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A METHOD FOR CONSTRUCTION OF SPLINE CURVES

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Abstract: This paper deals with the construction of so-called $L_{Q,p}$-splines based on Lienhard's interpolation method [see Lienhard].

1. Generalization of Lienhard's interpolation method

Let $Q \geq 1$ be an integer. In the space R^m ($m > 1$ integer) let n distinct points $P_i = x_j^{(i)}$ ($i = 1, \dots, n; j = 1, \dots, m$) be given. The symbol $x_j^{(i)}$ denotes also the corresponding ordered m-tuple of coordinates, or rather the vector which has these coordinates. Thus, the elements of the set $\mathbf{R}^m$ are either pionts or vectors, according to which of the notions corresponds more to our conception in the given context. As a rule, we use the notion of a point in situations when location in the space $\mathbf{R}^m$ is discussed while the notion of a vector indicates that we are interested in the direction.

We shall look for polynomials in the real variable t of degree at most K (not determined as yet)

$$P_{x_j}^{(i)}(t) = \sum_{k=0}^{K} a_{jk}^{(i)} t^k \quad (i = 1, ..., n-1) \tag{1.1}$$

such that

$$P_{x_j}^{(i)}(-1) = x_j^{(i)}, P_{x_j}^{(i)}(1) = x_j^{(i+1)} \tag{1.2}$$

$$\frac{d^q}{dt^q} P_{x_j}^{(i)}(1) = \frac{d^q}{dt^q} P_{x_j}^{(i+1)}(-1) \quad (q = 1, ..., Q). \tag{1.3}$$

Conditions (1.2) guarantee that the interpolation arc parametrized with the aid of the functions $P_{x_j}^{(i)}(t)$ $(j = 1, \dots, m)$ passes through the supporting points (nodes) P_i, P_{i+1}. Conditions (1.3) guarantee the fluent transition from arc to arc, in the first till the Q derivatives. To satisfy conditions (1.3) we have to know the values of the first till Q-th derivatives of the functions $P_{x_j}^{(i)}(t)$ at the points P_i, P_{i+1} :

$$\frac{d}{dt} P_{x_j}^{(i)}(-1) = Dx_j^{(i)}, \frac{d}{dt} P_{x_j}^{(i)}(1) = Dx_j^{(i+1)},$$

(1.4) ....

$$\frac{d^Q}{dt^Q} P_{x_j}^{(i)}(-1) = D^Q x_j^{(i)}, \frac{d^Q}{dt^Q} P_{x_j}^{(i)}(1) = D^Q x_j^{(i+1)};$$

by convention, $Dx_j^{(i)} = D^1 x_j^{(i)}, D^2 x_j^{(i)}, ..., D^Q x_j^{(i)}$, $Dx_j^{(i+1)} = D^1 x_j^{(i+1)}$, $D^2 x_j^{(i+1)}$,..., $D^Q x_j^{(i+1)}$ is the notation of these values. The manner of their determination will be discussed in Section 2. By (1.2), (1.3) we have $2Q + 2$ definite conditions for every polynominal (1.1). With their aid each of these polynomials is thus uniquely determined as a polynomial of degree at most $K = 2Q+1$:

$$P_{x_j}^{(i)}(t) = \sum_{k=0}^{2Q+1} a_{jk}^{(i)} t^k. \tag{1.5}$$

For the q-th derivative of the function $P_{x_j}^{(i)}(t)$ we have

$$\frac{d^q}{dt^q} P_{x_j}^{(i)}(t) = \sum_{k=q}^{2Q+1} k(k-1)\ \ldots\ (k-q+1) a_{jk}^{(i)} t^{k-q}. \tag{1.6}$$

If we substitute the values $t = -1, 1$ into (1.5), (1.6), we obtain (taking into account (1.2), (1.4)) the following system of $2Q+2$ linear equations for the $2Q+2$ unknown coefficients $a_{jk}^{(i)}$ of the polynomial (1.5):

$$\begin{aligned}
&\sum_{k=0}^{2Q+1} (-1)^k a_{jk}^{(i)} = x_j^{(i)},\\
&\sum_{k=q}^{2Q+1} (-1)^{k-q} k(k-1)\ \ldots\ (k-q+1) a_{jk}^{(i)} = D^q x_j^{(i)},\\
&\qquad (q = 1, 2, \ldots, Q)\\
&\sum_{k=0}^{2Q+1} a_{jk}^{(i)} = x_j^{(i+1)},\\
&\sum_{k=q}^{2Q+1} k(k-1)\ \ldots\ (k-q+1) a_{jk}^{(i)} = D^q x_j^{(i+1)},\\
&\qquad (q = 1, 2, \ldots, Q).
\end{aligned} \tag{1.7}$$

