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WELL COVERED AND WELL DOMINATED BLOCK GRAPHS AND UNICYCLIC GRAPHS

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Abstract: A graph is called well covered if every maximal independent set is a maximum independent set. Analogously, a well dominated graph is one in which every minimal dominating set is a minimum dominating set. In this paper, characterizations of well dominated and well covered block graphs and unicyclic graphs are given.

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In this paper, we discuss finite undirected simple graphs. For any undefined term see [2] and [10]. For a graph G, we denote by $V(G)$ and $E(G)$ the vertex set and edge set of G, respectively. For $v \in V(G)$, let $N_G(v)$ be the set of vertices (neighbours) adjacent to v in G and, more generally, $N_G(S) = \bigcup_{v \in S} N_G(v)$ and $\overline{N}_G(S) = N_G(S) \cup S$ for $S \subseteq V(G)$. If $X \subseteq V(G)$, then $[X]$ (resp., $G - X$) denotes the subgraph of G induced by X (resp., $V(G) - X$). We write $G — x$ instead of $G — \{x\}$ if $x \in V(G)$.

The vertex v of G is an end vertex of G if $d_G(v) = 1$, where $d_G(x) = |N_G(x)|$ is the degree of $x \in V(G)$. An edge incident with an end vertex of G is called an end edge of G. For a graph G, let $\Omega(G)$ ($E_e(G)$, resp.) be the set of end vertices (end edges, resp.) of G. A vertex v of a connected graph G is called a cut vertex of G if $G - v$ contains more components than G. Let $C(G)$ be the set of cut vertices of G. For $v \in V(G)$, let $NC_G(v) = N_G(v) - C(G)$. A connected graph with no cut vertices is called a block. A block of a graph G is a subgraph of G which is itself a block and which is maximal with respect to that property. A graph G is called a block graph if every block of G is a complete graph. In this paper, we define an exterior block of a graph G as a block containing at least one non-cut vertex of G. For a graph G, the corona $G \circ K_1$ of G and K_1 is the supergraph of G obtained from G by adding, for every vertex x of G, exactly one new vertex adjacent to x only. Note that a graph H is the corona of some graph G and K_1 if and only if $E_e(H)$ is a perfect matching of H.

A set $D \subseteq V(G)$ is a dominating set of G if $N_G(v) \cap D \neq \emptyset$ for every $v \in V(G) - D$, and is an independent set of G if $N_G(v) \cap D = \emptyset$ for every $v \in D$. Let $i(G)$ and $\alpha(G)$ ($\gamma(G)$ and $\Gamma(G)$, resp.) denote the minimum and maximum cardinalities of a maximal independent set (a minimal dominating set, resp.) in G. A graph G is said to be well covered if every maximal independent set in G is a maximum independent set. A graph G is said to be well dominated [7] if every minimal dominating set in G is a minimum dominating set. Equivalently, G is a well covered (dominated, resp.) graph if $i(G) = \alpha(G)$ ($\gamma(G) = \Gamma(G)$, resp.).

Well covered graphs were introduced by Plummer in 1970 [11]. Until now, however, only a few classes of well covered graphs have been characterized. For example, Ravindra [12] gave a characterization of well covered bipartite graphs. Recently Finbow, Hartnell, and Nowakowski in [7], [8], and [9] have completely described well covered and well dominated graphs of girth at least 5, well dominated bipartite graphs, and well covered graphs containing neither a cycle C_4 nor a cycle C_5 as a subgraph. For related results the reader is referred to [1], [3-6], and [13-15]. In this paper, it is shown that for a block graph G one of the four equations $\gamma(G) = \alpha(G)$, $\gamma(G) = \Gamma(G)$, $i(G) = \alpha(G)$, $i(G) = \Gamma(G)$ holds if and only if the other three hold. Structural characterizations of well covered and well dominated block graphs are given. Similar results for unicyclic graphs are presented.

In the sequel, we will need the following simple results and observations.

Proposition 1. *For any graph* G,

$$\gamma(G) \le i(G) \le \alpha(G) \le \Gamma(G).$$

Proof. It follows at once from the simple observation that every maximal independent set in G is a minimal dominating set in G. $\diamondsuit$

This proposition implies that every well dominated graph is well covered. The converse implication is not necessarily true (see, for example, Theorems 2 and 3 below). The next proposition implies that the corona of any graph G (and K_1) is a well dominated graph. Theorem 1, among other things, proves that every well covered block graph is well dominated.

