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Fully *-prime rings with involution

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Abstract:
Some known results on fully prime rings and almost fully prime rings are extended to the category of rings with involution. In particular, various properties of fully *-prime involution rings are presented, a classification of fully *-prime involution rings which satisfy a polynomial identity is given, and almost fully *-prime involution rings are characterized. The structure of the additive groups of these involution rings is also studied.

1. Introduction

Throughout the present paper all rings are associative and do not necessarily have identity. A ring with involution, or involution ring, is a ring with an additional unary operation * (called involution) such that, for all $a, b \in R$, $(a+b)^* = a^* + b^*$, $(ab)^* = b^*a^*$ and $(a^*)^* = a$. Let us recall that an ideal (biideal, subring, subfield) I of an involution ring R is called a *-ideal (*-biideal, *-subring, *-subfield) of R if I is closed under involution, that is, $I^* = \{a^* \in R : a \in I\} \subseteq I$. As usual, we say that involution rings R and T are *-isomorphic (shortly written as $R \cong_* T$) if there exists a ring isomorphism $\varphi : R \to T$ such that $\varphi(r^*) = \varphi(r)^*$ for each $r \in R$.

Rings in which every (nonzero) proper ideal satisfies a given property have been studied by several authors. For instance, in [12], Hirano studied rings in which every (nonzero) proper ideal I is completely prime

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(that is, for any $a, b \in R$, $ab \in I$ implies that $a \in I$ or $b \in I$). Blair and Tsutsui, in [4], studied fully prime rings (rings in which every proper ideal is prime). In [17], Tsutsui considered almost fully prime rings (rings in which every nonzero proper ideal is prime).

Let us recall that a *-ideal P of an involution ring R is said to be a **-prime* (**-semiprime*) *-ideal of R if $IJ \subseteq P$ ($I^2 \subseteq P$) implies that $I \subseteq P$ or $J \subseteq P$ ($I \subseteq P$), where I and J are *-ideals of R. These notions were studied in [2], where the connection between the prime ideals and the *-prime *-ideals of an involution ring was illustrated and several characterizations presented. In particular, it was shown that a *-ideal P of R is *-prime if, for a, $b \in R$, $aRb \subseteq P$ implies that $a \in P$ or $b \in P$. A *-ideal P of an involution ring R is *completely *-prime* if $ab \in P$ and $ab^* \in P$ implies that $a \in P$ or $b \in P$ (see [1]and [15]). Clearly, every completely *-prime *-ideal is a *-prime *-ideal.

In [17], Tsutsui studied *fully *-prime* involution rings, that is, involution rings in which every proper *-ideal is *-prime. He characterized these rings and considered, in particular, those which are right fully bounded right Noetherian. In this note, we continue to study fully *-prime involution rings. We also consider, in particular, involution rings in which every *-ideal is completely *-prime. These are characterized in Proposition 2.4. Examples of fully *-prime involution rings which are not fully prime are provided in Proposition 2.6. Some known results on fully prime rings are studied in the category of rings with involution. For example, the structure of fully *-prime involution rings which satisfy a polynomial identity is determined in Corollary 2.10. Involution rings in which every nonzero proper *-ideal is *-prime, called *almost fully *-prime*, are also characterized in Proposition 4.1 and some examples are presented. The additive groups of both fully *-prime and almost fully *-prime involution rings are classified in Proposition 2.13 and Proposition 4.6, respectively.

2. Some properties and related concepts

Let R be an involution ring and $a \in R$. Henceforth, the ideal of R generated by a will be denoted by $\langle a \rangle$ and the *-ideal of R generated by a will be denoted by $\langle a \rangle_*$. Clearly, $\langle a \rangle_* = \langle a \rangle + \langle a^* \rangle$.

We note below that, in an involution ring, every *-ideal is idempotent if and only if every ideal is idempotent.

Proposition 2.1. For an involution ring R, the following statements are equivalent.