We introduce the matrices

$$A_{ij} = (a_{j0}^{(i)}, \ldots, a_{j,2Q+1}^{(i)}), \tag{1.8}$$

$$X_{ij} = (x_j^{(i)}, Dx_j^{(i)}, \ldots, D^Q x_j^{(i)}, x_j^{(i+1)}, Dx_j^{(i+1)}, \ldots, D^Q x_j^{(i+1)}) = \tag{1.9}$$

$$= (x_j^{(i)}, X_{ij}^*, x_j^{(i+1)}, X_{i+1,j}^*),$$

(1.10)
$$X_{ij}^* = (Dx_j^{(i)}, \dots, D^Q x_j^{(i)}),$$
$$X_{i+1,j}^* = (Dx_j^{(i+1)}, \dots, D^Q x_j^{(i+1)}).$$

The matrix of the coeffients of system (1.7), which is necessarily regular in consequence of the uniqueness of the determination of the desired polynomials, is denoted by the symbol A_Q; it is a matrix of type $(2Q+ +2, 2Q+2)$. Then the solution of system (1.7) is expressed in matrix notation by the relation

(1.11)
$$A_{ij}^T = A_Q^{-1} \circ X_{ij}^T.$$

Here the superscript T denotes the transposed matrices to matrices (1.8), (1.9), and A_Q^{-1} is the inverse matrix of A_Q. By (1.11) we then have for polynomials (1.5) the expression

(1.12)
$$P_{x_j}^{(i)}(t) = (P_{x_j}^{(i)}(t)) = (1, t, \dots, t^{2Q+1}) \circ A_{ij}^T.$$

Here we have identified the type (1,1) matrix $(P_{x_j}^{(i)}(t))$ with the element $P_{x_j}^{(i)}(t)$.

2. Determination of the values $\mathrm{D}^q\mathrm{x}_j^{(i)}$, $\mathrm{D}^q\mathrm{x}_j^{(i+1)}$

Values of the first till the Q-th derivatives of the functions $P_{x_j}^{(i)}(t)$ at the points P_i, P_{i+1} (cf. (1.4)) are determinated as follows: In the plane with the rectangular coordinate system t, s_j we construct the points $(2h, x_j^{(i+h)})$, $-Q+p \le h \le Q-p$, h integer.
Here the fixed chosen integer p satisfies the inequality $0 \le p \le \le Q-1$. According to Fig. 1 the points determine uniquely the following polynomial of degree at most $2Q-2p$:

(2.1)
$${}_{Q,p}R_{x_j}^{(i)}(t) = \sum_{k=0}^{2Q-2p} {}_{Q,p}b_{jk}^{(i)} t^k.$$

With the aid of this polynomial we put

$$(2.2)\qquad \begin{aligned} Dx_j^{(i)} &= \frac{d}{dt}{}_{Q,p}R_{x_j}^{(i)}(0) = {}_{Q,p}b_{j1}^{(i)}, \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ D^Q x_j^{(i)} &= \frac{d^Q}{dt^Q}{}_{Q,p}R_{x_j}^{(i)}(0) = Q(Q-1)\ \ldots\ 3.2_{Q,p}b_{jQ}^{(i)}. \end{aligned}$$

