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CHARACTERIZATION OF BINARY OPERATIONS ON THE UNIT INTERVAL SATISFYING THE GENERALIZED MODUS PONENS INFERENCE RULE

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Abstract: The generalized modus ponens inference rule is examined in a formal way and a characterization of the $[0,1]^2$-$[0,1]$ mappings, especially fuzzy implication operators, is given according to their behaviour with respect to the sup-triangular norm inference rule. The analogies between triangular norms and their dual triangular conorms on the one hand, and fuzzy implication operators on the other hand are described. Finally it is proven that a fuzzy implication operator that is an extension of the classical formula (NOT P) OR Q in the sense that the negation is replaced by a strict complement operator and the disjunction by a triangular conorm, never satisfies the sup-minimum inference rule.

1. Introduction

The compositional rule of inference was introduced by L. A. Zadeh [17] as an extension of the classical reasoning scheme "modus ponens". Its main purpose is to infer a possibility distribution, given a relationship between two linguistic variables modelled as possibility distributions on their respective universes of discourse and a possibility distribution which represents the vague knowledge about the matching of the antecedent. This inference rule can be represented as

$$\begin{array}{l} x \text{ is } A \Rightarrow y \text{ is } B \\ x \text{ is } A' \\ \hline \qquad\qquad y \text{ is } B'. \end{array}$$

Here, A and A' are possibility distributions on a universe of discourse $\mathcal{U}$, B and the derived distribution B' are possibility distributions on a universe $\mathcal{V}$. The distribution B' is calculated as

$$B' : \mathcal{V} \to [0,1] : v \mapsto \sup_{u\in\mathcal{U}} \min(A'(u), \mathcal{F}(A(u), B(v))),$$

where $\mathcal{F}$ is a $[0,1]^2$–$[0,1]$ mapping representing the relationship between A and B. This generalized inference scheme is very powerful as it is able to deduce knowledge from incomplete and uncertain information [15,18]. As mentioned in [4] and [7] the formalism for deriving the distribution B' can be easily generalized by replacing the minimum operator by a general triangular norm T (Definition 1.2, [14]). When $\mathcal{F}$ is a fuzzy implication operator (Definition 1.1) the proposed inference scheme is an extension of the classical "modus ponens" inference scheme. In [7,8,9] extensive case studies are presented by Martin-Clouaire and Mizumoto where the minimum operator is replaced by a general t-norm and $\mathcal{F}$ is a fuzzy implication operator. The results of these studies and the idea of modus ponens generating functions by Trillas and Valverde [15] suggest a strong connection between the choice of the fuzzy implication operator used to model the linguistic rule and fuzzy relation "IF x is A THEN y is B" and the triangular norm in the generalized modus ponens inference rule or compositional rule of inference. With regard to other inference schemes like "modus tollens" and "syllogism" the same remark can be made. In the sequel a formal treatment of the properties of $[0,1]^2$–$[0,1]$ mappings and especially fuzzy implication operators, w.r.t. their relationship with the triangular norm in the sup-triangular norm generalized inference scheme is presented. The family of equations

$$(\forall y \in [0,1])(y = \sup_{x\in[0,1]} T(x, \mathcal{F}(x,y)))$$

is examined and the mapping $\mathcal{F}$ is characterized w.r.t. the triangular norm in the sup-triangular norm inference rule. In every example $\mathcal{F}$ is a fuzzy implication operator. This section is mainly concerned with definitions and notations. In section 2 some properties of $[0,1]^2$–$[0,1]$ mappings, based on properties of fuzzy implication operators are presented. In section 3, the restrictions for the modus ponens inference rule yield three possible classes of mappings. Section 4 deals with the properties of one of these classes. Section 5 deals with the similarities of triangular conorms and the properties of the fuzzy implication operator, and in section 6 it is proven that the classical formula (NOT P) OR Q cannot be generalized without loss of the sup-min modus ponens inference rule.

Definition 1.1. A $[0,1]^2$–$[0,1]$ mapping $\mathcal{I}$ satisfying the *boundary conditions*

$$\mathcal{I}(0,0) = \mathcal{I}(0,1) = \mathcal{I}(1,1) = 1 \text{ and } \mathcal{I}(1,0) = 0$$

is a *fuzzy implication operator.*

Definition 1.1 of a fuzzy implication operator is weaker than Weber's definition [16]. These conditions are the weakest that can be imposed: a fuzzy implication operator is a $[0,1]^2$–$[0,1]$ mapping that is an extension of the material, binary implication operator.