Proposition 2. *For any graph* G,

$$\gamma(G \circ K_1) = i(G \circ K_1) = \alpha(G \circ K_1) = \Gamma(G \circ K_1) = |V(G)|.$$

Proof. After Proposition 1, it is enough to show that every minimal dominating set in $G \circ K_1$ has exactly $|V(G)|$ vertices. Let D be a minimal dominating set in $G \circ K_1$. For a vertex $x \in V(G)$, let $\overline{x}$ be the only neighbour of x in $\Omega(G \circ K_1)$. It is clear from the definition of $G \circ K_1$ that the sets $\{x, \overline{x}\}$, $x \in V(G)$, form a disjoint partition of $V(G \circ K_1)$. Therefore the minimality of D implies that $|D \cap \{x, \overline{x}\}| = 1$ for every $x \in V(G)$. Hence $|D| = |V(G)|$. $\diamondsuit$

Proposition 3. [2]. *An independent set I of a graph G is maximum if and only if*

$$|N_G(J) \cap I| \ge |J|$$

for every independent subset J of $V(G) - I$. $\Diamond$

Corollary 1. *Every vertex of a well covered block graph G belongs to at most one exterior block of G.*

Proof. Suppose, to the contrary, that a vertex v belongs to at least two exterior blocks of G, say B_1 and B_2. Let I be a maximal independent set which contains v, and let $v_i \in V(B_i) - C(G)$ for $i = 1, 2$. Then $|N_G(\{v_1, v_2\}) \cap I| = |\{v\}| < |\{v_1, v_2\}|$ and therefore, by Proposition 3, I is not a maximum independent set in G which is impossible in a well covered graph. $\Diamond$

Corollary 2. *Let v be a cut vertex in a well covered block graph G. If the set $NC_G(v)$ is not empty, then every two vertices of $NC_G(v)$ are adjacent.*

Proof. Suppose, to the contrary, that two vertices v_1 and v_2 of $NC_G(v)$ are not adjacent. Then they belong to different exterior blocks of G, say B_1 and B_2. Clearly, v belongs to B_1 and B_2 which (according to Corollary 1) is impossible in a well covered graph. $\Diamond$

Proposition 4. *If G is a well covered graph and I is an independent set in G, then $G - \overline{N}(I)$ is well covered.*

Proof. Immediate by contradiction. $\Diamond$

Now we are prepared to give characterizations of well covered and well dominated block graphs.

Theorem 1. *For a block graph G, the following statements are equivalent:*

(i) $\gamma(G) = \Gamma(G)$;
(ii) $\gamma(G) = \alpha(G)$;
(iii) $i(G) = \alpha(G)$;
(iv) $i(G) = \Gamma(G)$;
(v) *The vertex sets $V(G_1), \ldots, V(G_k)$ of the exterior blocks of G form a disjoint partition of $V(G)$;*
(vi) *The induced subgraph $[NC_G(v)]$ of G is nonempty and complete every cut vertex v of G.*

Proof. The implications (i) $\Rightarrow$ (ii), (ii) $\Rightarrow$ (iii), (i) $\Rightarrow$ (iv), and (iv) $\Rightarrow$ (iii) follow at once from Proposition 1. We will show the implications (iii) $\Rightarrow$ (v), (v) $\Rightarrow$ (i), (v) $\Rightarrow$ (vi), and (vi) $\Rightarrow$ (v).