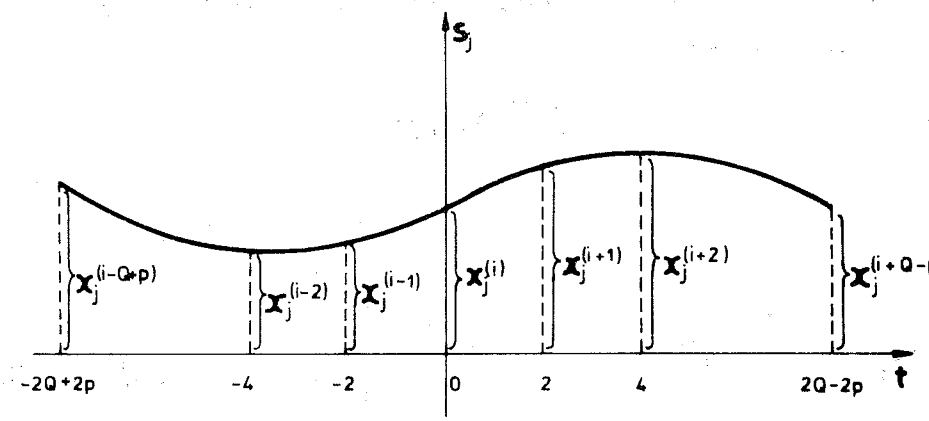

Fig. 1

The originality of Lienhard's interpolation method consists precisely in this approach to the determination of the values (2.2). The method yields the "missing" values of the derivatives in the mentioned manner from the auxiliary polynomials (2.1). In brief, we will speak of the $L_{Q,p}$ interpolation method.

Since every coefficient of the polynomial (2.1) is a certain linear combination of the values $x_j^{(i-Q+p)}, \ldots, x_j^{(i+Q-p)}$, every derivative $Dx_j^{(i)}, \ldots, D^Q x_j^{(i)}$ is also a certain linear combination of the same values. Therefore, there exists a matrix $B_{Q,p}$ of type $(Q, 2Q-2p+1)$ such that we have (see (1.10))

$$(2.3)\qquad X_{ij}^* = (x_j^{(i-Q+p)}, \ldots, x_j^{(i+Q-p)}) \circ B_{Q,p}^T.$$

Then

$$(2.4)\qquad (x_j^{(i)}, X_{ij}^*) =$$

$$
= \left(x_j^{(i-Q+p)}, \dots, x_j^{(i+Q-p)}, x_j^{(i+Q-p+1)}\right) \circ
\begin{pmatrix}
0 & \\
\vdots & \\
\vdots & B^T_{Q,p} \\
1 & \\
\vdots & \\
0 & 0\dots0
\end{pmatrix}
\begin{matrix} \\ \\ \\ Q-p+1, \\ \\ \\ \end{matrix}
$$

where the unit in the first column stands in the $(Q-p+1)$-st row. Analogously we have

$$
(2.5) \qquad (x_j^{(i+1)}, X^*_{i+1,j}) =
$$

$$
= \left(x_j^{(i-Q+p)}, \dots, x_j^{(i+Q-p)}, x_j^{(i+Q-p+1)}\right) \circ
\begin{pmatrix}
0 & 0\dots0 \\
\vdots & \\
\vdots & B^T_{Q,p} \\
1 & \\
\vdots & \\
0 &
\end{pmatrix}
\begin{matrix} \\ \\ \\ Q-p+2, \\ \\ \\ \end{matrix}
$$

where the unit in the first column stands now in the $(Q-p+2)$-nd row. Employing (2.4), (2.5) it is then possible to express (1.9) in the form

$$
X_{ij} = \left(x_j^{(i-Q+p)}, \dots, x_j^{(i+Q-p)}, x_j^{(i+Q-p+1)}\right) \circ
$$

$$
Q+2
$$

$$
(2.6) \qquad \circ
\begin{pmatrix}
0 & & 0 & 0\dots0 \\
\vdots & & \vdots & \\
\vdots & B^T_{Q,p} & \vdots & \\
1 & & 0 & B^T_{Q,p} \\
0 & & 1 & \\
\vdots & & \vdots & \\
0 & 0\dots0 & 0 &
\end{pmatrix}
\begin{matrix} \\ \\ \\ Q-p+1 \\ Q-p+2 \\ \\ \\ \end{matrix}
$$

Substitution of (2.6) into (1.11) yields

$$A_{ij}^T = C_{Q,p} \circ \begin{pmatrix} x_j^{(i-Q+p)} \\ \vdots \\ x_j^{(i+Q-p)} \\ x_j^{(i+Q-p+1)} \end{pmatrix}, \tag{2.7}$$

where

$$C_{Q,p} = A_Q^{-1} \circ \begin{pmatrix} 0 & \dots & \overset{Q-p+1}{1} & & \overset{Q-p+2}{0} & \dots & 0 \\ & & & & & & 0 \\ & & & B_{Q,p} & & & \vdots \\ & & & & & & \vdots \\ 0 & \dots & 0 & & 1 & \dots & 0 \\ 0 & & & & & & \\ \vdots & & & B_{Q,p} & & & \\ 0 & & & & & & \end{pmatrix} Q+2. \tag{2.8}$$