Definition 1.2. [14]. A $[0,1]^2$–$[0,1]$ mapping T satisfying

(T1) *boundary conditions:*

$$(\forall x \in [0,1])(T(1,x) = x),$$

(T2) *symmetry property:*

$$(\forall (x,y) \in [0,1]^2)(T(x,y) = T(y,x)),$$

(T3) *associative property:*

$$(\forall (x,y,z) \in [0,1]^3)(T(x,T(y,z)) = T(T(x,y),z)),$$

(T4) *monotonicity:*

$$(\forall (x,y) \in [0,1]^2)(\forall (x',y') \in [0,1]^2) \\ ((x \le x') \wedge (y \le y') \Rightarrow T(x,y) \le T(x',y')),$$

is a *triangular norm* (shortly *t-norm*).

Definition 1.3. A $[0,1]^2$–$[0,1]$ mapping S satisfying

(T1') *boundary conditions:*

$$(\forall x \in [0,1])(S(0,x) = x)$$

and (T2)–(T4) is a *triangular conorm* (shortly *t-conorm*).

Definition 1.4. A $[0,1]$–$[0,1]$ mapping C that is strictly decreasing and involutive and satisfies $C(0) = 1$ and $C(1) = 0$ is a *strict complement operator.*

Remark. The following two well-known properties can be easily proved [3]:

1. If T is a t-norm and C a strict complement operator then S_T^C is a t-conorm, where

$$S_T^C : [0,1]^2 \to [0,1] : (x,y) \mapsto C(T(C(x),C(y))).$$

2. If S is a t-conorm and C is a strict complement operator then T_S^C is a t-norm, where

$$T_S^C : [0,1]^2 \to [0,1] : (x,y) \mapsto C(S(C(x),C(y))).$$

Definition 1.5. Let T be a t-norm and $\mathcal{F}$ a $[0,1]^2$–$[0,1]$ mapping then $\mathcal{F}$ satisfies the *sup-T modus ponens inference rule* iff

$$(\forall y \in [0,1])(y = \sup_{x\in[0,1]} T(x, \mathcal{F}(x,y)))$$

This definition is a natural extension of the sup-min modus ponens inference rule.

Definition 1.6. Let f and g be two $[0,1]^2 - [0,1]$ mappings then $f \leq g$ iff $(\forall (x,y) \in [0,1]^2)(f(x,y) \leq g(x,y))$.

2. Potential properties of the fuzzy implication operator

Let $\mathcal{I}$ be a fuzzy implication operator and C a strict complement operator. The following potential properties for $\mathcal{I}$ are defined [6].

Definition 2.1. $\mathcal{I}$ satisfies the *contrapositive symmetry* iff

$$(\forall (x,y) \in [0,1]^2)(\mathcal{I}(x,y) = \mathcal{I}(C(y), C(x))).$$

Definition 2.2. $\mathcal{I}$ satisfies the *exchange principle* iff

$$(\forall (x,y,z) \in [0,1]^3)(\mathcal{I}(x, \mathcal{I}(y,z)) = \mathcal{I}(y, \mathcal{I}(x,z)))).$$

Definition 2.3. $\mathcal{I}$ is *hybrid monotonous* [2,6] iff

$$(\forall (x,y) \in [0,1]^2)\ (\forall (x',y') \in [0,1]^2)$$
$$((x \leq x') \wedge (y \geq y') \Rightarrow \mathcal{I}(x,y) \geq \mathcal{I}(x',y')).$$

Although the definition of the hybrid monotonicity of $\mathcal{I}$ seems rather strange, it satisfies the intuitive idea that the less the antecedent is true and the more the consequence is true, the more the implication should be true. The following property is easily verified.

Property 2.1. *If $\mathcal{I}$ is a hybrid monotonous fuzzy implication operator then*

$$(\forall x \in [0,1])(\mathcal{I}(0,x) = 1).$$

Definition 2.4. $\mathcal{I}$ satisfies the *neutrality principle* iff

$$(\forall x \in [0,1])(\mathcal{I}(1,x) = x).$$

Remark. Obviously these definitions can be extended to general $[0,1]^2$–$[0,1]$ mappings.

