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AN INITIAL AND BOUNDARY VALUE PROBLEM FOR NONLINEAR COMPOSITE TYPE SYSTEMS OF THREE EQUATIONS*

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Abstract: Boundary value problems for systems of composite type were investigated by A. Dzhuraev, see [1]. Using the theory of singular integral equations in [1] linear problems for linear systems of three and of four equa-

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tions having one and two real characteristics, respectively are treated. Here a nonlinear problem for a nonlinear system of three equations is studied by utilizing a method from the theory of elliptic systems (see e.g. [3],[4]) based on Schauder imbedding. The case of three equations is important in particular because every elliptic second order equation in two independent variables may be reduced to a first order composite type system of three equations.

1. Formulation of the initial and boundary value problem

In this paper, we consider the nonlinear system of first order composite type equations

$$\text{(1.1)} \qquad \begin{cases} \omega_{\bar{z}} = F(z,\omega,\omega_z,s), \\ F = Q_1\omega_z + Q_2\overline{\omega_{\bar{z}}} + A_1\omega + A_2\overline{\omega} + A_3 s + A_4, \end{cases}$$

$$\text{(1.2)} \qquad \begin{cases} s_y = G(z,\omega,s), \\ G = B_1\omega + B_2\overline{\omega} + B_3 s + B_4, \end{cases}$$

in a bounded simply connected domain D, where

$$\begin{aligned} &Q_j = Q_j(z,\omega,\omega_z,s), \qquad j=1,2, \quad A_j = A_j(z,\omega,s), \\ &B_j = B_j(z,\omega,s), \qquad j=1,\dots,4, \end{aligned}$$

and $\omega(z), Q_j, A_j, B_j (j=1,2), A_4$ are complex valued functions, $B_2 = \overline{B}_1$, $s(z)$, A_3, $B_j(j=3,4)$ are real valued functions. For the sake of convenience, we may assume that D is the unit disk and the lower boundary of D is $\gamma = \{|z|=1,\, y\le 0\}$. We suppose that system (1.1) and (1.2) satisfy the following condition.

Condition C

(1) $Q_j(z,\omega,U,s), j=1,2,\ A_j(z,\omega,s),\ j=1,\dots,4$ are measurable in $z\in D$ for all continuous functions $\omega(z)$, $s(z)$ and all measurable functions $U(z)$ on $\overline{D}$, satisfying

$$\text{(1.3)} \qquad \begin{aligned} &L_p[A_j(z,\omega(z),s(z)),\overline{D}] \le k_0 < \infty, \quad j=1,2,4, \\ &L_p[A_3(z,\omega(z),s(z)),\overline{D}] \le \varepsilon, \end{aligned}$$

where $p(>2)$, $k_0(>0)$ and $\varepsilon(>0)$ are positive constants.

(2) The above mentioned functions are continuous in $\omega \in \mathbb{C}$ (the complex plane) and $s \in \mathbb{R}$ (the real axis) for almost every point $z \in D$ and $U \in \mathbb{C}$.

(3) The complex equation (1.1) satisfies the uniform ellipticity condition

$$|F(z,\omega,U_1,s) - F(z,\omega,U_2,s)| \le q_0|U_1 - U_2|, \tag{1.4}$$

for almost every point $z \in D$ and $\omega, U_1, U_2 \in \mathbb{C}$, $s \in \mathbb{R}$, in which $q_0(<1)$ is a non-negative constant.

(4) $B_j(z,\omega,s)$ $(j = 1,\dots,4)$, $G(z,\omega,s)$ are continuous for $z \in \overline{D}$ for all Hölder continuous functions $\omega_j(z), s_j(z) \in C_\beta(\overline{D})$ $(j = 1,2)$ satisfying

$$\begin{cases} C_\beta[B_j(z,\omega_1,s_1),\overline{D}] & \le\ k_0 < \infty, \quad j = 1,\dots,4, \\ G(z,\omega_1,s_1) - G(z,\omega_2,s_2) & =\ B_1^*(\omega_1-\omega_2) + B_2^*(\overline{\omega_1-\omega_2}) + \\ & \quad + B_3^*(s_1 - s_2), \end{cases} \tag{1.5}$$

in which $C_\beta[B_j^*,\overline{D}] \le k_0, \beta\,(0<\beta<1)$ is real, for $j = 1,2,3$.

For system (1.1) and (1.2) we discuss the following nonlinear initial and boundary value problem.

