\varepsilon}(n-1)(e^{2\alpha-\sigma} - e^{\sigma})\xi^t\alpha_t] + e^{2p\alpha-\sigma}\bar{\varepsilon}(n-1)\xi^t\alpha_t\} + \\ +\{e^{(2p-1)\alpha}[\nabla_j A_i(g^{ij} - \xi^i\xi^j) + (4p+n-3)\alpha^t A_t - 2(2p-1)\xi^t\alpha_t\xi^p A_p - \\ -\bar{\varepsilon}(n-1)(e^{2\alpha-\sigma} - e^{\sigma})\xi^p A_p] + e^{2p\alpha-\sigma}\bar{\varepsilon}(n-1)\xi^p A_p\}.\end{gathered} \tag{3.5}$$

If we choose the constant $p = -\frac{n-3}{4}$ and take into account (1.4), the expression in the second bracket of (3.5) reduces to

$$e^{2(p-1)\alpha}[\nabla_j A_i(g^{ij} - \xi^i\xi^j) + \bar{\varepsilon}(n-1)\xi^t A_t],$$

and so, (3.5) becomes

$$\begin{gathered}({}^*g^{ji} - {}^*\xi^j{}^*\xi^i)^*\nabla_j{}^*A_i + \bar{\varepsilon}(n-1){}^*A_p{}^*\xi^p = \\ = 2pA\{e^{2(p-1)\alpha}[\nabla_j\alpha_i(g^{ji} - \xi^j\xi^i) + \tfrac{n-3}{2}\alpha_t\alpha^t + \tfrac{n+1}{2}(\xi^t\alpha_t)^2 - \\ -\bar{\varepsilon}(n-1)(e^{2\alpha-\sigma} - e^{\sigma})\xi^t\alpha_t] + \bar{\varepsilon}(n-1)e^{2p\alpha-\sigma}\xi^t\alpha_t\} + \\ +e^{2(p-1)\alpha}[\nabla_j A_i(g^{ij} - \xi^j\xi^i) + \bar{\varepsilon}(n-1)\xi^t A_t],\end{gathered}$$

from which we find that (3.2) follows from (3.1) if and only if

$$e^{2(p-1)\alpha}[\nabla_j\alpha_i(g^{ji}-\xi^j\xi^i)+\tfrac{n-3}{2}\alpha_t\alpha^t]+ \\ +\tfrac{n+1}{2}e^{2(p-1)\alpha}(\xi^t\alpha_t)^2-\bar{\varepsilon}(n-1)e^{2(p-1)\alpha}(e^{2\alpha-\sigma}-e^{\sigma})\xi^t\alpha_t+ \\ +\bar{\varepsilon}(n-1)e^{2p\alpha-\sigma}\xi^t\alpha_t=0.$$

In view of (1.4), the last condition can be written as follows

$$e^{(2p-1)\alpha}[\nabla_j\alpha_i(g^{ji}-\xi^j\xi^i)+\tfrac{n-3}{2}\alpha_t\alpha^t]+ \\ +\tfrac{n+1}{2}e^{2(p-1)\alpha}(1-e^{\sigma})^2-(n-1)(e^{2p\alpha-\sigma}-e^{2(p-1)\alpha+\sigma)})(1-e^{\sigma})+ \\ +(n-1)e^{2p\alpha-\sigma}(1-e^{\sigma})=0,$$

from which we get (2.12). Thus, we have

Theorem. *Let A be a function in M and let $^*A=e^{-\frac{n-3}{2}\alpha}A$. Then the conditions* (3.1) *and* (3.2) *are equivalent if and only if the D-conformal change* (1.3) *with $\varepsilon=+1$ is D-conharmonic.*

4. D-conharmonic curvature tensor

Let us denote by $^*R_{kji}{}^h$, $^*R_{ji}$ and $^*R$ the Riemannian curvature tensor, the Ricci tensor and the scalar curvature of metric $^*g$ respectively. Then we have ([1], (3.18) and (3.19)):

$$(4.1)\qquad {}^*R_{kji}{}^h-{}^*g_{ki}\delta^h_j+{}^*g_{ji}\delta^h_k=R_{kji}{}^h-g_{ki}\delta^h_j+g_{ji}\delta^h_k+ \\ +\alpha_{ki}(\delta^h_j-\eta_j\xi^h)-\alpha_{ji}(\delta^h_k-\eta_k\xi^h)+(g_{ki}-\eta_k\eta_i)\alpha^h_j-(g_{ji}-\eta_j\eta_i)\alpha^h_k,$$

$$(4.2)\qquad {}^*R_{ji}+(n-1){}^*g_{ji}=R_{ji}+(n-1)g_{ji}-(n-3)\alpha_{ji}-\alpha_t{}^t(g_{ji}-\eta_j\eta_i),$$

where

$$(4.3)\qquad \alpha_{ji}=\nabla_j\alpha_i-\alpha_j\alpha_i-\bar{\varepsilon}e^{\sigma}(\alpha_j\eta_i+\alpha_i\eta_j)+ \\ +\tfrac{1}{2}(\alpha_p\alpha^p-e^{2\sigma}+1)(g_{ji}-\eta_j\eta_i)+(\bar{\varepsilon}e^{\sigma}\sigma_p\xi^p-e^{2\sigma}+1)\eta_j\eta_i.$$