(iii) $\Rightarrow$ (v). Suppose that the implication (iii) $\Rightarrow$ (v) is false and let G be a well covered block graph with minimum number of vertices in which the vertex sets $V(G_1), \ldots, V(G_k)$ of the exterior blocks of G do not form a disjoint partition of $V(G)$. According to Corollary 1, the sets $V(G_1), \ldots, V(G_k)$ are mutually disjoint. The choice of G implies that G is connected and its diameter d is greater than three. Let $P = (v_0, v_1, \ldots, v_d)$ be any longest path without triangular chords in G, and let B_i be that block of G which contains the vertices v_{i-1} and v_i of $P (i = 1, \ldots, d)$. From Corollary 1 and the choice of P it follows that the blocks $B_1, \ldots, B_d$ are different, $\{v_1, \ldots, v_{d-1}\} \subseteq C(G)$, B_1 is an exterior block of G, and B_1 and B_2 are the only blocks of G which contain the vertex v_1. In addition, the choice of G makes it obvious that v_1 and v_2 are the only vertices of B_2. Let us consider the connected block graph $H = G - \overline{N}_G(v_0) = G - V(B_1)$. Since H is well covered (by Proposition 4) and has fewer vertices than G, the vertex sets $V(H_1), \ldots, V(H_l)$ of the exterior blocks $H_1, \ldots, H_l$ of H form a disjoint partition of $V(H)$.

We now claim that v_2 is not a cut vertex of H. For if not, then $B_1, H_1, \ldots, H_l$ are the exterior blocks of G and their vertex sets $V(B_1)$, $V(H_1), \ldots, V(H_l)$ form a disjoint partition of $V(G)$, a contradiction. This implies the desired claim. In a similar manner, we find that every vertex of $B_3 - v_2$ is a cut vertex of H. From the above it follows that B_3 is one of the exterior blocks of H, say $B_3 = H_l$, and B_3 is not an exterior block of G. Hence $B_1, H_1, \ldots, H_{l-1}$ are the exterior blocks of G and the sets $V(B_1)$, $V(H_1), \ldots, V(H_{l-1})$ form a disjoint partition of $V(G) - V(B_3)$.

We now show that the graph G has maximal independent sets of different cardinalities. Take exactly one vertex u_i from the set $V(H_i) - C(H)$ $(i = 1, \ldots, l)$. From the properties of the blocks $B_1, H_1, \ldots, H_l$ it follows that $u_l = v_2$ and $I = \{v_0, v_2, u_1, \ldots, u_{l-1}\}$ is a maximal independent set in G. On the other hand, let $V(B_3) - \{v_2\} = \{x_1, \ldots, x_p\}$. Since B_3 is an exterior block of H and each x_i is a cut vertex of H, there exists a nonexterior block F_i of H that contains $x_i (i = 1, \ldots, p)$. Let z_i be any vertex of $F_i - x_i$ and let H_{j_i} be the exterior block of H that contains $z_i (i = 1, \ldots, p)$. Without loss of generality, we may assume that $\{i_1, \ldots, i_p\} = \{1, \ldots, p\}$. It is not hard to observe that the set $I' = \{v_1, z_1, \ldots, z_p, u_{p+1}, \ldots, u_{l-1}\}$ is a maximal independent set in G. (The graph in Figur 1 illustrates these constructions.) Since $|I'| \neq |I|$, G is not a well covered graph, a contradiction. This proves the implication (iii) $\Rightarrow$ (v).

(v) $\Rightarrow$ (i). Assume that (v) holds. Since $V(G_i) - C(G) \neq \emptyset$, we may choose exactly one vertex x_i from the set $V(G_i) - C(G)$ $(i = 1, \ldots, k)$ and form the set $D = \{x_1, \ldots, x_k\}$. (v) implies that D is a dominating set in G. We claim that $\gamma(G) = |D| = k$. Suppose, to the contrary, that there exists a dominating set D_1 in G such that $|D_1| < k$. Then it follows from (v) that $D_1 \cap V(G_{i_0}) = \emptyset$ for some $i_0 \in \{1, \ldots, k\}$, which implies that $x_{i_0} \notin D_1$ and $N_G(x_{i_0}) \cap D_1 = \emptyset$ (since $N_G(x_{i_0}) \subset V(G_{i_0})$), a contradiction. This proves that $\gamma(G) = k$. Similarly, we claim that $\Gamma(G) = |D| = k$. Suppose indirectly that there is a minimal dominating set D_2 in G such that $|D_2| > k$. Then (v) implies that $|D_2 \cap V(G_{j_0})| \geq 2$ for some $j_0 \in \{1, \ldots, k\}$ and, in addition, $D_2 \cap V(G_i) \neq \emptyset$ for each $i \in \{1, \ldots, k\}$. Let v be any vertex of $D_2 \cap V(G_{j_0})$, and let $D_2' = D_2 - \{v\}$. Clearly, D_2' is a dominating set in G and contains one vertex less than D_2 which is impossible since D_2 was a minimal dominating set in G. Therefore $\Gamma(G) = k$. Consequently $\gamma(G) = \Gamma(G)$.