Problem A

$$\mathrm{Re}[\overline{\lambda(t)}\omega(t)] = P(t,w,s)t, \in \Gamma = \partial D, \tag{1.6}$$

$$a(t)s(t) = Q(t,\omega,s), t \in \gamma. \tag{1.7}$$

Here $\lambda(t), P(t,\omega,s)$ are Hölder continuous functions, $|\lambda(t)| = 1$, and $\lambda(t), P_0(t) = P(t,0,0), P\,(t,\omega,s)$ satisfy

$$\begin{cases} C_\alpha[\lambda\,[t\,(\zeta)],L] \ \le\ k_0, C_\alpha\,[P_0\,[t\,(\zeta)],L] \ \le\ k_1, L = \zeta\,(\Gamma), \\ C_\alpha[P\,(t\,(\zeta),\omega_1,s_1) - P\,(t\,(\zeta),\omega_2,s_2),L] \le \\ \le \varepsilon\{C_\alpha[\omega_1-\omega_2,L] + C_\alpha[s_1-s_2,\ell]\}, \ell = \zeta(\gamma), \end{cases} \tag{1.8}$$

for all $\omega_j[t\,(\zeta)] \in C_\alpha\,(L), s_j\,(t) \in C_\alpha(\ell)$, $j = 1,2$, where $\zeta\,(z)$ is the homeomorphic solution to the Beltrami equation $\zeta_{\overline{z}} = q\,(z)\zeta_z$ with a proper $q\,(|q\,(z)| \le q_0 < 1)$ which maps D onto the unit disk H such that $\zeta\,(0) = 0$, $\zeta\,(1) = 1; z\,(\zeta)$ is the inverse function of $\zeta\,(z)$, k_1 and ε are positive constants. Moreover, $|a\,(t)| = 1$, $Q_0\,(t) = Q\,(t,0,0)$ and $Q\,(t,\omega,s)$ satisfy

$$(1.9)\qquad \begin{cases} C_\beta[Q_0(t),\gamma] \le k_2, \\ C_\beta[Q(t,\omega_1,s_1) - Q(t,\omega_2,s_2),\gamma] \le \\ \le k_2 C_\beta(\omega_1-\omega_2,\gamma) + \varepsilon C_\beta(s_1-s_2,\gamma), \end{cases}$$

in which k_2 is a positive constant. Obviously Problem A is not necessarily solvable. Hence we consider the modified initial-boundary value problem (**Problem B**) where (1.6) is replaced by

$$(1.10)\qquad \mathrm{Re}\,[\overline{\lambda(t)}\,\omega(t)] = P(t,\omega,s) + h(t),\ t\in\Gamma,$$

with

$$(1.11)\qquad h(t) = \begin{cases} 0, t\in\Gamma, \text{ if } K\ge 0, \quad K = \frac{1}{2\pi}\Delta_\Gamma \arg\lambda(t), \\ h_0 + \mathrm{Re}\sum_{m=1}^{-K-1}(h_m^+ + ih_m^-)\,t^m,\ t\in\Gamma, \text{ if } K<0, \end{cases}$$

where $h_0, h_m^\pm$ $(m = 1,\dots,-K-1)$ are unknown real constants to be determined appropriately. If $K\ge 0$, we assume that the solution $\omega(z)$ to Problem A satisfies the side conditions

$$(1.12)\qquad \mathrm{Im}\,[\overline{\lambda(a_j)}\omega(a_j)] = b_j,\quad j=1,\dots,2K+1,$$

where a_j $(j=1,\dots,2K+1)$ are distinct points on Γ, and $b_j (j = 1,\dots,2K+1)$ are real constants with the condition $|b_j|\le k_1$.

In the following, we first give an a priori estimate of solutions to Problem B. Afterwards, we prove Problem B and Problem A to be solvable by using the *Schauder fixed-point theorem*. Under some more restrictions, we can discuss the uniqueness of the solution to Problem B.

2. A propri estimate of solutions to the initial and boundary value problem

First of all, we discuss the system of first order composite type equations

$$(2.1)\qquad \begin{cases} \omega_{\bar z} = F^*(z,\omega,\omega_z,s), \\ F^* = Q_1\omega_z + Q_2\overline{\omega_{\bar z}} + A_1\omega + A_2\overline{\omega} + A, \end{cases}$$

$$(2.2)\qquad \begin{cases} s_y = G^*(z,\omega,s), \\ G^* = B_3 s + B, \end{cases}$$

together with the following linear initial and boundary value problem.
Problem B*

$$\text{Re}[\overline{\lambda(t)}\omega(t)] = P_0(t) + h(t),\ t \in \Gamma, \tag{2.3}$$

$$Im[\overline{\lambda(a_j)}\omega(a_j)] = b_j,\ j = 1, \ldots, 2K+1,\ K \geq 0, \tag{2.4}$$

$$a(t)s(t) = Q_0(t),\ t \in \gamma, \tag{2.5}$$

where Q_j, A_j $(j = 1, 2)$, $B_3, \lambda, P_0, h, b_j, a, Q_0$ are defined as in 1, and $A = A(z, \omega, s)$, $B = B(z, \omega, s)$ are similar to A_4, B_4, but satisfsying the conditions