3. Natural restrictions for the generalized modus ponens inference rule

In this section some natural restrictions on the mapping

$$\sup T : [0,1] \to [0,1] : y \mapsto \sup_{x\in[0,1]} T(x, \mathcal{F}(x,y))$$

are introduced, where T, $\mathcal{F}$ respectively, is a t-norm, a $[0,1]^2$–$[0,1]$ mapping respectively. These restrictions generate three disjoint classes of mappings that are examined to determine whether or not these mappings satisfy the generalized modus ponens.

Property 3.1. *If $\mathcal{F}$ is a $[0,1]^2$–$[0,1]$ mapping and T a t-norm then*

$$(\forall y \in [0,1])(\mathcal{F}(1,y) \leq \sup_{x\in[0,1]} T(x, \mathcal{F}(x,y)) \leq \sup_{x\in[0,1]} \min(x, \mathcal{F}(x,y))).$$

Proof. As $(\forall y \in [0,1])(\mathcal{F}(1,y) = T(1, \mathcal{F}(1,y)))$ and $T \leq \min$ for every t-norm T the result is immediately obtained. $\diamondsuit$

Considering Property 3.1 three disjoint classes of mappings can be defined:

Class I: $(\forall y \in [0,1])(\mathcal{F}(1,y) = y)$, i.e. $\mathcal{F}$ satisfies the neutrality principle,

Class II: $(\forall y \in [0,1])(\mathcal{F}(1,y) \leq y)$ and $(\exists y_0 \in [0,1])(\mathcal{F}(1,y_0) < y_0)$,

Class III: $(\exists y_0 \in [0,1])(\mathcal{F}(1,y_0) > y_0)$.

Remarks.

1. The three disjoint classes are a partition of the set of the $[0,1]^2$–$[0,1]$ mappings.
2. For the mappings of classes I and II the following property is easily proven:

Property 3.2. *Let T, T_1 and T_2 be t-norms. If $\mathcal{F}$ satisfies the sup-T_1 and the sup-T_2 modus ponens inference rule and if $T_1 \leq T \leq T_2$ then $\mathcal{F}$ satisfies the sup-T modus ponens inference rule.*

Proof.

$$(\forall (x,y) \in [0,1]^2)(T_1(x, \mathcal{F}(x,y)) \leq T(x, \mathcal{F}(x,y)) \leq T_2(x, \mathcal{F}(x,y)))$$

and thus

$$(\forall y \in [0,1])(\sup_{x\in[0,1]} T_1(x, \mathcal{F}(x,y)) \leq \sup_{x\in[0,1]} T(x, \mathcal{F}(x,y)) \leq$$

$$\leq \sup_{x\in[0,1]} T_2(x, \mathcal{F}(x,y))).\Diamond$$

3. From Property 3.1 it follows that the mappings of class III never satisfy the generalized modus ponens.

In section 4 some results on mappings satisfying the neutrality principle are presented.

4. $[0,1]^2$–$[0,1]$ mappings satisfying the neutrality principle

In this section T is an arbitrary t-norm and $\mathcal{F}$ a $[0,1]^2$–$[0,1]$ mapping that satisfies the neutrality principle.

Definition 4.1 ([4,16]):

$$\triangleright_T : [0,1]^2 \rightarrow [0,1] : (x,y) \mapsto \sup\{z \mid T(x,z) \leq y\}.$$

The following theorem is based on some theorems proved by Dubois and Prade [3,4] and deals with the existence of a maximal solution $\mathcal{F}$ of the family of equations

$$(\forall y \in [0,1])(y = \sup_{x\in[0,1]} T(x, \mathcal{F}(x,y)));$$

it gives a sufficient and necessary condition to determine whether or not $\mathcal{F}$, satisfying the neutrality principle, is a solution of the above family of equations.

Theorem 4.1. *Let T be a t-norm such that every partial mapping of T is infra-semicontinuous and $\mathcal{F}$ a $[0,1]^2$–$[0,1]$ mapping satisfying the neutrality principle. $\mathcal{F}$ satisfies the sup-T modus ponens inference rule iff $\mathcal{F} \leq \triangleright_T$.*

Proof.1. If $\mathcal{F}$ satisfies the sup-T modus ponens then $\mathcal{F} \leq \triangleright_T$ [4].

