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# SEQUENCES OF DOMINATING SETS

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**Abstract:** It is proved for any $0 < \beta < 1$ and any graph $G = (V, E)$ there exists an ordering $v_1, v_2, \ldots, v_{|V|}$ of vertices of $G$ such that either for every $i \in \{1, \ldots, |V|\}$ the set $\{v_1, \ldots, v_i\}$ dominates in $G$ all but at most $|V| \cdot \beta^i$ vertices, or for every $j \in \{1, \ldots, |V|\}$ the set $\{v_1, \ldots, v_j\}$ dominates in the complement $\bar{G}$ of $G$ all but at most $|V|(1-\beta)^j$ vertices.

Let $X$ be a subset of the vertex-set of a graph $G = (V, E)$ and $N_G(X) = X \cup \{y \in V | \exists x \in X : (x, y) \in E\}$. Let us say that $X$ is $\beta$-dominating in $G$, if $|V \setminus N_G(X)| \leq |V| \cdot \beta^{|X|}$. By $\bar{G}$ we denote the complement of $G$.

Erdős and Hajnal [1] conjectured that for any positive integer $t$ and any graph $G = (V, E)$ with $|V| \geq t$ either there exists a 0.5-dominating set $X$ in $G$ with $|X| = t$ or there exists a (1-0.5)-dominating set $Y$ in $\bar{G}$ with $|Y| = t$. Erdős, Faudree, Gyárfás and Schelp [2] proved that this conjecture remains true even if we put any $0 < \beta < 1$ instead of 0.5.

The aim of the present note is to prove the following somewhat stronger statement, which was obtained independently of [2].

**Proposition 1.** *For any $0 < \beta < 1$ and any graph $G = (V, E)$ either there exists a numbering $v_1, v_2, \dots, v_{|V|}$ of the vertices of $G$ such that the set $\{v_1, v_2, \dots, v_i\}$ is $\beta$-dominating in $G$ for every $i \in \{1, \dots, |V|\}$, or there exists a numbering $u_1, u_2, \dots, u_{|V|}$ of the vertices of $G$ such that the set $\{u_1, u_2, \dots, u_j\}$ is $(1-\beta)$-dominating in $\bar{G}$ for every $j \in$ $\in \{1, \dots, |V|\}$.*

Proposition 1 is a consequence of the following proposition. (To see this, apply Proposition 2 to the bipartite graph $\widetilde{G} = (X, Y; \widetilde{E})$, where $\widetilde{G}$ is obtained from $G = (V, E)$ as follows: $|X| = |Y| = |V|$ and $(x_i, y_j) \in \widetilde{E}$ iff $(v_i, v_j) \in E$).

**Proposition 2.** *Let $G = (X, Y; E)$ be a bipartite graph with parts $X$ and $Y$, and $0 < \beta < 1$. Then at least one of the following assertions is true:*

(a) *there is a numbering $x_1, x_2, \dots, x_{|X|}$ of the vertices of $X$ such that*
$$|Y \setminus N_G(\{x_1, x_2, \dots, x_i\})| \le |Y| \cdot \beta^i$$
*for every $i \in \{1, \dots, |X|\}$;*

(b) *there is a numbering $y_1, y_2, \dots, y_{|Y|}$ of the vertices of $Y$ such that*

$$|X \setminus N_{\bar{G}}(\{y_1, y_2, \dots, y_j\})| < |X|(1-\beta)^j \tag{1}$$

*for every $j \in \{1, \dots, |Y|\}$.*

**Proof.** We try to construct the proper numbering of vertices of $X$, using the following *Procedure 1*.
BEGIN. Let $X_0 := \emptyset$; $i := 1$;
*Step* i. If $i = |X| + 1$, then END. If there is $x \in X \setminus X_{i-1}$ such that $|(Y \setminus N_G(X_{i-1})) \setminus N_G(\{x\})| \le \beta |Y \setminus N_G(X_{i-1})|$ (in particular, if $Y \setminus N_G(X_{i-1}) = \emptyset$), then set $x_i := x$, $X_i := X_{i-1} \cup \{x_i\}$ and go to Step $i+1$. Else END.

If the Procedure stops on Step $t$ and $t = |X| + 1$, then Assertion (a) of our Proposition 2 is true. Let $t \le |X|$ and $Y_0 = Y \setminus N_G(X_{t-1})$. Then $Y_0 \neq \emptyset$ and, by the construction, for $x \in X$ we have

$$|Y_0 \cap N_G(x)| < (1-\beta)|Y_0|. \tag{2}$$

The following *Procedure 2* will make it possible to number the vertices of $Y_0$ properly.
BEGIN. *Step* $k(1 \leq k \leq |Y_0|)$. Before Step $k$ the vertices $y_1, \dots, y_{k-1} \in \in Y_0$ are chosen that the inequality (1) is fullfilled for $j = 1, 2, \dots, k-1$ and, denoting $Y_{k-1} := Y_0 \setminus \{y_1, \dots, y_{k-1}\}$, for any $x \in \in X \setminus N_{\bar{G}}(\{y_1, \dots, y_{k-1}\}) = \bigcap_{j=1}^{k-1} N_G(y_j) \cap X$ the inequality

$$(3) \qquad |N_G(\{x\}) \cap Y_{k-1}| < (1-\beta)|Y_{k-1}|$$

holds. Note that for $k = 1$, (3) follows from (2). If $X \subset N_{\bar{G}}(\{y_1, \dots, y_{k-1}\})$, then choose an arbitrary $y \in Y_{k-1}$, $y_k := y$, $Y_k := Y_{k-1} \setminus \{y_k\}$ and go to Step $k+1$. Suppose $\bigcap_{j=1}^{k-1} N_G(\{y_j\}) \cap X \neq \emptyset$. Due to (3), there exists $y \in Y_{k-1}$ such that

$$(4) \qquad |\bigcap_{j=1}^{k-1} N_G(\{y_j\}) \cap X \cap N_G(\{y\})| < (1-\beta)|\bigcap_{j=1}^{k-1} N_G(\{y_j\}) \cap X|.$$

Set $y_k := y$, $Y_k := Y_{k-1} \setminus \{y_k\}$. Notice that (4) implies the validity of (1) for $j = k$. Because of (3), we have

$$(5) \qquad |N_G(\{x\}) \cap Y_k| = |N_G(\{x\}) \cap Y_{k-1}| - 1 < (1-\beta)|Y_{k-1}| - 1 <$$

$$< (1-\beta)|Y_k|$$

for every $x \in \bigcap_{j=1}^{k} N_G(\{y_j\}) \cap X$. To to Step $k+1$, knowing that for that Step inequality (3) holds, since now (5) holds. END.

Thus, on completion of Procedure 2 the vertices of $Y_0$ will be numbered properly. But, according to (2), $N_{\bar{G}}(Y_0) \supset X$. Thus, the vertices of $Y \setminus Y_0$ we can number by $|Y_0| + 1, \dots, |Y|$ in an arbitrary order.

**Remark.** Evidently, a polynomial time via $|X| + |Y|$ is sufficient for numbering $X$ or $Y$.

# References

[1] ERDŐS, P. and HAJNAL, A.: Ramsey type theorems, Preprint, 1987.

[2] ERDŐS, P.; FAUDREE, R.; GYÁRFÁS, A. and SCHELP, R.H.: Domination in colored complete graphs, *Journal of Graph Theory* (to appear).