(v) $\Rightarrow$ (vi). Assume that (v) holds and let v be a cut vertex of G. By (v), the vertex v belongs to $V(G_i)$ for some $i \in \{1, \ldots, k\}$. Since the set $V(G_i) - C(G)$ is nonempty and v is adjacent to every vertex of $V(G_i) - C(G)$, the set $NC_G(v)$ is nonempty. Hence, the subgraph $[NC_G(v)]$ is nonempty and complete (by Corollary 2 and the equivalence of (v) and (iii)).

(vi) $\Rightarrow$ (v). Assume that (vi) holds. First let us observe that the sets $V(G_1), \ldots, V(G_k)$ are disjoint. For if not, then there exist $i, j \in \{1, \ldots, k\}$, $i \neq j$, and a vertex v such that $v \in V(G_i) \cap V(G_j)$. Certainly, v is a cut vertex of G and since the sets $NC_G(v) \cap V(G_i)$, $NC_G(v) \cap V(G_j)$ are nonempty, the subgraph $[NC_G(v)]$ is not complete. This contradicts our assumption. Hence, the sets $V(G_1), \ldots, V(G_k)$ are disjoint and it remains to show that $V(G) = \bigcup_{i=1}^k V(G_i)$. To prove this it is sufficient to show that $C(G) \subset \bigcup_{i=1}^k V(G_i)$, since $\bigcup_{i=1}^k V(G_i) \subset V(G)$ and $V(G) - C(G) \subset \bigcup_{i=1}^k V(G_i)$ from the definition of the graphs $G_1, \ldots, G_k$. It follows from (vi) that for every $v \in C(G)$, $[NC_G(v)]$ is a subgraph of exactly one of the graphs $G_1, \ldots, G_k$. This implies that every $v \in C(G)$ belongs to exactly one of the graphs $G_1, \ldots, G_k$ and therefore $C(G) \subset \bigcup_{i=1}^{k} V(G_i)$. This proves the implication (vi) $\Rightarrow$ (v) and completes the proof of the theorem. $\diamondsuit$

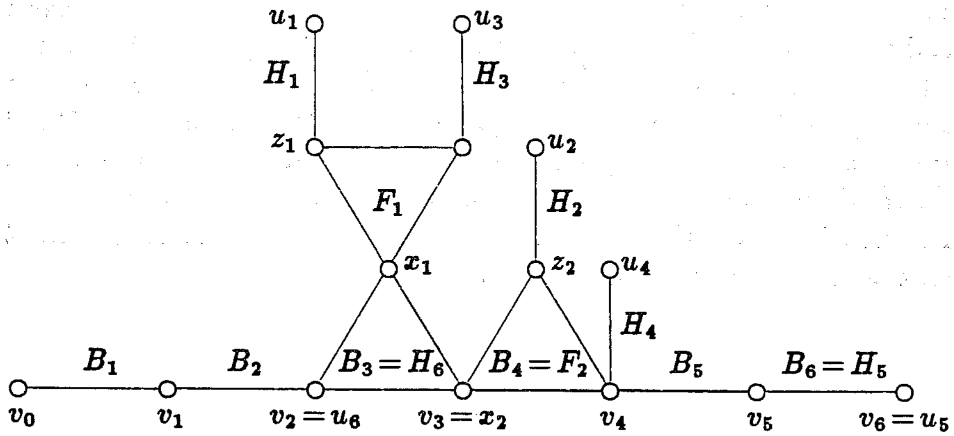

Figure 1

From Theorem 1 we can immediately deduce the following corollary for trees.

Corollary 3. *For a tree T, each of the statements* (i) – (vi) *of Theorem 1 is equivalent to the statement*

(vii) $T = K_1$ *or* $T = R \circ K_1$ *for some tree* R. $\diamondsuit$

Theorem 1 and Proposition 1 imply that for a block graph G, each of the equations $\gamma(G) = \Gamma(G)$, $\gamma(G) = \alpha(G)$, $i(G) = \alpha(G)$, $i(G) = \Gamma(G)$ implies each of the equations $\gamma(G) = i(G)$ and $\alpha(G) = \Gamma(G)$. The converse is not true. This can be seen with the aid of the graph $K_{1,2}$.