2. $\triangleright_T$ is a solution of the family of equations

$$(\forall y \in [0,1])(y = \sup_{x\in[0,1]} T(x, \mathcal{F}(x,y))).$$

Although this has already been proven for a continuous t-norm T this property holds for every t-norm which has infra-semicontinuous partial mappings. The proof is entirely based on the exchange of supremum and T. This property is proven in the Appendix.

3. It has already been proven that if $\mathcal{F}$ satisfies the generalized sup-T modus ponens then $\mathcal{F} \leq \triangleright_T$. The reverse implication is established now. Let $\mathcal{F}$ be a mapping that satisfies the condition $\mathcal{F} \leq \triangleright_T$ and T a t-norm with infra-semicontinuous partial mappings then

$$(\forall y \in [0,1])(\mathcal{F}(1,y) \leq \sup_{x\in[0,1]} T(x,\mathcal{F}(x,y)) \leq \sup_{x\in[0,1]} T(x, x \triangleright_T y))$$

or

$$(\forall y \in [0,1])(y \leq \sup_{x\in[0,1]} T(x,\mathcal{F}(x,y)) \leq y).\diamond$$

Remarks.

1. If the partial mappings of T are not infra-semicontinuous Theorem 4.1 cannot be generalized. Counterexample: let T be $\mathbb{Z}$ then

$$\triangleright_{\mathbb{Z}} : [0,1]^2 \to [0,1] : (x,y) \mapsto \begin{cases} 1 & ; \quad \forall(x,y) \in [0,1[\times[0,1] \\ y & ; \quad x = 1. \end{cases}$$

It is easily verified that $\sup_{x\in[0,1]} \mathbb{Z}(x, x \triangleright_{\mathbb{Z}} y_0) = 1$ or the sup-$\mathbb{Z}$ modus ponens inference rule does not hold when $\mathcal{F} = \triangleright_{\mathbb{Z}}$.

2. Consider for $a \in]0,1[$ the mapping

$$T_a : [0,1]^2 \to [0,1] : (x,y) \mapsto \begin{cases} 0 & ; \quad \text{if } \max(x,y) \leq a \\ \min(x,y) & ; \quad \text{elsewhere.} \end{cases}$$

Then every partial mapping of T_a is infra-semicontinuous and T_a is a t-norm. Hence, there exists t-norms that are not continuous and that have infra-semicontinuous partial mappings.

Corollary 4.1. *If T is an arbitrary t-norm and $\mathcal{F}$ satisfies the sup-T modus ponens then $\mathcal{F} \leq \triangleright_T$.*

Proof. This is an immediate consequence of the first part of the proof of Theorem 4.1. $\diamond$

Corollary 4.2. *If $\mathcal{F}$ is a $[0,1]^2$–$[0,1]$ mapping satisfying the neutrality principle and the sup-min modus ponens then $\mathcal{F}$ satisfies the sup-T modus ponens inference rule, where T is an arbitrary t-norm.*

Proof. Obvious considering Property 3.1. $\diamond$

Corollary 4.3. *Let T_1 and T_2 be arbitrary t-norms then T_2 satisfies the sup-T_1 modus ponens inference rule.*

Proof. For every t-norm T_2 the inequality $T_2 \leq \min \leq \triangleright_{\min}$ holds. As $(\forall x \in [0,1])(T_2(1,x) = x)$ considering Theorem 4.1 and Corollary 4.2, T_2 satisfies the sup-T_1 modus ponens. $\diamond$

Example 4.1. In the examples only $[0,1]^2$–$[0,1]$ mappings that are implication operators are considered. Let T be W (Lucasiewicz t-norm [14]) or explicitly

$$W : [0,1]^2 \to [0,1] : (x,y) \mapsto \max(0, x+y-1)$$

and $\mathcal{F}$ be the Kleene-Dienes implication operator[13]

$$\mathcal{F} : [0,1]^2 \to [0,1] : (x,y) \mapsto \max(1-x, y).$$

Considering Theorem 4.1 and the inequality $\mathcal{F} \leq \triangleright_W$, $\mathcal{F}$ satisfies the sup-W modus ponens inference rule.