$$L_p[A, \overline{D}] \leq k_3, C_\beta[B, \overline{D}] \leq k_4, \tag{2.6}$$

for any $\omega(z), s(z) \in C_\beta(\overline{D})$, in which k_3, k_4 are non-negative constants.
Lemma 2.1. *If* $[\omega(z), s(z)]$ *is a solution to* Problem B* *for the system* (1.1), (2.2), *then* $[\omega(z), s(z)]$ *satisfies the estimates*

$$C_\beta[\omega, \overline{D}] \leq M_1(k_1 + k_3),\ L_{p_0}[|\omega_z| + |\omega_{\overline{z}}|, \overline{D}] \leq M_2(k_1 + k_3), \tag{2.7}$$

$$C^*_\beta[s, \overline{D}] := C_\beta[s, \overline{D}] + C[s_y, \overline{D}] \leq M_3(k_2 + k_4), \tag{2.8}$$

where $M_j = M_j(q_0, p_0, k_0, \alpha, k, K)$, $j = 1, 2, 3$, $k = (k_1, k_2, k_3, k_4)$, $\beta = \min(\alpha, 1 - \frac{2}{p_0})$, $p_0 = \min(p, \frac{1}{1-\alpha})$.
Proof. Substituting the solution $[\omega, s]$ to Problem B* into the complex system (2.1), (2.2), and assuming that $k' = \max(k_1, k_3) > 0$, $k'' = \max(k_2, k_4) > 0$, we put

$$W(z) = \frac{\omega(z)}{k'}, S(z) = \frac{s(z)}{k''}. \tag{2.9}$$

It is clear that $W(z)$ is a solution to the boundary value problem

$$W_{\overline{z}} = Q_1 W_z + Q_2 \overline{W}_{\overline{z}} + A_1 W + A_2 \overline{W} + \frac{A}{k'}, \tag{2.10}$$

$$\text{Re}[\overline{\lambda(t)}\, W(t)] = \frac{P_0(t) + h(t)}{k'},\ t \in \Gamma, \tag{2.11}$$

$$\text{Im}[\overline{\lambda(a_j)} W(a_j)] = \frac{b_j}{k'},,\quad j = 1, \ldots, 2K+1, K \geq 0. \tag{2.12}$$

Noting that

$$L_p\left[\frac{A}{k'}, \overline{D}\right] \leq 1,\ C_\alpha\left[\frac{P_0(t(\zeta))}{k'}, L\right] \leq 1,\quad |\tfrac{b_j}{k'}| \leq 1, \tag{2.13}$$

and according to Theorem 5.6 of Chapter 5 in [3] or Theorem 4.3 of Chapter 2 in [4], we know that $W(z)$ satisfies the estimate

$$C_\beta[W, \overline{D}] \leq M_1,\ L_{p_0}[|W_{\overline{z}}| + |W_z|, \overline{D}] \leq M_2. \tag{2.14}$$

Moreover, $S(z)$ is a solution to the initial value problem

$$S_y = B_3 S + \frac{B}{k''}, \tag{2.15}$$

$$a(t)S(t) = \frac{Q_0(t)}{k''},\ t \in \gamma, \tag{2.16}$$

where $C_\beta\left[\frac{B}{k''}, \overline{D}\right] \le 1$, $C_\beta\left[\frac{Q_0}{k''}, \gamma\right] \le 1$. On the basis of Theorem 2.4 in [2], $S(z)$ can be seen to satisfy the estimate

$$C^*_\beta\left[S, \overline{D}\right] \le M_3. \tag{2.17}$$

From (2.14), (2.17) it follows that (2.7), (2.8) for $k' > 0$, $k'' > 0$ are true. If $k' = 0$ or $k'' = 0$, then (2.7), (2.8) for $k' = \varepsilon > 0$ or $k'' = \varepsilon > 0$ hold. Letting ε tend to 0, we obtain (2.7), (2.8) for $k' = 0$ or $k'' = 0$. $\diamondsuit$

Theorem 2.2. *Let the complex system* (1.1) *and* (1.2) *satisfy* Condition C *and the constant* ε *in* (1.3), (1.8) *and* (1.9) *be small enough. Then the solution* $[\omega(z), s(z)]$ *to* Problem B *for* (1.1), (1.2) *satisfies the estimate*

$$U = C_\beta[\omega, \overline{D}] + L_{p_0}\left[|\omega_{\overline{z}}| + |\omega_z|, \overline{D}\right] \le M_4, \tag{2.18}$$

$$V = C^*_\beta\left[s, \overline{D}\right] \le M_5, \tag{2.19}$$

where $M_j = M_j(q_0, p_0, k_0, a, k, K)$, $j = 4, 5$.

