

NOTE ON PROJECTIVE MODULES

Daniel Katz

Department of Mathematics, University of Kansas, 217 Strong Hall, Lawrence, Kansas 66045-2142, USA.

Received August 1988

AMS Subject Classification: 13 C 10, 18 G 05

Keywords: projective module, rank, trace ideal.

Abstract: We give an elementary proof of the fact that over a commutative ring a projective module of constant finite rank is finitely generated.

Let P be a projective module over a commutative ring; a well-known theorem of Kaplansky asserts that P is locally free (see [1]). This allows one to define the rank of P at any prime ideal q – it's just the cardinality of any basis for the free module P localized at q . A fact which apparently is not as well known is that P must be finitely generated if the rank of P is finite and *constant* as q varies. Vasconcelos gives a proof of this in [2] where he uses the wedge product to reduce to the rank one case and employs an idempotent argument from there. The purpose of this note is to give an elementary proof of this fact which avoids the wedge product. We use induction and a familiar localization argument to reduce to the rank zero case where the result is obvious.

Theorem. ([2], Proposition 1.3): *Let R be a commutative ring and P a projective R -module of constant finite rank. Then P is finitely generated.*

Proof. Induct on n , the constant rank of P . If $n = 0$, then P is locally zero, hence zero and therefore finitely generated (by the empty set).

Suppose $n > 0$ and the result is true for all projective modules (over all commutative rings) of constant rank less than n . Since P is projective, there exist R -modules F and K , with F free, such that $F = P \oplus K$.

Let I be the trace ideal of P i.e., the ideal of R generated by the set

$$\{y \in R \mid y = f(x) \text{ for some } x \in P \text{ and } f \in \text{Hom}(P, R)\}.$$

Then $I = R$. Indeed, suppose not. Then $I \subseteq M$ for some maximal ideal $M \subseteq R$. It readily follows that $P \subseteq MF$. Hence $P \subseteq MP \oplus MK$ so $P \subseteq MP$. Thus $P = MP$. Since P_M , i.e. P localized at M , is finitely generated, Nakayama's Lemma implies that $P_M = 0$. Thus the rank of P at M is zero, a contradiction. Therefore $I = R$.

It follows that there exist $f_i \in \text{Hom}(P, R)$ and $x_i \in P$ satisfying

$$(*) \quad 1 = f_1(x_1) + \dots + f_r(x_r).$$

We may assume that no $f_i(x_i)$ is nilpotent by shortening $(*)$ and replacing 1 by a unit in R .

Let S_i be the multiplicatively closed subset of R generated by $f_i(x_i)$. Then for all i we have induced homomorphisms on the localizations

$$\widehat{f}_i : P_{S_i} \rightarrow R_{S_i}$$

(defined by $\widehat{f}_i(P/s_i) = f_i(P)/s_i$) which, by construction, are surjective. Thus for each i , there exists a projective R_{S_i} -module K_i satisfying

$$P_{S_i} = R_{S_i} \oplus K_i.$$

Since P has constant rank n , each K_i has constant rank $n - 1$. By induction, each K_i is finitely generated, thus P_{S_i} is finitely generated (as an R_{S_i} -module). By $(*)$, P is finitely generated.

Remark. (i) The idea of using the trace ideal comes from Vasconcelos' paper [2].

(ii) Let R be a countable direct product of fields and P the corresponding countable direct sum. Then P is a projective R -module, its rank at each prime is either zero or one and P is not finitely generated.

References

- [1] KAPLANSKY, I: Projective Modules, *Ann. Math.* **68** (1958), 372 – 377.
- [2] VASCONCELOS, W: On projective modules of finite rank, *Proc. AMS* **22** (1969), 430 – 473.