From (4.2) we get

$$(4.4)\qquad \alpha_t{}^t=\frac{1}{2(n-2)}\{R+n(n-1)-e^{2\sigma}[{}^*R+n(n-1)]\}.$$

But from (4.3) we find

$$\alpha_t{}^t=\alpha_{ji}g^{ji}=g^{ji}(\nabla_j\alpha_i-\alpha_i\alpha_j)-2\bar{\varepsilon}e^{\sigma}\xi^i\alpha_i+ \\ +\tfrac{1}{2}(n-1)(\alpha_p\alpha^p-e^{2\sigma}+1)+(\bar{\varepsilon}e^{\sigma}\sigma_p\xi^p-e^{2\sigma}+1),$$

or, using the second equation (1.4),

$$\alpha_t{}^t = g^{ji}(\nabla_j\alpha_i - \alpha_j\alpha_i) + (e^\sigma - 1)^2 + \tfrac{1}{2}(n-1)(\alpha_p\alpha^p - e^{2\sigma} + 1) + \bar{\varepsilon}e^\sigma\sigma_p\xi^p.$$

Thus, for a D-conharmonic change, according to (2.11) we have $\alpha_t{}^t = 0$, and so by (4.2)

$$\alpha_{ji} = \tfrac{1}{n-3}[R_{ji} + (n-1)g_{ji}] - \tfrac{1}{n-3}[{}^*R_{ji} + (n-1){}^*g_{ji}].$$

Substituting this into (4.1) we find

$$\begin{aligned}
&{}^*R_{kji}{}^h - {}^*g_{ki}\delta_j^h + {}^*g_{ji}\delta_k^h + \tfrac{1}{n-3}\{[{}^*R_{ki} + (n-1){}^*g_{ki}](\delta_j^h - {}^*\eta_j{}^*\xi^h) - \\
&\quad -[{}^*R_{ji} + (n-1){}^*g_{ji}](\delta_k^h - {}^*\eta_k{}^*\xi^h) + \\
&+({}^*g_{ki} - {}^*\eta_k{}^*\eta_i)[{}^*R_j^h + (n-1)\delta_j^h] - ({}^*g_{ji} - {}^*\eta_j{}^*\eta_i)[{}^*R_k^h + (n-1)\delta_k^h]\} = \\
&\quad = R_{kji}{}^h - g_{ki}\delta_j^h + g_{ji}\delta_k^h + \\
&+\tfrac{1}{n-3}\{[R_{ki} + (n-1)g_{ki}](\delta_j^h - \eta_j\xi^h) - [R_{ji} + (n-1)g_{ji}](\delta_k^h - \eta_k\xi^h) + \\
&\quad +(g_{ki} - \eta_k\eta_i)[R^h{}_j + (n-1)\delta_j^h] - (g_{ji} - \eta_j\eta_i)[R^h{}_k + (n-1)\delta_k^h].
\end{aligned}$$

Let us put

$$\begin{aligned}
&E_{kji}{}^h = R_{kji}{}^h - g_{ki}\delta_j^h + g_{ji}\delta_k^h + \\
&+\tfrac{1}{n-3}\{[R_{ki} + (n-1)g_{ki}](\delta_j^h - \eta_i\xi^h) - [R_{ji} + (n-1)g_{ji}](\delta_k^h - \eta_k\xi^h) + \\
&\quad +[R^h{}_j + (n-1)\delta_j^h](g_{ki} - \eta_k\eta_i) - [R^h{}_k + (n-1)\delta_k^h](g_{ji} - \eta_j\eta_i)\},
\end{aligned}$$

or

$$\begin{aligned}
(4.5)\qquad &E_{kji}{}^h = R_{kji}{}^h + \tfrac{n+1}{n-3}(g_{ki}\delta_j^h - g_{ji}\delta_k^h) - \\
&-\tfrac{n-1}{n-3}(g_{ki}\eta_j\xi^h - g_{ji}\eta_k\xi^h + \delta_j^h\eta_k\eta_i - \delta_k^h\eta_j\eta_i) + \\
&+\tfrac{1}{n-3}[R_{ki}(\delta_j^h - \eta_j\xi^h) - R_{ji}(\delta_k^h - \eta_k\xi^h) + R_j^h(g_{ki} - \eta_k\eta_i) - R_k^h(g_{ji} - \eta_j\eta_i)].
\end{aligned}$$

Then we have the

Theorem. *Let M be an n-dimensional special para-Sasakian manifold and $n \geq 4$. Then the tensor field $E_{kji}{}^h$ is invariant under a D-conharmonic change.*

The tensor field $E_{kji}{}^h$ is called *the D-conharmonic curvature tensor field* in a special para-Sasakian manifold $n > 3$.