Proof. Let the solution $[\omega(z), s(z)]$ be inserted into the complex system (1.1), (1.2), the boundary condition (1.10), the side condition (1.12) and the initial condition (1.7). We see that $A = A_3 s + A_4$, $B = B_1\omega + B_2\overline{\omega} + B_4$, $P(t, \omega, s)$, $Q(t, \omega, s)$, b_j satisfy

$$L_p[A, \overline{D}] \le \varepsilon C[s, \overline{D}] + L_p[A_4, \overline{D}] \le \varepsilon C[s, \overline{D}] + k_0, \tag{2.20}$$

$$C_\beta[B, \overline{D}] \le C_\beta[B_1\omega + B_2\overline{\omega}, \overline{D}] + C_\beta[B_4, \overline{D}] \le 2k_0 C_\beta[\omega, \overline{D}] + k_0, \tag{2.21}$$

$$\begin{aligned} C_\alpha[P, L] \le\ & C_\alpha[P_0(t(\zeta)), L] + C_\alpha[P[t(\zeta), \omega, s] - P_0[t(\zeta)], L] \le \\ \le\ & k_1 + \varepsilon\{C_\alpha[\omega, L] + C_\beta[s, \ell]\}, \end{aligned} \tag{2.22}$$

$$|b_j| \le k_1,\ j = 1, \ldots, 2K+1, K \ge 0, \tag{2.23}$$

$$\begin{aligned} C_\beta[Q, \gamma] &\le C_\beta[Q_0(t), \gamma] + k_0 C_\beta[\omega, \gamma] + \varepsilon C_\beta[s, \gamma] \le \\ &\le k_2 + k_2 C_\beta[\omega, \overline{D}] + \varepsilon C_\beta[s, \overline{D}]. \end{aligned} \tag{2.24}$$

Using (2.7) and (2.8) we have

$$U \le (M_1 + M_2)\{\varepsilon C[s, \overline{D}] + k_0 + k_1 + \varepsilon[C_\alpha(\omega, L) + C_\alpha[s, \ell]\} \le$$
$$(2.25) \le (M_1 + M_2)[k_0 + k_1 + \varepsilon C_\beta(\omega, \overline{D}) + \varepsilon C_\beta(s, \overline{D})] \le$$
$$\le (M_1 + M_2)(k_0 + k_1 + \varepsilon U + \varepsilon V),$$

$$(2.26) \quad \begin{aligned} V &\le M_3[2k_0 C_\beta(\omega, \overline{D}) + k_0 + k_2 + k_2 C_\beta(\omega \overline{D}) + \varepsilon C_\beta(s, \overline{D}] \le \\ &\le M_3[k_0 + k_2 + (2k_0 + k_2)U + \varepsilon V]. \end{aligned}$$

Choosing the constant ε so small that

$$(M_1 + M_2)\varepsilon \le \tfrac{1}{2}, \quad M_3[1 + 2(2k_0 + k_2)(M_1 + M_2)]\varepsilon \le \tfrac{1}{2},$$

one can show

$$(2.27) \qquad U \le \tfrac{(M_1+M_2)(k_0+k_1+\varepsilon V)}{1-(M_1+M_2)\varepsilon} \le 2(M_1 + M_2)(k_0 + k_1 + \varepsilon V),$$

$$V \le M_3[k_0 + k_2 + 2(2k_0 + k_2)(M_1 + M_2)(k_0 + k_1 + \varepsilon V) + \varepsilon V] \le$$
$$(2.28) \le \frac{M_3[k_0 + k_2 + 2(2k_0 + k_2)(k_0 + k_1)(M_1 + M_2)]}{1 - M_3[1 + 2(2k_0 + k_2)(M_1 + M_2)]\varepsilon} \le$$
$$\le 2M_3[k_0 + k_2 + 2(2k_0 + k_2)(k_0 + k_1)(M_1 + M_2)] = M_5,$$

$$(2.29) \qquad U \le 2(M_1 + M_2)(k_0 + k_1 + \varepsilon M_5) = M_4. \ \diamondsuit$$

3. Solvability of the initial and boundary value problem

First we prove the existence of solutions to `Problem B` for the system

$$(3.1) \quad \begin{cases} \omega_{\overline{z}} = F(z, \omega, \omega_z), \ F = Q_1\omega_z + Q_z\overline{\omega}_{\overline{z}} + A_1\omega + A_z\overline{\omega} + A_3, \\ Q_j = Q_j(z, \omega_z), \ j = 1, 2, A_j = A_j(z), \ j = 1, 2, 3, \end{cases}$$

and (1.2) by using the parameter extension method, and then verify the existence of solutions to `Problem B` for the system (1.1) and (1.2) by using Theorem 2.2 and the *Schauder fixed point theorem.* Finally, we give conditions for `Problem A` for (1.1), (1.2) to be solvable.

Theorem 3.1. *Let the system* (3.1), (1.2) *satisfy* `Condition C` *and the constant* ε *be small enough. Then* `Problem B` *for* (3.1), (1.2) *is solvable.*

Proof. We consider the following initial boundary value problem with parameter $t\,(0 \le t \le 1)$.

