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A NOTE ON CYCLE DOUBLE COVERS IN CAYLEY GRAPHS

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Abstract: We show that every finite Cayley graph of degree at least two has a cycle double cover.

All graphs in this note are finite. Lovász ([4], Problem 11) conjectured that every connected vertex-transitive graph has a Hamilton path. One well-studied class of vertex-transitive graphs is the class of Cayley graphs. Tarsi [5] and Goddyn [1] proved that every 2-edge-connected graph with a Hamilton path has a cycle double cover. Combining these ideas led the authors to consider cycle double covers in Cayley graphs.

We prove that every Cayley graph of degree at least two has a cycle double cover.

Lemma 1. *Let G be a graph whose edge set can be decomposed into a collection of* 1*-factors and* 2*-factors, where the number of* 1*-factors is not* 1*. Then G has a cycle double cover.*

Proof. This is virtually trivial. We have a decomposition of the edge set of G into a set of cycles $\mathcal{C}$ and a set of 1-factors $F_1, F_2, \ldots, F_k$. Each of the graphs induced by $F_i \cup F_{i+1}$, $i = 1, \ldots, k$ (addition modulo n) is a 2-factor $\mathcal{F}_i$ of G. The collection of cycles in the $\mathcal{F}_i$ together with two copies of each of the cycles in $\mathcal{C}$, constitutes a cycle double cover of G. ◊

Now suppose that Γ is a group, and $S \subseteq \Gamma$, where

(i) $id \notin S$,

(ii) $v \in S \Rightarrow v^{-1} \in S$, and

(iii) S generates Γ

We write Cay(Γ, S) for the Cayley graph of Γ with respect to this set of generators. Cay(Γ, S) has vertex set Γ, and its edge set is $\{(g, gs) : g \in \Gamma, s \in S\}$.

Theorem. *If $G =$ Cay(Γ, S), with $|S| \geq 2$, then G has a cycle double cover.*

Proof. Each element g of S corresponds to a set of edges E_g of G. If order$(g) = 2$, then E_g is a 1-factor of G. If order$(g) \neq 2$, then E_g is a 2-factor of G.

Case 1. If every element of S has order greater than two, or if at least two elements of S have order two, then Lemma 1 applies.

Case 2. If order$(x) = 2$, for exactly one $x \in S$, then there is some $y \in S$ with order$(y) > 2$. The sets E_g, $g \in S' = S - \{x, y, y^{-1}\}$, correspond to 2-factors of G. We note that $G - \cup\{E_g : g \in S'\}$ may not be connected, and may consist of several disjoint copies of Cay$(<x, y, y^{-1}>, \{x, y, y^{-1}\})$. Thus, we need only consider the case in which $|S| = 3$. If Γ is abelian, then G has a Hamilton cycle (see Holsztyński and Strube [2]) and therefore a cycle double cover. Thus, we may assume that Γ is non-abelian.

Since $\{x, y\}$ generates Γ, $\{y, xy\}$ generates Γ. Suppose that $y(xy)^j = id$, for some j. Then $y = (xy)^{-j}$, and $\{xy\}$ generates Γ, contradicting the assumption that Γ is non-abelian. Therefore, we may assume that $y(xy)^j \neq id$, for any j. Similarly, we may assume that

$x(yx)^j \neq id$, for any j.

We now describe a cycle double cover of G. For $g \in \Gamma$, let

$$C_g = g_0 g_1 g_2 \cdots g_{2k-1} g_0,$$

denote the closed trail of G with vertices $g_{2i} = g(xy)^i$, $g_{2i+1} = g_{2i}x$, $0 \leq i < k = \text{order}(xy)$. We claim that C_g is, in fact, a cycle of G. If C_g is not a cycle, then $g_\alpha = g_\beta$, for some α, β, where, without loss of generality, $0 \leq \alpha < \beta < 2k$. We consider four cases depending on the parities of α and β.

(i) If α and β are both even, then $(xy)^{(\beta-\alpha)/2} = id$, which contradicts $\text{order}(xy) = k$.

(ii) If α and β are both odd, then $g_{\alpha+1} = g_{\beta+1}$, and case (i) applies.

(iii) If α is odd and β is even, then $y(xy)^{(\beta-\alpha-1)/2} = id$, contradicting the fact that $y(xy)^j \neq id$, for any j.

(iv) If α is even and β is odd, then $y(yx)^{(\beta-\alpha-1)/2} = id$, contradicting the fact that $x(yx)^j \neq id$, for any j.

Therefore, C_g must be a cycle. We note that the cycles C_g and C_{gxy} are really the same cycle, and thus that each cycle has k names.

Let g and h be adjacent vertices in $\text{Cay}(< x, y, y^{-1} >, \{x, y, y^{-1}\})$. Thus, $h = gx$ or $h = gy$ or $h = gy^{-1}$. If $h = gx$, the edge (g, h) is contained in the cycles C_g and C_h. If $h = gy$, the edge (g, h) is contained in the cycle C_{gx} and in E_y. If $h = gy^{-1}$, the edge (g, h) is contained in the cycle C_{hx} and in E_y. We also note that every vertex g is on exactly two of the cycles of the form C_w, specifically, C_g and C_{gx}.

Thus the set $\mathcal{C} = \{C_g : g \in \Gamma\} \cup E_y$ is a cycle double cover of G. ◇

We leave open the question of whether or not every vertex-transitive graph has a cycle double cover. There have been many papers establishing that certain classes of vertex-transitive graphs have Hamilton paths. We mention only one such result. Lipman [3] proves that every graph with a transitive nilpotent automorphism group, and every vertex-transitive graph on a prime power number of vertices must have a Hamilton path. Thus, by [1,5], each of these graphs has a cycle double cover. It is easy to see that if $|S|$ is even, then the Cayley graph has a cycle cover.

References

[1] GODDYN, L.A.: Cycle double covers of graphs with Hamilton paths, *Journal of Combinatorial Theory*, Series B **46** (1989), 253 – 254.

[2] HOLSZTYŃSKI, W. and STRUBE, R.F.E.: Paths and circuits in finite groups, *Discrete Mathematics* **22** (1978), 263 – 272.

[3] LIPMAN, M.J.: Hamilton cycles and paths in vertex-transitive graphs with abelian and nilpotent groups, *Discrete Mathematics* **54** (1985), 15 – 21.

[4] LOVÁSZ, L.: Combinatorial Structures and their Applications in: *(Proceedings of the Calgary International Conference, Calgary, Alberta, 1969)* Gordon and Breach, London (1970), 497.

[5] TARSI, M.: Semi-duality and the cycle double cover conjecture, *Journal of Combinatorial Theory,* Series B **41** (1986), 332 – 340.