The final section of this paper is devoted to characterizations of well dominated and well convered unicyclic graphs. Let us recall that a unicyclic graph is a connected graph with exactly one cycle. Let $\mathcal{U}$ denote the set of all unicyclic graphs. For $G \in \mathcal{U}$, we denote by C_G the unique cycle of G, and by $g(G)$ the length of C_G, i.e., $g(G)$ is the girth of G. Let $\mathcal{KU}$ be the subfamily of $\mathcal{U}$, where $G \in \mathcal{KU}$ if and only if $G = H \circ K_1$ for some $H \in \mathcal{U}$. In what follows, it is helpful to note that a graph G belongs to the set $\mathcal{KU}$ if and only if G is a unicyclic graph and the sets of the family $\{\{v,u\} : vu \in E_e(G)\}$ form a disjoint partition of the set $V(G)$. Similarly we define the subfamilies $\mathcal{S}_3, \mathcal{S}_4$, and $\mathcal{S}_5$ of $\mathcal{U}$. A graph G is in the family of $\mathcal{S}_3$ if G is a unicyclic graph of girth 3 in which the unique cycle C_G has 1 or 2 vertices of degree three or more and the sets of the family $\{V(C_G)\} \cup \{\{v,u\} : vu \in E_e(G)\}$ form a disjoint partition of the set $V(G)$. A graph G is in the family $\mathcal{S}_4$ if G is a unicyclic graph of girth 4, the unique cycle C_G of G contains exactly two adjacent vertices of degree two (in G), say a and b, and the set $\{ab\} \cup E_e(G)$ is a perfect matching of G. Finally, a graph G is in the family $\mathcal{S}_5$ if G is a unicyclic graph of girth 5, the unique cycle C_G of G does not contain two adjacent vertices of degree three or more, and the sets of the family $\{V(C_G)\} \cup \{\{v,u\} : vu \in E_e(G)\}$ form a disjoint partition of the set $V(G)$. (The graphs G_2, G_3, and G_4 in Figure 2 belong to $\mathcal{S}_3$, $\mathcal{S}_4$, and $\mathcal{S}_5$, respectively.)

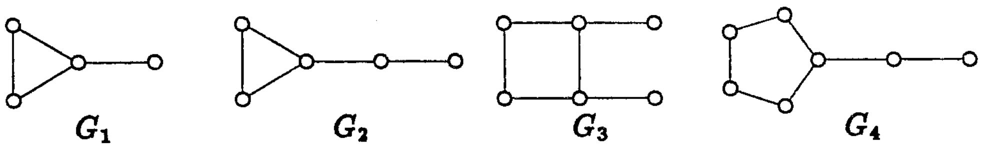

Figure 2

Proposition 5. *For any* $G \in \mathcal{S}_3$, $\gamma(G) = \Gamma(G)$.

Proof. Since every graph G in $\mathcal{S}_3$ is a block graph in which the unique cycle $C_G = C_3$ of G and the subgraphs generated by the end edges of G are the exterior blocks of G and their vertex sets form a disjoint partition of $V(G)$, the result follows from Theorem 1. $\diamondsuit$

Proposition 6. *For any* $G \in \mathcal{S}_4$, $\gamma(G) = |E_e(G)|$ *and* $i(G) = \alpha(G) = = \Gamma(G) = |E_e(G)| + 1$.

Proof. For $G \in \mathcal{S}_4$, let C_G be the unique cycle of G, and let a and b be the adjacent vertices of degree two (in G). Let D be any minimal dominating set in G. It follows from the minimality of D that $|D \cap \cap \{v,u\}| = 1$ for each $vu \in E_e(G)$ and $|D \cap \{a,b\}| \leq 1$. Therefore, since the sets of the family $\{\{a,b\}\} \cup \{\{v,u\} : vu \in E_e(G)\}$ form a disjoint partition of $V(G)$, $|E_e(G)| \leq \gamma(G) \leq |D| \leq \Gamma(G) \leq |E_e(G)| + 1$. From this and from the fact that the sets $N_G(\Omega(G))$ (of cardinality $|E_e(G)|$) and $\Omega(G) \cup \{a\}$ (of cardinality $|E_e(G)| + 1$) are minimal dominating sets in G, we obtain $\gamma(G) = |E_e(G)|$ and $\Gamma(G) = |E_e(G)| + 1$. Similar analysis shows that every maximal independent set of G has exactly $|E_e(G)| + 1$ vertices. Thus, $i(G) = \alpha(G) = |E_e(G)| + 1$. ◊