Problem B′

$$\omega_{\bar z} = tF(z,\omega,\omega_z) + A(z) \text{ in } D,\ A \in L_{p_0}(\overline{D}), \tag{3.2}$$

$$\operatorname{Re}[\overline{\lambda(z)}\,\omega(z)] = tP(z,\omega,s) + p(z) + h(z), \text{ on } \Gamma, p \in C_\beta(\Gamma), \tag{3.3}$$

$$\operatorname{Im}[\overline{\lambda(a_j)}\,\omega(a_j)] = b_j,\ j = 1,\dots,2K+1, K \ge 0, \tag{3.4}$$

$$s_y = tG(z,\omega,s) + B(z) \text{ in } D,\ B \in C_\beta(\overline{D}), \tag{3.5}$$

$$a(z)s(z) = tQ(z,\omega,s) + q(z) \text{ on } \gamma, q \in C_\beta(\gamma). \tag{3.6}$$

When $t = 0$, Problem B′ has a unique solution $[\omega(z), s(z)]$ with $\omega \in C_\beta(\overline{D})$, $s \in C^*_\beta(\overline{D})$ - see [2], [3] and [4].

Assuming that Problem B′ for $t_0\,(0 \le t_0 \le 1)$ is solvable, we will prove that there exists a positive constant δ such that Problem B′ on

$$E = \{t \mid |t - t_0| \le \delta, 0 \le t \le 1\} \tag{3.7}$$

for any $A \in L_{p_0}(\overline{D})$, $B \in C_\beta(\overline{D})$, $p \in C_\beta(\Gamma)$ and $q \in C_\beta(\gamma)$ has a unique solution $[\omega(z), s(z)]$, $\omega \in C_\beta(\overline{D}) \cap W^1_{p_0}(D)$, $s \in C^*_\beta(\overline{D})$.

We rewrite (3.2) – (3.6) as

$$\omega_{\bar z} - t_0F(z,\omega,\omega_z) = (t - t_0)F(z,\omega,\omega_z) + A(z), \tag{3.8}$$

$$\operatorname{Re}[\overline{\lambda(z)}\,\omega(z)] - t_0P(z,\omega,s) = (t - t_0)P(z,\omega,s) + p(z) + h(z), \tag{3.9}$$

$$\operatorname{Im}[\overline{\lambda(a_j)}\,\omega(a_j)] = b_j,\ j = 1,\dots,2K+1, K \ge 0, \tag{3.10}$$

$$s_y - t_0G(z,\omega,s) = (t - t_0)G(z,\omega,s) + B(z), \tag{3.11}$$

$$a(z)s(z) - t_0Q(z,\omega,s) = (t - t_0)Q(z,\omega,s) + q(z). \tag{3.12}$$

Choosing arbitrary functions $\omega_0 \in C_\beta(\overline{D}) \cap W^1_{p_0}(D)$, $s_0 \in C^*_\beta(\overline{D})$, for instance $\omega_0(z) \equiv 0$, $s_0(z) \equiv 0$, we substitute $\omega_0(z), s_0(z)$ into the corresponding positions of the right hand sides in (3.8) – (3.12). By assumption, for t_0 the initial-boundary value problem (3.8) – (3.12) has a unique solution $[\omega_1(z), s_1(z)]$, $\omega_1 \in C_\beta(\overline{D}) \cap W^1_{p_0}(D), s_1 \in C^*_\beta(\overline{D})$. Let us substitute $\omega_1(z), s_1(z)$ into the right hand sides of (3.8) – (3.12) and find unique solution $[\omega_2(z), s_2(z)], \omega_2 \in C_\beta(\overline{D}) \cap W^1_{p_0}(D)$, $s_2 \in C^*_\beta(\overline{D})$ to this system. Thus, we obtain $[\omega_n(z), s_n(z)], n = 1, 2, \dots$, satisfying

$$\omega_{n+1\bar z} - t_0F(z,\omega_{n+1},\omega_{n+1\,z}) = (t - t_0)F(z,\omega_n,\omega_{nz}) + A(z), \tag{3.13}$$

$$\text{(3.14)}\qquad \begin{aligned} &\text{Re}[\overline{\lambda}\,\omega_{n+1}] - t_0 P(z,\omega_{n+1},\, s_{n+1}) = \\ &= (t-t_0)\,P(z,\omega_n s_n) + p(z) + h(z), \end{aligned}$$

$$\text{(3.15)}\qquad \text{Im}[\overline{\lambda(a_j)}\,\omega_{n+1}(a_j)] = b_j,\ j=1,\ldots,2K+1, K\geq 0,$$

$$\text{(3.16)}\qquad s_{n+1y} - t_0 G(z,\omega_{n+1},s_{n+1}) = (t-t_0)\,G(z,\omega_n,s_n) + B(z),$$

$$\text{(3.17)}\qquad a(z)s_{n+1} - t_0 Q(z,\omega_{n+1},s_{n+1}) = (t-t_0)\,Q(z,\omega_n,s_n) + q(z).$$