Example 4.2. Let T be the well-known algebraic product $\times$, then $\triangleright_\times$ is the operator $G43$ of [1,5,13]:

$$\triangleright_\times : [0,1]^2 \to [0,1] : (x,y) \mapsto \begin{cases} 1 & ; \quad \text{if} \quad x \leq y \\ y/x & ; \quad \text{elsewhere.} \end{cases}$$

Consider as defined in $\mathcal{F}$ [10]

$$\mathcal{F} : [0,1]^2 \to [0,1] : (x,y) \mapsto \max(\min(x,y), 1-x).$$

Let $x_0 = 1/3$ and $y_0 = 1/8$ then $\mathcal{F}(x_0, y_0) = 2/3$ and $\triangleright_\times(x_0, y_0) = 3/8$. Considering Theorem 4.1 $\mathcal{F}$ does not satisfy the sup-$\times$ modus ponens inference rule.

Example 4.3. Let T be any t-norm, then define

$$T_\mathcal{I} : [0,1]^2 \to [0,1] : \begin{cases} (0,0) \mapsto 1 \\ (0,1) \mapsto 1 \\ (x,y) \mapsto T(x,y) & ; \quad \text{elsewhere.} \end{cases}$$

Obviously $T_\mathcal{I}$ is a fuzzy implication operator that satisfies every sup-T modus ponens inference rule, whatever T is (Theorem 4.1 and Corollary 4.2).

5. On the extension of the classical formula (NOT P) OR Q

The properties of the mappings that are extensions of the classical formula (NOT P) OR Q are examined. The negation is fuzzified by a strict complement operator C and the disjunction by a t-conorm S.

These extensions are the implication operators of type I of Dubois and Prade [3].

Definition 5.1. Let S be a t-conorm and C a strict complement operator. The mapping $\mathcal{I}_S^C$ is defined as

$$\mathcal{I}_S^C : [0,1]^2 \to [0,1] : (x,y) \mapsto S(C(x),y).$$

Obviously the mapping $\mathcal{I}_S^C$ is a fuzzy implication.

Properties 5.1.

1. $\mathcal{I}_S^C$ *satisfies the contrapositive symmetry,*
2. $\mathcal{I}_S^C$ *satisfies the exchange principle,*
3. $\mathcal{I}_S^C$ *is hybrid monotonous,*
4. $\mathcal{I}_S^C$ *satisfies the neutrality principle,*
5. *If S is continuous then $\mathcal{I}_S^C$ is continuous.*

Proof. Immediate from the symmetric and associative properties, the monotonicity and the boundary conditions of S and the involutive property of C. Property 5. is proven by the chain rule for continuous functions. $\diamondsuit$

Definition 5.2. Let $\mathcal{I}$ be a fuzzy implication operator and C a strict complement operator. The mapping $S_\mathcal{I}^C$ is defined by

$$S_\mathcal{I}^C : [0,1]^2 \to [0,1] : (x,y) \mapsto \mathcal{I}(C(x),y).$$

It is easily proven that $S_\mathcal{I}^C$ is an extension of the classical union operator; i.e. $S_\mathcal{I}^C \mid_{\{0,1\}^2}$ is the classical union operator $\cup$.

Properties 5.2.

1. *if $\mathcal{I}$ satisfies the contrapositive symmetry then $S_\mathcal{I}^C$ is symmetric,*
2. *if $\mathcal{I}$ satisfies the exchange principle and the contrapositive symmetry then $S_\mathcal{I}^C$ is associative,*
3. *if $\mathcal{I}$ is monotonous then $S_\mathcal{I}^C$ is increasing,*
4. *if $\mathcal{I}$ satisfies the neutrality principle then $S_\mathcal{I}^C$ satisfies the condition*

$$(\forall y \in [0,1])(S_\mathcal{I}^C(0,y) = y),$$

5. *if $\mathcal{I}$ is continuous then $S_\mathcal{I}^C$ is continuous.*

Proof. As an example 1. is proved. Let $(x,y) \in [0,1]^2$ then

$$S_\mathcal{I}^C(x,y) = \mathcal{I}(C(x),y)$$

and as C is involutive and $\mathcal{I}$ satisfies the contrapositive symmetry

$$\mathcal{I}(C(x),y) = \mathcal{I}(C(y),C(C(x))) = \mathcal{I}(C(y),x) = S_\mathcal{I}^C(y,x)\diamondsuit$$

Corollary 5.1. *Let S be a t-conorm and $\mathcal{I}$ a fuzzy implication operator then*

$$S^C_{\mathcal{I}^C_S} = S \quad ; \quad \mathcal{I}^C_{S^C_{\mathcal{I}}} = \mathcal{I}.$$

Corollary 5.2.

