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ON AN INTEGRAL INEQUALITY FOR CERTAIN ANALYTIC FUNCTIONS

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Abstract: Let g be an analytic function on the unit disc $U = \{z; |z| < 1\}$, with $g(0) = g'(0) - 1 = 0$ and let $f(z) = \int_0^z [g(t)/t]dt$. It is shown that if g satisfies the inequality $|g'(z) - 1| < 8/(2+\sqrt{15}) = 1.362\ldots$ for $z \in U$, then $|zf'(z)/f(z) - 1| < 1$, which is equivalent to $Re \int_0^1 [g(uz)/ug(z)]du > 1/2$, for $z \in U$.

1. Introduction

Let A denote the class of functions f, which are analytic on the unit disc $U = \{z; |z| < 1\}$, with $f(0) = 0$ and $f'(0) = 1$. In a recent paper we obtained the following result [3, Corollary 4.2].

If $g \in A$ satisfies $|g'(z) - 1| < 1$, for $z \in U$, then

$$Re \int_0^1 \frac{g(uz)}{ug(z)}\, du > \frac{1}{2}, \quad \text{for } z \in U.$$

If we let

$$f(z)=\int_0^1 \frac{g(uz)}{u}\,du,$$

then this last inequality is equivalent to

$$\left|\frac{zf'(z)}{f(z)}-1\right|<1, \quad \text{for } z\in U.$$

In the present paper we improve the above result, by showing that the same conclusion holds under the less restrictive condition $|g'(z)-1|<8/(2+\sqrt{15})=1.362\ldots$

2. Preliminaries

If f and g are analytic functions on U, then we say that f is subordinate to g, written $f\prec g$, or $f(z)\prec g(z)$, if g is univalent, $f(0)=g(0)$ and $f(U)\subset g(U)$.

We shall use the following lemmas to prove our results.

Lemma 1 [1,p.192]. *Let h be a convex function on U (i.e. h is univalent and $h(U)$ is a convex domain). If p is analytic in U and satisfies the differential subordination*

$$p(z)+zp'(z)\prec h(z),$$

then

$$p(z)\prec \frac{1}{z}\int_0^z h(t)\,dt.$$

Lemma 2 [2,p.201]. *Let E be a set in the complex plane $\mathbb{C}$ and let q be an analytic and univalent function on U. Suppose that the function $H:\mathbb{C}^2\times U\to\mathbb{C}$ satisfies*

$$H[q(\zeta),m\zeta q'(\zeta);z]\notin E,$$

whenever $m\geq 1$, $|\zeta|=1$ and $z\in U$. If p is analytic on U, and satisfies $p(0)=q(0)$ and

$$H[p(z),zp'(z);z]\in E, \quad \text{for } z\in U,$$

then $p \prec q$.

For use in Section 4 we need the following elementary sharp inequalities.

Lemma 3. *If* $z \in \mathbb{C}$ *then* $|\sin z| \leq \operatorname{sh} |z|$; *if* $z \in \mathbb{C}$ *and* $|z| < \pi/2$ *then* $|\tan z| \leq \tan |z|$.

3. Main results

Theorem 1. *If* $f \in A$ *satisfies*

$$|f'(z) + zf''(z) - 1| < M, \quad z \in U, \tag{1}$$

where $M \leq M_0 = 8/(2+\sqrt{15}) = 1.362\ldots$, *then*

$$\left|\frac{zf'(z)}{f(z)} - 1\right| < 1, \quad z \in U. \tag{2}$$

Proof. Since the inequality (1) can be rewritten as

$$f'(z) + zf''(z) \prec 1 + Mz,$$

by using Lemma 1, we deduce $f'(z) \prec 1 + Mz/2$ and

$$\frac{f(z)}{z} \prec 1 + \frac{M\,z}{4}. \tag{3}$$

Let $p(z) = zf'(z)/f(z)$ and $P(z) = f(z)/z$. Since (3) implies $P(z) \neq 0$, the function p is analytic in U and the inequality (1) becomes

$$|P(z)[zp'(z) + p^2(z)] - 1| < M, \quad z \in U. \tag{4}$$

The inequality (2) is equivalent to

$$p(z) \prec 1 + z \tag{5}$$

and in order to show that (5) holds, by Lemma 2, it is sufficient to check the inequality

$$|P(z)[m\zeta + (1+\zeta)^2] - 1| \geq M, \tag{6}$$

for all $m \geq 1$, $|\zeta| = 1$ and $z \in U$.