Let us suppose that M is a manifold of constant curvature -1, i.e.

$$R_{kji}{}^h = g_{ki}\delta_j^h - g_{ji}\delta_k^h.$$

Then

$$R_{ji} = -(n-1)g_{ji}.$$

Substituting this into (4.5), we find $E_{kji}{}^h = 0$. Thus, if the special para-Sasakian manifold is a manifold of constant curvature -1, then its D-conharmonic curvature tensor vanishes.

Now, we shall investigate the reverse case. To do that, we note first that contracting (4.5) with respect to h and k, we find

$$E_{pji}{}^{p} = E_{ji} = -\tfrac{1}{n-3}[n(n-1)+R](g_{ji}-\eta_j\eta_i). \tag{4.6}$$

Using (4.5) and (4.6), we can express the D-conformal curvature tensor (1.7), as follows

$$\begin{aligned} B_{kji}{}^{h} &= E_{kji}{}^{h} + \tfrac{1}{n-2}(E_{ki}\delta_j^h - E_{ji}\delta_k^h)+ \\ &+\tfrac{1}{(n-2)(n-3)}[n(n-1)+R](g_{ki}\eta_j\xi^h - g_{ji}\eta_k\xi^h). \end{aligned} \tag{4.7}$$

Now, suppose that $E_{kji}{}^{h} = 0$. Then $E_{ji} = 0$ too, and from (4.6) we get

$$n(n-1)+R = 0. \tag{4.8}$$

Therefore, $B_{kji}{}^{h} = 0$, from which follows, according Theorem A, that M can be transformed into a manifold $^*M$ of constant curvature -1 by a D-conformal change (1.3). This change is conharmonic one. In fact, for $^*M$ we have

$$^*R_{kji}{}^{h} = {}^*g_{ki}\delta_j^h - {}^*g_{ji}\delta_k^h,$$

from which

$$^*R_{ji} = -(n-1)^*g_{ji} \qquad \text{and} \qquad ^*R = -n(n-1).$$

Substituting this and (4.8) into (4.4), we find $\alpha_t{}^t = 0$. But the D-conformal change (1.3) satisfying $\alpha_t{}^t = 0$ is also D-conharmonic. Thus, we have

Theorem. *A necessary and sufficient condition that an n-dimensional $(n > 4)$ special para-Sasakian manifold may be transformed into a manifold of constant curvature -1 by a suitable conharmonic transformation, is $E_{kji}{}^{h} = 0$.*

Let us denote by $K_{\omega\nu\mu}{}^{\lambda}$ and $K_{\nu\mu}$ the curvature tensor and Ricci tensor of N respectively. Between tensors of M and N, the following relations are known (cf. [3]):

$$B_k^\omega B_j^\nu B_i^\mu B_\lambda^h K_{\omega\nu\mu}{}^{\lambda} = R_{kji}{}^{h} - g_{ki}\delta_j^h + g_{ji}\delta_k^h, \tag{4.9}$$

$$B_j^\nu B_i^\mu K_{\nu\mu} = R_{ji} + (n-1)g_{ji}. \tag{4.10}$$

Also, in view of (2.2), (2.3) and (2.6), we have

$$g^{\mu\nu} = g^{ij}B_i^\mu B_j^\nu, \quad g_{\nu\mu}B_i^\nu B_j^\mu = g_{ij} - \eta_i\eta_j \tag{4.11}$$

and

$$K^{\lambda}{}_{\mu} B^{\mu}_{i} B^{h}_{\lambda} = R^{h}_{i} + (n-1)\delta^{h}_{i}. \tag{4.12}$$

Now, using (2.6), (4.9), (4.10), (4.11) and (4.12), it is easily proved that

$$Z_{\omega\nu\mu}{}^{\lambda} B^{\omega}_{k} B^{\nu}_{j} B^{\mu}_{i} B^{h}_{\lambda} = E_{kji}{}^{h}, \tag{4.13}$$

where $Z_{\omega\nu\mu}{}^{\lambda}$ is the conharmonic curvature tensor of level surface N, i.e. (cf. [6])

$$Z_{\omega\nu\mu}{}^{\lambda} = K_{\omega\nu\mu}{}^{\lambda} + \frac{1}{n-3}(g_{\omega\nu} K^{\lambda}{}_{\nu} - g_{\nu\mu} K^{\lambda}{}_{\omega} + \delta^{\lambda}_{\nu} K_{\omega\mu} - \delta^{\lambda}_{\omega} K_{\nu\mu}).$$

From (4.13), we have

Theorem. *The tensor field $E_{kji}{}^{h}$ of a special para-Sasakian manifold vanishes if and only if $Z_{\omega\nu\mu}{}^{\lambda} = 0$ in every level surface, i.e. if and only if each level surface is conharmonicly flat.*

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