Proposition 7. *For any $G \in \mathcal{S}_5$, $\gamma(G) = \Gamma(G) = |E_e(G)| + 2$.*

Proof. For $G \in \mathcal{S}_5$, let C_G be the unique cycle of G, and let D be any minimal dominating set in G. We need only observe that $|D| = |E_e(G)| + 2$. Because the sets of the family $\{V(C_G)\} \cup \{\{v, u\} : vu \in E_e(G)\}$ form a disjoint partition of $V(G)$ and D is a minimal dominating set in G, we find $|D| = |D \cap V(C_G)| + \Sigma_{vu \in E_e(G)} |D \cap \{v, u\}| = |D \cap V(C_G)| + |E_e(G)|$. Simple observations show that $|D \cap V(C_G)| = 2$, and so, $|D| = |E_e(G)| + 2$, as required. ◊

Proposition 8. *Let G be a unicyclic graph with $g(G) \geq 5$. Then the following statements are equivalent:*

(i) $\gamma(G) = \Gamma(G)$;
(ii) $\gamma(G) = \alpha(G)$;
(iii) $i(G) = \Gamma(G)$;
(iv) $i(G) = \alpha(G)$;
(v) $G \in \{C_5, C_7\} \cup \mathcal{S}_5 \cup \{H \circ K_1 : H \in \mathcal{U}$ *and* $g(H) \geq 5\}$.

Proof. The implications (i) ⇒ (ii), (i) ⇒ (iii), (ii) ⇒ (iv) and (iii) ⇒ (iv) immediately follow from Proposition 1. The implication (v) ⇒ (i) is obvious if $G \in \{C_5, C_7\}$ and follows from Propositions 7 and 2 if $G \in \mathcal{S}_5 \cup \{H \circ K_1 : H \in \mathcal{U}$ with $g(H) \geq 5\}$. Finally, it is a simple matter to obtain the implication (iv) ⇒ (v) from [8, Corollary 4] (see also [7]). ◊

Theorem 2. *For a unicyclic graph G, the following statements are equivalent:*

(i) $\gamma(G) = \Gamma(G)$;
(ii) $\gamma(G) = \alpha(G)$;
(iii) $G \in \{C_3, C_4, C_5, C_7\} \cup \mathcal{KU} \cup \mathcal{S}_3 \cup \mathcal{S}_5$.

Proof. The implication (i) $\Rightarrow$ (ii) follows from Proposition 1. The equivalence of (ii) and (iii) has been proved in [15]. By Propositions 2, 5 and 7, the implication (iii) $\Rightarrow$ (i) is true for every graph $G \in \mathcal{KU} \cup \mathcal{S}_3 \cup \mathcal{S}_5$. Finally, it is straightforward to verify that the cycles C_3, C_4, C_5 and C_7 are well dominated. ◇

As a consequence of Theorem 2 and Proposition 1 we see that for a unicyclic graph G, each of the equations $\gamma(G) = i(G)$, $i(G) = \alpha(G)$, $i(G) = \Gamma(G)$, $\alpha(G) = \Gamma(G)$ follow from each of the equations $\gamma(G) = \Gamma G)$ and $\gamma(G) = \alpha(G)$. The graphs G_1 and G_3 (shown in Figure 2) prove that the converse is not necessarily true.

The next theorem presents necessary and sufficient conditions for a unicyclic graph to be well covered. The proof is based on the following proposition.