3. Grouping of nodes

In the case of a unclosed interpolation curve we construct groups consisting of $2Q-2p+2$ points each from the given nodes $P_1, \dots, P_n$:

$$\begin{array}{ll} \text{Group 1:} & P_{Q-p+1}, \dots, P_3, P_2, P_1, P_2, P_3, \dots, P_{Q-p+2}, \\ \text{Group 2:} & P_{Q-p}, \dots, P_2, P_1, P_2, P_3, P_4, \dots, P_{Q-p+3}, \\ \cdots\cdots & \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \text{Group (n-1):} & P_{n-Q+p-1}, \dots, P_{n-3}, P_{n-2}, P_{n-1}, P_n, \\ & P_{n-1}, \dots, P_{n-Q+p}. \end{array} \tag{3.1}$$

In this grouping we replace the eventually "missing" points by those points which are obtained by the "mirror image" of the given points with respect to the point P_1, or to the point P_n.
For instance, for $n = 3$, $Q = 2$, $p = 0$ we form the following groups of six points each:

$$\text{Group 1: } P_3, P_2, P_1, P_2, P_3, P_2,$$
$$\text{Group 2: } P_2, P_1, P_2, P_3, P_2, P_1.$$

If formula (2.7) is applied to group 1 now, we obtain by (1.12) the polynomials $P_{x_j}^{(1)}(t)$ $(j = 1, \ldots, m)$ which parametrize the arc P_1P_2. Similarly we obtain the other arcs.
Then the unclosed interpolation curve $P_1P_2 \ldots P_{n-1}P_n$ is composed of these arcs.
In the case of a closed interpolation curve $P_1P_2 \ldots P_nP_1$ we construct the nodes as follows:

$$\begin{array}{ll} \text{Group 1: } & P_{n-Q+p+1}, \ldots, P_{n-1}, P_n, P_1, P_2, P_3, \ldots, P_{Q-p+2}, \\ \text{Group 2: } & P_{n-Q+p+2}, \ldots, P_n, P_1, P_2, P_3, P_4, \ldots, P_{Q-p+3}, \\ \text{Group 3: } & P_{n-Q+p+3}, \ldots, P_1, P_2, P_3, P_4, P_5, \ldots, P_{Q-p+4}, \\ \ldots\ldots & \ldots\ldots\ldots\ldots\ldots\ldots \\ \text{Group n: } & P_{n-Q+p}, \ldots, P_{n-2}, P_{n-1}, P_n, P_1, P_2, \ldots, P_{Q-p+1}. \end{array} \tag{3.2}$$

In this grouping we replace the eventually "missing" points by those points which follow in one or other direction the point P_1, or P_n.

4. The interpolation method $L_{1,0}$

In formula (1.11) we have in this case

$$A_1^{-1} = \frac{1}{4} \begin{pmatrix} 2 & 1 & 2 & -1 \\ -3 & -1 & 3 & -1 \\ 0 & -1 & 0 & 1 \\ 1 & 1 & -1 & 1 \end{pmatrix}. \tag{4.1}$$

To determine matrix $B_{1,0}$ of type (1.3) (cf. (2.3)) we fit the polynomial ${}_{1,0}R_{x_j}^{(i)}(t)$ (cf. (2.1)) to the points $(2h, x_j^{(i+h)})$, $h = -1, 0, 1$, and put $Dx_j^{(i)} ={}_{1,0}\, b_{j1}^{(i)}$ in accordance with (2.2). Then we obtain (cf. (2.3))

$$X_{ij} = (x_j^{(i-1)}, x_j^{(i)}, x_j^{(i+1)}) \circ B_{1,0}^T, \tag{4.2}$$

where

$$B_{1,0} = \frac{1}{4}(-1, 0, 1). \tag{4.3}$$

By (2.8) and with the aid of (4.3) we than have

$$C_{10} = \frac{1}{4}\begin{pmatrix} 2 & 1 & 2 & -1 \\ -3 & -1 & 3 & -1 \\ 0 & -1 & 0 & 1 \\ 1 & 1 & -1 & 1 \end{pmatrix} \circ \frac{1}{4}\begin{pmatrix} 0 & 4 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix}. \tag{4.4}$$