If we let $\zeta = e^{i\theta}$, then

$$\begin{aligned} L(m,\theta,z) &\equiv |P(z)[m\zeta + (1+\zeta)^2] - 1|^2 = \\ &= |P(z)\zeta(\zeta + \bar{\zeta} + m + 2) - 1|^2 = \\ &= (2\cos\theta + m + 2)\{(2\cos\theta + m + 2)|P(z)|^2 - \\ &\quad - 2Re[e^{i\theta}P(z)]\} + 1. \end{aligned}$$

From (3) we deduce $|P(z) - 1| < M/4$ and $|P(z)| > 1 - M/4$. For $m \geq 1$ we have

$$\begin{aligned} \frac{\partial L}{\partial m} &= (2\cos\theta + m + 2)|P(z)|^2 - Re[e^{i\theta}P(z)] = \\ &= (m+2)|P(z)|^2 + Re\{e^{i\theta}P(z)[2\overline{P(z)} - 1]\} \geq \\ &\geq |P(z)|\{(3|P(z)| - |2P(z) - 1|\} \geq |P(z)|(2 - \frac{5M}{4}) > 0, \end{aligned}$$

which shows that L is an increasing function of m. Hence we deduce

$$\begin{aligned} L(m,\theta,z) \geq L(1,\theta,z) &= (2\cos\theta + 3)[3|P|^2 - 2Re[e^{i\theta}P(\bar{P} - 1)] + 1 \\ &\geq (2\cos\theta + 3)|P|[3|P| - 2|P - 1|] + 1 \geq \\ &\geq \left(1 - \frac{M}{4}\right)\left[3\left(1 - \frac{M}{4}\right) - \frac{M}{2}\right] + 1 \equiv K(M). \end{aligned}$$

Since $0 < M \leq M_0$, where M_0 is the positive root of the equation $K(M) = M^2$, we deduce $L(m,\theta,z) \geq M^2$, which yields (6). Hence the subordination (5) holds and we obtain (2), which completes the proof of Theorem 1.

The following two theorems are integral versions of Theorem 1.

Theorem 2. *If $g \in A$ satisfies $|g'(z) - = 1| < M_0 = 8/(2 + \sqrt{15})$ then*

$$\left|\frac{zf'(z)}{f(z)} - 1\right| < 1, \quad \text{for } z \in U,$$

where

$$f(z) = \int_0^z \frac{g(t)}{t}\,dt = \int_0^1 \frac{g(uz)}{u}\,du.$$

Theorem 3. *If $g \in A$ satisfies $|g'(z) - 1| < M_0 = 8/(2 + \sqrt{15}$ then*

$$Re \int_0^1 \frac{g(uz)}{ug(z)}\, du > \frac{1}{2}, \quad \text{for } z \in U.$$

4. Examples

Example 1. If we let $g(z) = (\sin \lambda z)/\lambda$, where

$$|\lambda| \le ln[1 + M_0 + \sqrt{M_0(M_0 + 2)}] = 1.504\ldots$$

then, by using Lemma 3, we have

$$|g'(z) - 1| = 2|\sin^2 \frac{\lambda z}{2}| \le 2\ sh^2 \frac{|\lambda z|}{2} < 2\ sh^2 \frac{|\lambda|}{2} \le M_0,$$

for $z \in U$ and by Theorem 3 we deduce

$$Re \frac{\mathrm{Si}(z)}{\sin z} > \frac{1}{2}, \quad \text{for} \quad |z| < 1.504\ldots$$

where

$$\mathrm{Si}(z) = \int_0^1 \frac{\sin uz}{u}\, du = \int_0^z \frac{\sin t}{t}\, dt.$$

Example 2. If we let $g(z) = (e^{\lambda z} - 1)/\lambda$, where

$$|\lambda| \le ln(1 + M_0) = 0.859\ldots$$

then $|g'(z) - 1| \le M_0$, for $z \in U$ and by Theorem 3 we deduce

$$Re \int_0^1 \frac{e^{uz} - 1}{u(e^z - 1)}\, du > \frac{1}{2}, \quad \text{for} \quad |z| < 0.859\ldots$$

Example 3. If we let $g(z) = [ln(1 + \lambda z)]/\lambda$, where

$$|\lambda| \le \frac{M_0}{1 + M_0} = 0.576\ldots$$

then $|g'(z) - 1| < M_0$, for $z \in U$ and by Theorem 3 we deduce

$$Re \int_0^1 \frac{\ln(1+uz)}{u\, ln(1+z)}\, du > \frac{1}{2}, \quad \text{for} \quad |z| < 0.576\ldots$$

Example 4. If we let $g(z) = (\tan \lambda z)/\lambda$, where

$$|\lambda| \leq \arctan \sqrt{M_0} = 0.862\ldots$$

then, by Lemma 3, we have

$$|g'(z) - 1| = |\tan^2 \lambda z| \leq \tan^2 |\lambda z| < \tan^2 |\lambda| \leq M_0,$$

for $z \in U$ and by Theorem 3 we deduce

$$Re \int_0^1 \frac{\tan\, uz}{u\, \tan\, z}\, du > \frac{1}{2}, \quad \text{for} \quad |z| < 0.862\ldots$$

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