Proposition 9 [12]. *A bipartite graph G without isolated vertices is well covered if and only if G has a perfect matching M and, for every edge $vu \in M$, the subgraph induced by the set $N_G(\{v,u\})$ is a complete bipartite graph.* ◇

Theorem 3. *For a unicyclic graph G, the following statements are equivalent:*

(i) $i(G) = \Gamma(G)$;
(ii) $i(G) = \alpha(G)$;
(iii) $G \in \{C_3, C_4, C_5, C_7\} \cup \mathcal{KU} \cup \mathcal{S}_3 \cup \mathcal{S}_4 \cup \mathcal{S}_5$.

Proof. The implications (i) $\Rightarrow$ (ii) and (iii) $\Rightarrow$ (i) easily follow from Theorem 2 and Propositions 1 and 6. Thus it remains to prove the implication (ii) $\Rightarrow$ (iii).

Assume that G is a unicyclic graph and $i(G) = \alpha(G)$. Let C_G be the unique cycle of G, and let $E_e(G)$ be the set of end edges of G. We split the proof into three parts, based on the girth $g(G)$ of G.

Case 1. If $g(G) \geq 5$, then $G \in \{C_5, C_7\} \cup \mathcal{S}_5 \cup \mathcal{KU}$ (by Proposition 8).

Case 2. Assume that $g(G) = 4$. Then G is bipartite and therefore by Proposition 9, G has a perfect matching M such that for every edge $vu \in M$, the subgraph induced by $N_G(\{v,u\})$ is a complete bipartite graph. We will show that either $G = C_4$ of $G \in \mathcal{KU} \cup \mathcal{S}_4$. In order to prove this, let us assume that $G \neq C_4$. It is clear that $E_e(G) \subseteq M$ and, in addition, $G \in \mathcal{KU}$ if (and only if) $M = E_e(G)$. Thus assume

that $M \neq E_e(G)$. We claim that $M \subseteq E(C_G) \cup E_e(G)$. For if not, then $M - (E(C_G) \cup E_e(G)) \neq \emptyset$ and for any edge $vu \in M - (E(C_G) \cup \cup E_e(G))$, the sets $N_G(v) - \{u\}$ and $N_G(u) - \{v\}$ are not empty and no vertex of $N_G(v) - \{u\}$ is adjacent to a vertex of $N_G(u) - \{v\}$, so $[N_G(\{v,u\})]$ is not a complete bipartite graph, a contradiction. We therefore henceforth suppose that $M \subseteq E(C_G) \cup E_e(G)$ and $M \cap E(C_G) \neq \emptyset$. Certainly, $|M \cap E(C_G)| = 1$; otherwise $|M \cap E(C_G)| = 2$, say $M \cap E(C_G) = \{xy, wz\}$, and then, since $G \neq C_4$, at least one of the subgraphs $[N_G(\{x,y\})]$ and $[N_G(\{w,z\})]$ is not a complete bipartite graph, a contradiction. Let vu be the only edge of $M \cap E(C_G)$. Then $M = \{vu\} \cup E_e(G)$ and, moreover, $|N_G(v)| = |N_G(u)| = 2$; otherwise $|N_G(v)| \geq 3$ or $|N_G(u)| \geq 3$ and then $[N_G(\{v,u\})]$ would not be a complete bipartite graph, a contradiction. This implies that $G \in \mathcal{S}_4$.
Case 3. If $g(G) = 3$, then G is a well covered block graph. We will show that either $G = C_3$ or $G \in \mathcal{KU} \cup \mathcal{S}_3$. Assume that $G \neq C_3$, and let $G_1, \ldots, G_k$ be the exterior blocks of G. By Theorem 1, the vertex sets $V(G_1), \ldots, V(G_k)$ form a disjoint partition of $V(G)$. If C_G is one of the blocks $G_1, \ldots, G_k$, say $C_G = G_1$, then $\{V(G_1), \ldots, V(G_k)\} = = \{V(C_G)\} \cup \{\{v,u\} : vu \in E_e(G)\}$ and $G \in \mathcal{S}_3$. If C_G is not an exterior block of G, then $\{V(G_1), \ldots, V(G_k)\} = \{\{v,u\} : vu \in E_e(G)\}$ and $G \in \mathcal{KU}$. This proves the implication (ii) $\Rightarrow$ (iii) and completes the proof of the theorem. $\diamondsuit$

In conclusion, let us note the according to Theorems 2 and 3, the well covered unicyclic graphs which are not well dominated are precisely those which belong to the family $\mathcal{S}_4$.

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