Setting $W_{n+1} = \omega_{n+1} - \omega_n$, $S_{n+1} = s_{n+1} - s_n$ from (3.13) – (3.17), we have

$$\text{(3.18)}\qquad \begin{aligned} &W_{n+1\overline{z}} - t_0[F(z,W_{n+1},W_{n+1z}) - F(z,W_n,W_{nz})] = \\ &= (t-t_0)[F(z,W_n,W_{nz}) - F(z,W_{n-1},W_{n-1z})], \end{aligned}$$

$$\text{(3.19)}\qquad \begin{aligned} &\text{Re}\,[\overline{\lambda}\,W_{n+1}] - t_0[P(z,\omega_{n+1},s_{n+1}) - P(z,\omega_n,s_n)] = \\ &= (t-t_0)[P(z,\omega_n,s_n) - P(z,\omega_{n-1},s_{n-1})] + h(z), \end{aligned}$$

$$\text{(3.20)}\qquad Im\,[\overline{\lambda(a_j)}W_{n+1}(a_j)] = 0, j=1,\ldots,2K+1, K\geq 0,$$

$$\text{(3.21)}\qquad \begin{aligned} &S_{n+1y} - t_0[G(z,\omega_{n+1},s_{n+1}) - G(z,\omega_n,s_n)] = \\ &= (t-t_0)\,[G(z,\omega_n,s_n) - G(z,\omega_{n-1},s_{n-1}], \end{aligned}$$

$$\text{(3.22)}\qquad \begin{aligned} &a(z)S_{n+1} - t_0[Q(z,\omega_{n+1},s_{n+1}) - Q(z,\omega_n,s_n)] = \\ &= (t-t_0)\,[Q(z,\omega_n,s_n) - Q(z,\omega_{n-1},s_{n-1})]. \end{aligned}$$

By Condition C

$$\text{(3.23)}\qquad \begin{aligned} &L_{p_0}[F(z,W_n,W_{nz}) - F(z,W_{n-1},W_{n-1z}),\overline{D}] \leq \\ &\leq L_{p_0}[W_{nz},\overline{D}] + 2k_0 C_\beta\,[W_n,\overline{D}], \end{aligned}$$

$$\text{(3.24)}\qquad \begin{aligned} &C_\alpha\{P[z(\zeta),\omega_n(z(\zeta)),s_n(z(\zeta))] - \\ &-P[z(\zeta),\omega_{n-1}(z(\zeta)),s_{n-1}(z(\zeta))],L \leq \\ &\leq \varepsilon\,\{C_\alpha[W_n(z(\zeta)),L] + C_\alpha[S_n(z(\zeta)),\ell]\}, \end{aligned}$$

$$\text{(3.25)}\qquad \begin{aligned} &C_\beta[G(z,\omega_n,s_n) - G(z,\omega_{n-1},s_{n-1}),\overline{D}] \leq \\ &\leq 2k_0 C_\beta[W_n,\overline{D}] + k_0 C_\beta[S_n,\overline{D}], \end{aligned}$$

$$(3.26)\qquad \begin{aligned} C_\beta[Q(z,\omega_n,s_n)-Q(z,\omega_{n-1},s_{n-1}),\gamma] \le \\ \le k_2 C_\beta[W_n,\gamma]+\varepsilon C_\beta[S_n,\gamma] \end{aligned}$$

can be obtained.
According to the method in the proof of Theorem 2.2, we can conclude that

$$(3.27)\qquad \begin{aligned} U_{n+1} := C_\beta[W_{n+1},\overline{D}] + L_{p_0}[|W_{n+1\bar z}|+|W_{n+1\,z}|,\overline{D}] \le \\ \le |t-t_0|\, M_6 U_n, \end{aligned}$$

$$(3.28)\qquad V_{n+1} := C^*_\beta[S_{n+1},\overline{D}] \le |t-t_0|\, M_6 V_n,$$

where $M_6 = M_6(q_0,p_0,k_0,\alpha,k,K,\varepsilon) \ge 0$.
Choosing $\delta = \frac{1}{2(M_6+1)}$, then for $|t-t_0| \le \delta$, $0 \le t \le 1$, and $n > > N+1 > 1$, we can derive the inequality

$$U_{n+1} \le \tfrac{1}{2}U_n \le \tfrac{1}{2^N}U_1, V_{n+1} \le \tfrac{1}{2^N}V_1.$$

Moreover, if $n, m > N+1$, then

$$(3.30)\qquad \begin{aligned} C_\beta[\omega_n-\omega_m,\overline{D}] + L_{p_0}[|\omega_n-\omega_m)_{\bar z}|+|(\omega_n-\omega_m)_z|,\overline{D}] \le \\ \le \tfrac{1}{2^N}\textstyle\sum_{j=0}^{\infty}\tfrac{1}{2^j}U_1 = \tfrac{1}{2^{N-1}}U_1, \\ C^*_\beta[s_n-s_m,\overline{D}] \le \tfrac{1}{2^{N-1}}C^*_\beta[s_1,\overline{D}]. \end{aligned}$$