Multiplication of these matrices and substitution into (2.7) yield

$$16A_{ij}^T = \begin{pmatrix} -1 & 9 & 9 & -1 \\ 1 & -11 & 11 & -1 \\ 1 & -1 & -1 & 1 \\ -1 & 3 & -3 & 1 \end{pmatrix} \circ \begin{pmatrix} x_j^{(i-1)} \\ x_j^{(i)} \\ x_j^{(i+1)} \\ x_j^{(i+2)} \end{pmatrix}. \tag{4.5}$$

Example 1. In the plane $\mathbf{R}^2$ let us consider the points $P_1 = (0,0)$, $P_2 = (2,3)$, $P_3 = (15,-6)$, $P_4 = (2,-10)$, $P_5 = (10,5)$. In Fig. 2 the unclosed interpolation curve $P_1\ P_2\ P_3\ P_4\ P_5$ is shown. The parametric equations of the individual arcs of this interpolation curve are obtained using formulas (4.5) and (1.12) (for $Q = 1$).

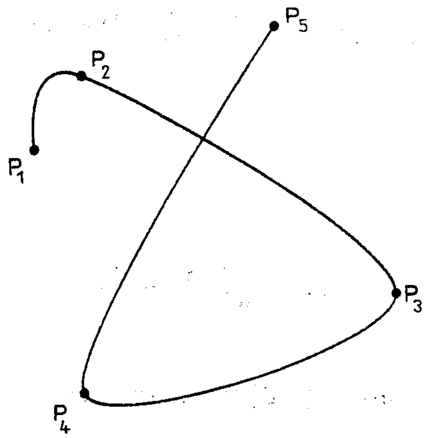

Fig. 2

It can be shown that in the cases of the interpolation methods $L_{2,0}$ and $L_{2,1}$ we have

$$(4.6)\qquad C_{2,0}=\frac{1}{768}\begin{pmatrix} 9 & -75 & 450 & 450 & -75 & 9\\ -13 & 81 & -562 & 562 & -81 & 13\\ -10 & 78 & -68 & -68 & 78 & -10\\ 18 & -106 & 228 & -228 & 106 & -18\\ 1 & -3 & 2 & 2 & -3 & 1\\ -5 & 25 & -50 & 50 & -25 & 5 \end{pmatrix},$$

and

$$(4.7)\qquad C_{2,1}=\frac{1}{32}\begin{pmatrix} -2 & 18 & 18 & -2\\ 3 & -25 & 25 & -3\\ 2 & -2 & -2 & 2\\ -4 & 12 & -12 & 4\\ 0 & 0 & 0 & 0\\ 1 & -3 & 3 & -1 \end{pmatrix}$$

respectively.

5. An alternative determination of the values $\mathbf{D}^q\mathbf{x}_j^{(i)}$, $\mathbf{D}^q\mathbf{x}_j^{(i+1)}$

Instead of (2.3) we now put

$$X_{ij}^{*} = \left(x_j^{(i-Q+p)}, \ldots, x_j^{(i+Q-p)}\right) \circ B_{Q,p}^{T} + M_Q b_j^{(i)}, \tag{5.1}$$

where $M_Q = (m_1, \ldots, m_Q)$ is a non-zero constant matrix while $b_j^{(1)}, \ldots, b_j^{(n+1)}$ are vectors which are undetermined as yet. It can be shown that

$$A_{ij}^{T} = \left(\begin{array}{|c|c|} \hline C_{Q,p} & D_Q \\ \hline \end{array} \right) \circ \begin{pmatrix} x_j^{(i-Q+p)} \\ \vdots \\ x_j^{(i+Q-p)} \\ x_j^{(i+Q-p+1)} \\ b_j^{(i)} \\ b_j^{(i+1)} \end{pmatrix}, \tag{5.2}$$

where $C_{Q,p}$ is the matrix (2.8), and further

$$D_Q = A_Q^{-1} \circ \begin{pmatrix} 0 & 0 \\ \boxed{M_Q^T} & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} \\ 0 & 0 \\ \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} & \boxed{M_Q^T} \end{pmatrix} Q+2 \tag{5.3}$$

Here A_Q^{-1} is the inverse to the matrix A_Q of the coefficients of the system (1.7).