1. $S^C_{\mathcal{I}}$ *is a t-conorm iff $\mathcal{I}$ satisfies the contrapositive symmetry, the exchange principle and neutrality principle and $\mathcal{I}$ is hybrid monotonous;*
2. $S^C_{\mathcal{I}}$ *is continuous iff $\mathcal{I}$ is continuous.*

Proof. Straightforward taking into account Properties 5.1, 5.2 and Corollary 5.1. ♢

6. A special case : the sup-min modus ponens inference rule

In this section the special case of the sup-min modus ponens inference rule is considered. As minimum is a continuous mapping Theorem 4.1 assures us that if $\mathcal{F}$ satisfies the neutrality principle then $\mathcal{F}$ satisfies the sup-min modus ponens inference rule iff $\mathcal{F} \leq \triangleright_{\min}$ where $\triangleright_{\min}$ is the Gödel implication [4,5,7,13,16]]. The condition $\mathcal{F} \leq \triangleright_{\min}$ can be easily transformed into the formula of Theorem 6.1.

Theorem 6.1. *If $\mathcal{F}$ is $[0,1]^2$–$[0,1]$ mapping satisfying the neutrality principle then $\mathcal{F}$ satisfies the sup-min modus ponens inference rule iff*

$$(\forall(x,y) \in [0,1]^2)(x > y \Rightarrow \mathcal{F}(x,y) \leq y)$$

Proof. Obvious considering Theorem 4.1 and the definition of the Gödel implication. ♢

Example 6.1. Let $\mathcal{F}$ be $\triangleright_W$ then

$$\mathcal{F} : [0,1]^2 \to [0,1] : (x,y) \mapsto \min(1, 1-x+y).$$

Let $x_0 = 0.9$ and $y_0 = 0.8$ then $\mathcal{F}(x_0, y_0) > y_0$. Considering Theorem 6.1 the sup-min modus ponens inference rule does not hold for $\triangleright_W$.

Example 6.2. Let $\mathcal{F}$ be the Kleene-Dienes implication operator [13]:

$$\mathcal{F} : [0,1]^2 \to [0,1] : (x,y) \mapsto \max(1-x, y).$$

Let $x_0 = 0.5$ and $y_0 = 0.3$ then $\mathcal{F}(x_0, y_0) > y_0$. Considering Theorem

6.1 the sup-min modus ponens inference rule does not hold for this implication operator.

Theorem 6.2. *If $\mathcal{F}$ is a $[0,1]^2$–$[0,1]$ mapping satisfying*

$$(\forall y \in [0,1])(\mathcal{F}(1,y) \leq y) \text{ and } (\exists y_0 \in [0,1])(\mathcal{F}(1,y_0) < y_0)$$

then the sup-min modus ponens inference rule holds iff

$$(\forall (x,y) \in [0,1]^2)((x > y \Rightarrow \mathcal{F}(x,y) \leq y) \textit{ and }$$
$$(\sup_{0\leq x\leq y} \min(x,\mathcal{F}(x,y)) = y \textit{ or } \sup_{y<x\leq 1} \min(x,\mathcal{F}(x,y)) = y)).$$

Proof. Considering Corollary 4.1, the inequality

$$(\forall (x,y) \in [0,1]^2)(x > y \Rightarrow \mathcal{F}(x,y) \leq y)$$

is immediately obtained as $\mathcal{F} \leq \triangleright_{\min}$ should be satisfied. The second part of the conjunction is proven as follows. Suppose

$$(\exists y_0 \in [0,1])(\sup_{0\leq x\leq y_0} \min(x,\mathcal{F}(x,y_0)) \neq y_0 \qquad \text{and}$$
$$\sup_{y_0<x\leq 1} \min(x,\mathcal{F}(x,y_0)) \neq y_0)$$

then

$$\sup_{0\leq x\leq 1} \min(x,\mathcal{F}(x,y_0)) = \max(\sup_{0\leq x\leq y_0} \min(x,\mathcal{F}(x,y_0)),$$
$$\sup_{y_0<x\leq 1} \min(x,\mathcal{F}(x,y_0))) \neq y_0$$

or the sup-min modus ponens inference rule does not hold, which is clearly a contradiction.