This shows that $C_\beta[\omega_n-\omega_m,\overline{D}] + L_{p_0}[|(\omega_n-\omega_m)_{\bar z}|+|(\omega_n-\omega_m)_z|, \overline{D}] \to 0$, $C^*_\beta[s_n-s_m,\overline{D}] \to 0$, if $n,m\to\infty$. Hence there exist $\omega_* \in \in C_\beta(\overline{D}) \cap W^1_{p_0}(D)$, $s_* \in C^*_\beta(\overline{D})$, such that $C_\beta[\omega_n-\omega_*,\overline{D}]+ +L_{p_0}\left[|(\omega_n-\omega_*)_{\bar z}|+|(\omega_n-\omega_*)_z|,\overline{D}\right] \to 0$, $C^*_\beta[s_n-s_*,\overline{D}] \to 0$, as $n \to \to \infty$, and $[\omega_n(z), s_n(z)]$ is just a solution to `Problem B'` on E for (3.2) – (3.6). Thus, we know that when $t = 0, 1, \dots, [\frac{1}{\delta}]\delta, 1$, Problem B′ for (3.2) – (3.6) is solvable. In particular, when $t=1$, $A=0$, $p=0$, $B=0$, $q=0$, `Problem B'` i. e. `Problem B` for (3.1), (1.2) is solvable. ◇

Theorem 3.2. *Under the same hypotheses as in Theorem 2.2,* `Problem B` *for* (1.1), (1.2) *has a solution.*

Proof. We introduce a bounded and closed convex set B_M in the Banach space $C(\overline{D})\times C(\overline{D})$, the elements of which are vectors of functions $w = [\omega, s]$ satisfying the condition

$$(3.31)\qquad C[\omega,\overline{D}] \le M_4, C[s,\overline{D}] \le M_5,$$

where M_4, M_5 are the constants stated in (2.18), (2.19). We choose an arbitrary vector of functions $\Omega = [W, S] \in B_M$ and insert $W(z), S(z)$ into the appropiate positions of the complex equation (1.1). Following Theorem 3.1, there exists a solution $[\omega(z), s(z)]$ to the initial boundary value `Problem B'`:

$$\begin{aligned} &\omega_{\bar z} = f(z, \omega, W, s, \omega_z),\\ &f = Q_1(z, W, \omega_z, s)\,\omega_z + Q_2(z, W, \omega_z, s)\,\overline{\omega_{\bar z}} + \\ &+ A_1(z, W, s)\,\omega + A_2(z, W, s)\,\overline{\omega} + A_3(z, W, s), \end{aligned} \tag{3.32}$$

and (1.2), (1.6), (1.10), (1.12), (1.7).
According to Theorem 2.2, the solution $[\omega(z), s(z)]$ satisfies the estimates (2.18) and (2.19), obviously $w = [\omega, s] \in B_M$. Denoting this mapping from $\Omega \in B_M$ onto w by $w = \underset{\sim}{S}\,[\Omega]$, it is clear that $\underset{\sim}{S}$ is an operator which maps B_M onto a compact set in B_M.
To prove that $\underset{\sim}{S}$ is continuous in B_M, we select a sequence of vectors $[W_n, S_n] (n = 0, 1, 2, \ldots)$ satisfying the condition

$$C[W_n - W_0, \overline{D}] \to 0, C[S_n - S_0, \overline{D}] \to 0 \text{ as } n \to \infty \tag{3.33}$$

and consider the difference $w_n - w_0 = \underset{\sim}{S}\,(\Omega_n) - \underset{\sim}{S}\,(\Omega_0)$. We have

$$[\omega_n - \omega_0]_{\bar z} = f(z, \omega_n, W_n, \omega_{nz}) - f(z, \omega_0, W_0, \omega_{0z}), \tag{3.34}$$

$$\mathrm{Re}[\overline{\lambda(t)}(\omega_n - \omega_0)] = P(z, \omega_n, s_n) - P(z, \omega_0, s_0) + h(t), t \in \Gamma, \tag{3.35}$$

$$\mathrm{Im}[\overline{\lambda(a_j)}(\omega_n(a_j) - \omega_0(a_j))] = 0,\ \ j = 1, \ldots, 2K+1, K \ge 0, \tag{3.36}$$

$$(s_n - s_0)_y = G(z, w_n, s_n) - G(z, w_0, s_0), \tag{3.37}$$

$$a(t)[s_n - s_0] = Q(t, w_n, s_n) - Q(t, w_0, s_0),\ t \in \gamma. \tag{3.38}$$

The complex equation (3.34) can be written as

$$\begin{aligned} &[\omega_n - \omega_0]_{\bar z} - [f(z, \omega_n, W_n, \omega_{nz}) - f(z, \omega_0, W_n, \omega_{0z})] = c_n,\\ &c_n = f(z, \omega_0, W_n, \omega_{0z}) - f(z, \omega_0, W_0, \omega_{0z}). \end{aligned} \tag{3.39}$$