6. The unclosed interpolation $\mathbf{L}_{Q,p}$ – spline

According to (5.2) the relation

$$P_{x_j}^{(i)}(t) = (1, t, t^2, \ldots, t^{2Q+1}) \circ A_{ij}^T \tag{6.1}$$

(see (1.12)) yields, upon differentiation and substitution of the values $t = 1, -1$, the relation

$$\frac{d^q}{dt^q} P_{x_j}^{(i)}(1) = \frac{d^q}{dt^q} P_{x_j}^{(i+1)}(-1) \tag{6.2}$$

for $q = 1, \ldots, Q$ (cf. (1.3)). Further differentiation and substitution of the values $t = 1, -1$ yield

$$\frac{d^{Q+1}}{dt^{Q+1}} P_{x_j}^{(i)}(1) = \tag{6.3}$$

$$= -G(x_j^{(i-Q+p)}, \ldots, x_j^{(i+Q-p+1)}) + f(M)b_j^{(i)} + g_1(M)b_j^{(i+1)},$$

$$\frac{d^{Q+1}}{dt^{Q+1}} P_{x_j}^{(i+1)}(-1) = \tag{6.4}$$

$$= F(x_j^{(i-Q+p+1)}, \ldots, x_j^{(i+Q-p+2)}) - g_2(M)b_j^{(i+1)} - h(M)b_j^{(i+2)};$$

here F and G are certain linear combinations of the values in the parantheses while f, g_1, g_2, h are certain linear forms of the point $M = (m_1, \ldots, m_Q) \in \mathbb{R}^Q$. Since we wish to construct an interpolation spline of degree $2Q + 1$, we compare expressions (6.3), (6.4). Then we obtain the following equality between vectors of the space $\mathbb{R}^m$:

$$f(M)b_j^{(i)} + g(M)b_j^{(i+1)} + h(M)b_j^{(i+2)} = p_j^{(i)}, \tag{6.5}$$

where $g = g_1 + g_2$, $F + G = p_j^{(i)}$. Let the form g differ from zero at least at one point, let us denote

$$Y_1 = \{M \in \mathbb{R}^Q | g(M) \neq 0\}. \tag{6.6}$$

For $i = 1, \ldots, n-1$, (6.5) is a system of n-1 linear equations in the unknown $b_j^{(1)}, \ldots, b_j^{(n+1)}$. Therefore we add to the system (6.5) the following boundary conditions (one as the first equation, the other as the last equation):

$$g(M)b_j^{(1)} + c_j b_j^{(2)} = z_j, d_j b_j^{(n)} + g(M)b_j^{(n+1)} = u_j, \tag{6.7}$$

where c_j, d_j, z_j and u_j are real numbers. Then the resulting system will be of the form

$$\begin{array}{lllllll} g(M)b_j^{(1)} & + & c_j b_j^{(2)} & & & & = z_j, \\ f(M)b_j^{(1)} & + & g(M)b_j^{(2)} & + & h(M)b_j^{(3)} & & = p_j^{(1)}, \\ & & f(M)b_j^{(2)} & + & g(M)b_j^{(3)} & + \; h(M)b_j^{(4)} & = p_j^{(2)}, \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ & & f(M)b_j^{(n-1)} & + & g(M)b_j^{(n)} & + \; h(M)b_j^{(n+1)} & = p_j^{(n-1)}, \\ & & & & d_j b_j^{(n)} & + \; g(M)b_j^{(n+1)} & = u_j. \end{array} \tag{6.8}$$

The matrix of the coefficients of system (6.8) is

$$E_{Q,p;j} = \begin{pmatrix} g(M) & c_j & 0 & \ldots & 0 & 0 & 0 \\ f(M) & g(M) & h(M) & \ldots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \ldots & f(M) & g(M) & h(M) \\ 0 & 0 & 0 & \ldots & 0 & d_j & g(M) \end{pmatrix} \tag{6.9}$$

Assume that the set

$$Y = \{M \in \mathbf{R}^Q |\ |g(M)| > |f(M)| + |h(M)|\} \tag{6.10}$$

is nonempty. Then this set is a part of the set (6.6). For an arbitrary point $M \in Y$ and for

$$|c_j| < |g(M)|, \ |d_j| < |g(M)| \tag{6.11}$$

the matrix (6.9) has a dominant main diagonal, i.e. it is regular. Then the system of equations (6.8) is uniquely solvable.