The reverse implication is obtained in a similar way. ◊

Example 6.3. Let $\mathcal{F}$ be $\mathcal{I}_\nabla$ of [9] or explicitly

$$\mathcal{I}_\nabla : [0,1]^2 \to [0,1[: (x,y) \mapsto \begin{cases} \min\left(1, \frac{y}{x}, \frac{1-x}{1-y}\right) & ; \forall (x,y) \in]0,1] \times [0,1] \\ 1 & ; \text{elsewhere.} \end{cases}$$

If $x_0 = 0.5$ and $y_0 = 0.5$ then $\mathcal{I}_\nabla(x_0,y_0) = 1 > y_0$ so sup-min modus ponens inference rule does not hold for $\mathcal{I}_\nabla$.

Example 6.4. Let

$$\mathcal{F} : [0,1]^2 \to [0,1] : (x,y) \mapsto \begin{cases} 1 & ; \quad x \leq y \\ \min(1-x,y) & ; \quad \text{elsewhere.} \end{cases}$$

By definition

$$(\forall(x,y) \in [0,1]^2)(x > y \Rightarrow \mathcal{F}(x,y) \leq y)$$

holds and $\sup\limits_{0\leq x\leq y} \min(x, \mathcal{F}(x,y)) = y$ and thus the sup-min modus ponens inference rule holds although $\mathcal{F}$ does not satisfy the neutrality principle.

Theorem 6.3. *Let S be a t-conorm and C a strict complement operator then $\mathcal{I}_S^C$ never satisfies the sup-min modus ponens inference rule.*

Proof.

1. As C is a continuous, strictly decreasing $[0,1]$–$[0,1]$ mapping and $C(0) = 1$, $C(1) = 0$,
$$(\exists p_0 \in]0,1[)(C(p_0) = p_0).$$
2. Let $y_0 < x_0 < p_0$, then $p_0 < C(x_0) < C(y_0)$ and hence
$$(\exists(x_0, y_0) \in [0,1]^2)(x_0 > y_0 \wedge C(x_0) > y_0).$$
3. S is a t-conorm, hence
$$S(C(x_0), y_0) \geq \max(C(x_0), y_0) = C(x_0).$$
4. Combining (2) and (3) yields
$$(\exists(x_0, y_0) \in [0,1]^2)(x_0 > y_0 \text{ and } \mathcal{I}_S^C(x_0, y_0) > y_0).$$

By Theorem 6.1, $\mathcal{I}_S^C$ does not satisfy the sup-min modus ponens inference rule. ◇

Corollary 6.1. *If $\mathcal{I}$ is a fuzzy implication operator satisfying the exchange principle, neutrality principle, contrapositive symmetry and which is hybrid monotonous then $\mathcal{I}$ does not satisfy the sup-min modus ponens inference rule.*

Proof. Immediate from Theorem 6.3 and Corollary 5.1. ◇

7. Conclusion

The interaction between $[0,1]^2$–$[0,1]$ mappings $\mathcal{F}$ and, as special cases the fuzzy implication operators, and the generalized sup-T modus ponens inference rule have been investigated. As indicated in the introduction there is a very strong connection between $\mathcal{F}$, in practice a fuzzy implication operator, and the t-norm T of the inference rule. Several authors already suggested this connection [2,3,7,15].

Weber [16] considers a $[0,1]^2$–$[0,1]$ mapping as a fuzzy implication operator iff there is some relationship with a t-norm or conorm and a strict complement operator and if the mapping is an extension of the binary, material implication. Theorem 6.3 proves that this relationship cannot be the fuzzification of the classical formula (NOT P) OR Q without loss of the sup-min modus ponens inference rule when negation and disjunction are fuzzified by a strict complement operator respectively a t-conorm.