Using the method in the proof of Theorem 2.2 of Chapter 4 in [3] or Theorem 2.6 of Chapter 2 in [4], we can verify that $L_{p_0}[c_n, \overline{D}] \to 0$ as $n \to \infty$. Hence, applying the method used in the proof of Theorem 2.1,

$$C_\beta[\omega_n - \omega_0, \overline{D}],\ C_\beta[s_n - s_0, \overline{D}] \le M_7\ L_{p_0}[c_n, \overline{D}] \tag{3.40}$$

can be concluded where M_7 is a non-negative constant. If $n \to \infty$, then $C[\omega_n - \omega_0, \overline{D}] \to 0$, $C[s_n - s_0, \overline{D}] \to 0$. Hence, $w = S(\Omega)$ is a continuous mapping from B_M onto a compact set in B_M. On the basis of the *Schauder fixed point theorem*, there exists a vector $w = [\omega, s] \in B_M$, so that $\omega = S(\omega)$, and $w = [\omega, s]$ is just a solution to `Problem B` for the system (1.1) and (1.2). $\diamondsuit$

Theorem 3.3. *Suppose that the system* (1.1), (1.2) *satisfies the same conditions as in Theorem 2.2, then the following statement holds*

(1) *If* $K \geq 0$, `Problem A` *for* (1.1), (1.2) *is solvable.*

(2) *If* $K < 0$, *there are* $-2K-1$ *conditions for* `Problem A` *to be solvable.*

Proof. Let us substitute the solution $[\omega(z), s(z)]$ to `Problem B` into the boundary condition (1.10). If $h(z) = 0, z \in \Gamma$, then $[\omega(z), s(z)]$ is also a solution to `Problem A` for (1.1), (1.2). The total number of real equalities in $h(z) = 0$ is just the total number of conditions stated in the theorem. $\diamondsuit$

Finally, in order to discuss the uniqueness of the solution to `Problem B` and `Problem A` for (1.1), (1.2) the following additional condition is imposed.

There exist $A_1^*, A_2^* \in L_{p_0}(\overline{D})$, with $L_{p_0}[A_2^*, \overline{D}]$ small enough, such that

$$(3.41)\quad F(z, \omega_1, U, s_1) - F(z, \omega_2, U, s_2) = A_1^*(\omega_1 - \omega_2) + A_2^*(s_1 - s_2),$$

for any functions $\omega_j, s_j \in C_\beta(\overline{D}), j = 1, 2$, and $U \in L_{p_0}(\overline{D})$ $(2 < p_0 < < p)$.

Theorem 3.4. (1.1), (1.2) *satisfies* `Condition C` *and* (3.41), *and the constant* ε *in* (1.3), (1.8), (1.9) *is small enough, then the solutions to* `Problem B` *are unique.*

Proof. Let $[\omega_1(z), s_1(z)]$, $[\omega_2(z), s_2(z)]$ be two solutions to `Problem B` for (1.1), (1.2). It is clear that $[\omega, s] = [\omega_1 - \omega_2, s_1 - s_2]$ is a solution to the initial-boundary value problem

$$\omega_{\overline{z}} = Q\omega_z + A_1^*\omega + A_2^* s,$$

$$Q = \begin{cases} \frac{F(z, \omega_1, \omega_{1z}, s_1) - F(z, \omega_1, \omega_{2z}, s_2)}{\omega_z}, & \omega_z \neq 0, \\ 0, \quad \omega_z = 0; \end{cases}$$

$$s_y = B_1^*\omega + B_2^*\overline{\omega} + B_3^* s,$$

$$\text{Re}[\overline{\lambda(t)}\omega(t)] = P(t, \omega_1, s_1) - P(t, \omega_2, s_2) + h(t), t \in \Gamma,$$

$$Im[\overline{\lambda(a_j)}\omega(a_j)] = 0,\ j = 1, 2, \ldots, 2K+1 (0 \leq K);$$

$$a(t)s(t) = Q(t, \omega_1, s_1) - Q(t, \omega_2, s_2), t \in \gamma.$$

With the method used in the proof of Theorem 2.2, we can show

$$C_\beta[\omega,\overline{D}] + L_{p_0}[|\omega_{\overline{z}}| + |\omega_z|,\overline{D}] = 0,$$
$$C^*_\beta(s,\overline{D}) = 0,$$

so that $\omega(z) \equiv 0, s(z) \equiv 0$, i.e. $\omega_1(z) \equiv \omega_2(z) s_1(z) \equiv s_2(z)$ in $\overline{D}$. ◊

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