Then the individual arcs of an unclosed interpolation spline

$P_1P_2\dots P_{n-1}P_n$ are constructed as follows: We form groups of $2Q-$ $-2p+2$ points each of the given nodes $P_1, P_2, \dots, P_n$ (see (3.1)). If formula (6.5) is applied to the first group of points and to the vectors $b_j^{(1)}, b_j^{(2)}$, we obtain, by (6.1), the polynomials $P_{x_j}^{(1)}(t)$ $(j = 1, \dots, m)$ which parametrize the arc P_1P_2. Similarly we obtain the other arcs. The desired interpolation spline $P_1P_2\dots P_{n-1}P_n$ is then composed of the mentioned arcs. To be more exact, we shall speak of the so-called $L_{Q,p}$-spline of degree $2Q+1$ since we have started from the $L_{Q,p}$ Lienhard interpolation method in the derivation (see Section 2). The behaviour of this spline may be modified by the choice of the feasible point $M \in Y$ (see (6.10)), the feasible numbers c_j, d_j (see (6.11)) and the numbers z_j, u_j.

In contradistinction to the $L_{Q+1,p}$ interpolation method which uses polynomials of degree at most $2Q+3$ and which guarantees the continuity of the first till $(Q+1)$-st derivatives (see (1.3) for $q = 1, \dots Q+1$) the $L_{Q,p}$-spline has identical properties (with reference to the derivatives) while polynomials of degree at most $2Q+1$ are sufficient.

Example 2. In the plane $\mathbf{R}^2$ let us consider the same nodes as in Example 1. The unclosed interpolation $L_{1,0}$-spline $P_1P_2P_3P_4P_5$ for $m_1 = 1/12$, $c_j = d_j = z_j = u_j = 0$ is shown in Fig. 3.

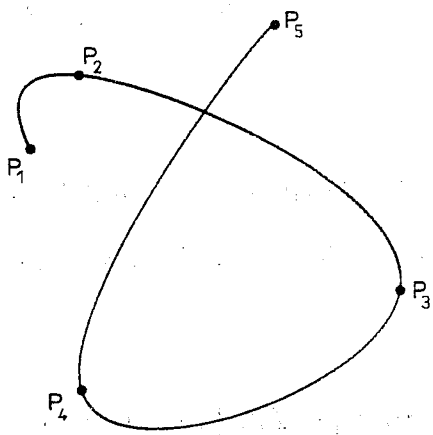

Fig. 3

7. The closed interpolation $L_{Q,p}$-spline

When constructing the closed interpolation $L_{Q,p}$-spline $P_1P_2\dots P_nP_1$ we group the nodes $P_1, P_2, \dots, P_n$ according to (3.2). To the equations (6.5) for $i = 1, \dots, n-1$ we add an additional equation for $i = n$ and two boundary conditions

$$b_j^{(1)} = b_j^{(n+1)},\ b_j^{(2)} = b_j^{(n+2)}. \tag{7.1}$$

According to (6.5), for $i = n$ and (7.1) we have

$$p_j^{(n)} =$$

$$= F(x_j^{(n-Q+p+1)}, \dots, x_j^{(n+Q-p+2)}) + G(x_j^{(n-Q+p)}, \dots, x_j^{(n+Q-p+1)}) =$$

$$= f(M)b_j^{(n)} + g_1(M)b_j^{(n+1)} + g_2(M)b_j^{(1)} + h(M)b_j^{(2)},$$

whence we obtain

$$-G(x_j^{(n-Q+p)}, \dots, x_j^{(n+Q-p+1)}) + f(M)b_j^{(n)} + g_1(M)b_j^{(n+1)} =$$
$$= F(x_j^{(n-Q+p+1)}, \dots, x_j^{(n+Q-p+1)}) - g_2(M)b_j^{(1)} - h(M)b_j^{(2)},$$

i.e. $d^{Q+1}P_{x_j}^{(n)}(1)/dt^{Q+1} = d^{Q+1}P_{x_j}^{(1)}(-1)/dt^{Q+1}$ (see (6.3), (6.4) for $i = n$).