Whenever T has infra-semicontinuous partial mappings, a mapping $\mathcal{F}$ that satisfies the neutrality principle and the inequality $\mathcal{F} \leq \triangleright_T$ satisfies the sup-T modus ponens inference rule : this is a remarkable fact since $\mathcal{F}$ should not even be an extension of the binary implication.

8. Appendix

Let F be a $[0,1] - [0,1]$ mapping then F is *infra-semicontinuous* iff

$$(\forall x_0 \in [0,1])(\forall \epsilon > 0)(\exists \delta > 0)(\forall x \in [0,1])(\mid x - x_0 \mid < \delta \Rightarrow \\ \Rightarrow F(x_0) - \epsilon < F(x)).$$

Theorem A1. *Let T be a t-norm, then T is completely distributive w.r.t. supremum iff every partial mapping of T is infra-semicontinuous.*

Proof.

1. The "if" part is established now for t-norms with infra-semicontinuous partial mappings. Let T be a t-norm with infra-semicontinuous partial mappings. First the inequality

$$\sup_{j \in J} T(x_j, y) \leq T(\sup_{j \in J} x_j, y) \quad ; \quad \forall y \in [0,1]$$

is proved, $(x_j)_{j \in J}$ being a family in $[0,1]$ and J an arbitrary index set. From $x_i \leq \sup_{j \in J} x_j$ and the non-decreasingness of $T(\cdot, y)$ it follows that

$$(\forall i \in J)(T(x_i, y) \leq T(\sup_{j \in J} x_j, y))$$

and hence

$$\sup_{i\in J} T(x_i, y) \le T(\sup_{j\in J} x_j, y).$$

To prove the equality, suppose $T(\sup_{j\in J} x_j, y) > \sup_{j\in J} T(x_j, y)$ and let

$$\varepsilon_0 = T(\sup_{j\in J} x_j, y) - \sup_{j\in J} T(x_j, y) > 0.$$

The infra-semicontinuity of $T(\cdot, y)$ in $\sup_{j\in J} x_j$ implies for $\varepsilon = \varepsilon_0$

$$(\exists \delta_0 > 0)(\forall x \in [0,1])(\sup_{j\in J} x_j - \delta_0 < x < \sup_{j\in J} x_j + \delta_0 \Rightarrow$$
$$\Rightarrow \sup_{j\in J} T(x_j, y) < T(x, y)).$$

From the characterization of supremum it is deduced that $\sup_{j\in J} x_j - \delta_0$ is no lower an upper bound for $(x_j)_{j\in J}$ and hence

$$(\exists i_0 \in J)(\sup_{j\in J} x_j - \delta_0 < x_{i_0} < \sup_{j\in J} x_j + \delta_0)$$

and so

$$\sup_{j\in J} T(x_j, y) < T(x_{i_0}, y),$$

a contradiction.

2. The reverse implication is proven now. Suppose the partial mapping $T(\cdot, y_0)$ is not infra-semicontinuous in x_0.

$$(\exists \varepsilon_0 > 0)(\forall \delta > 0)(\exists x \in [0,1])(\mid x - x_0 \mid < \delta \text{ and}$$
$$T(x, y_0) \le T(x_0, y_0) - \varepsilon_0).$$

Choose $\varepsilon_0 > 0$ and let $\delta_n = 1/n$; $\forall n \in \mathbb{N}^*$. From the previous formula it follows that for each δ_n there exists an x_n satisfying the condition $\mid x_n - x_0 \mid < 1/n$ and $T(x_n, y_0) \le T(x_0, y_0) - \varepsilon_0$. Obviously $\lim_{n\to\infty} x_n =$ $= x_0$. From the monotonicity of T it is deduced that

$$(\forall n \in \mathbb{N}^*)(x_n \le x_0)$$

and hence

$$\lim_{n\to\infty} x_n = \sup_{n\in\mathbb{N}^*} x_n = x_0$$

and

$$(\forall n \in \mathbb{N}^*)(T(x_n, y_0) \le T(x_0, y_0) - \varepsilon_0)$$

or

$$\sup_{n\in\mathbb{N}^*} T(x_n, y_0) \leq T(x_0, y_0) - \varepsilon_0 < T(x_0, y_0).$$

Thus $\sup_{n\in\mathbb{N}^*} T(x_n, y_0) \neq T(\sup_{n\in\mathbb{N}^*} x_n, y_0)$. ◇

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