Further, by (6.2) we have, for $i = n$, $d^qP_{x_j}^{(n)}(1)/dt^q = d^qP_{x_j}^{(1)}(-1)/dt^q$ for $q = 1, \dots, Q$. Instead of system (6.8) for the case of the unclosed interpolation $L_{Q,p}$-spline we now have the following system of equations:

$$\begin{array}{lllllll} g(M)b_j^{(2)} & + & h(M)b_j^{(3)} & & & & = p_j^{(1)} \\ f(M)b_j^{(2)} & + & g(M)b_j^{(3)} & + & h(M)b_j^{(4)} & & = p_j^{(2)}, \\ & & f(M)b_j^{(3)} & + & g(M)b_j^{(4)} & + \ h(M)b_j^{(5)} & = p_j^{(3)}, \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ & & f(M)b_j^{(n-1)} & + & g(M)b_j^{(n)} & + \ h(M)b_j^{(n+1)} & = p_j^{(n-1)}, \\ h(M)b_j^{(2)} & & & + & f(M)b_j^{(n)} & + \ g(M)b_j^{(n+1)} & = p_j^{(n)}. \end{array} \tag{7.2}$$

The matrix of coefficients of system (7.2) is

$$F_{Q,p;j} = \begin{pmatrix} g(M) & h(M) & 0 & \dots & 0 & f(M) \\ f(M) & g(M) & h(M) & \dots & 0 & 0 \\ 0 & f(M) & g(M) & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ h(M) & 0 & 0 & \dots & f(M) & g(M) \end{pmatrix}. \tag{7.3}$$

For an arbitrary point $M \in Y$ (see (6.10)) the matrix (7.3) has a dominant main diagonal, i.e. it is regular. Then the system of equations (7.2) is uniquely solvable.

Example 3. In the space $\mathbb{R}^3$ let us consider the points $P_1 = (5,0,0)$, $P_2 = (10,5,5)$, $P_3 = (0,10,15)$, $P_4 = (-5,3,8)$. The closed interpolation $L_{1,0}$-spline $P_1P_2P_3P_4P_1$ for $m_1 = 1/12$ is shown in axonometric projection in Fig. 4. For the sake of simplicity, the symbol P_i is also used here to denote the axonometric projection of a node while the symbol P_i' denotes its axonometric first projection.

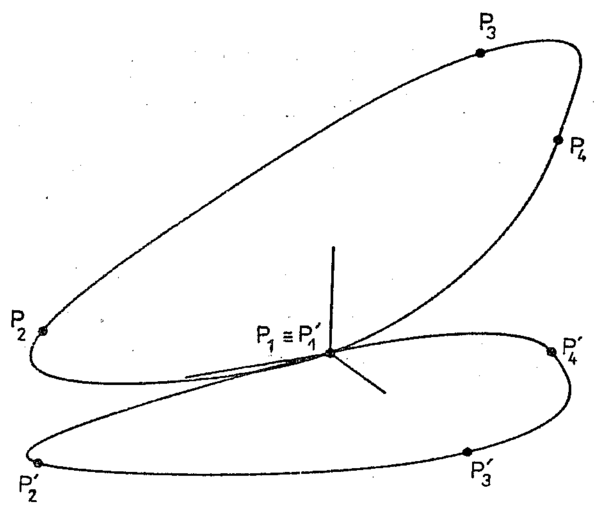

Fig. 4

Example 4. Let us construct a closed $L_{2,0}$-spline $P_1P_2P_3P_4P_1$ with the same nodes as in Example 3. For chosen $m_1 = 1/12$, $m_2 = 1/4$ the constructed curve is shown in axonometric projection in Fig. 5.

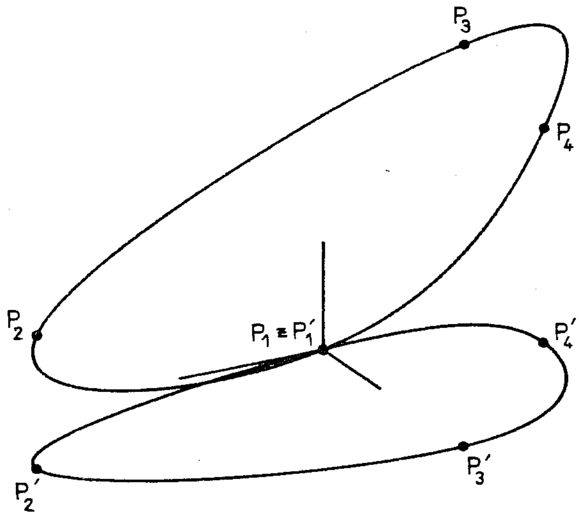

Fig. 5

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