(i) Every proper *-ideal of R is semiprime (that is, a semiprime *-ideal);

(ii) Every *-ideal of R is idempotent;

(iii) Every ideal of R is idempotent;

(iv) For arbitrary *-ideals I and J of R, $I \cap J = IJ$;

(v) For every $a \in R$, $a \in \langle a \rangle_*^2$.

Proof. (i) implies (ii). Suppose that $I \neq I^2$ for some *-ideal I of R. Then I/I^2 is a nonzero nilpotent *-ideal of R/I^2 and hence the proper *-ideal I^2 is not semiprime. (ii) implies (iii). If every *-ideal of R is idempotent, then, for any nonzero ideal I of R, $(I + I^*)/I^* \cong I/(I \cap I^*)$ is idempotent. Hence $I \subseteq I^2 + (I \cap I^*) = I^2 + (I \cap I^*)^2 \subseteq I^2 + II^* = I^2 + (II^*)^2 \subseteq I^2$.

It is easily seen that (iii) implies (iv) and that (iv) implies (v).

(v) implies (i). Let I be proper *-ideal of R which is not semiprime. Then there exists a *-ideal J of R such that $J^2 \subseteq I$ and $J \nsubseteq I$. Hence there exists $0 \neq a \in J$ such that $a \notin I$. Using (v), we have that $a \in \langle a \rangle_*^2 \subseteq J^2 \subseteq I$, which is a contradiction. $\lozenge$

The following corollary is an immediate consequence of the previous proposition and [17, Theorem 3.1].

Corollary 2.2. An involution ring R is fully *-prime if and only if every ideal of R is idempotent and the set of *-ideals of R is linearly ordered.

Proposition 2.3. Every *-ideal of an involution ring R is completely prime (completely semiprime) if and only if every ideal of R is completely prime (completely semiprime).

Proof. Suppose that every *-ideal of R is completely prime and let I be an ideal of R. Let $a, b \in R$ such that $ab \in I$. Then $ab \in I + I^*$, which implies that $a \in I + I^*$ or $b \in I + I^*$. If $a = i + j$ for some $i \in I$ and $j \in I^*$, then $ab = (i + j)b$ and so $jb \in I \cap I^*$. Therefore $j \in I \cap I^*$ or $b \in I \cap I^*$. So $a \in I$ or $b \in I$. If $b \in I + I^*$, we may conclude in a similar way that $a \in I$ or $b \in I$. The statements on completely semiprime are shown analogously. $\lozenge$

In [1], Aburawash introduced and studied the concept of *-zero-divisor and related concepts. In [15], the author of the present paper, unware of Aburawash's paper, studied involution rings without *-zero divisors, calling them *-domains and presented other characterizations and results concerning these involution rings. A **-domain* is an involution ring R such that for any nonzero $a, b \in R$, $ab \neq 0$ or $ab^* \neq 0$. It was shown, in particular, that an involution ring R is a *-domain if and only if R has neither symmetric zero-divisors nor skew-symmetric zero-divisors. Recall that an element $a \in R$ is said to be symmetric if $a^* = a$ and skew-symmetric if $a^* = -a$. An involution ring R is a *-domain if and only if R is reduced and *-prime. A proper *-ideal I of R is completely *-prime if and only if R/I is a *-domain. Henceforth, an involution ring R with the property that every proper *-ideal is completely *-prime, will be called a *strong *-domain.* A strong *-domain is a fully *-prime involution ring. Below, a strong domain is a ring in which every proper ideal is completely prime. Moreover, if a is an element of an involution ring R, then $RaR = \{\sum_{i=1}^{n} r_i a s_i : n \in \mathbb{N}, r_1, ..., r_n, s_1, ..., s_n \in R\}$.

Proposition 2.4. The statements below are equivalent for an involution ring R.

(i) R is a strong *-domain;

(ii) The set of *-ideals of R is linearly ordered and $a \in \langle a^2 \rangle_*$ for each $a \in R$;

(iii) The set of *-ideals of R is linearly ordered and $a \in Ra^2R + R(a^*)^2 R$ for each $a \in R$;

(iv) The set of *-ideals of R is linearly ordered and $a \in Ra^nR + R(a^*)^n R$ for each $a \in R$ and for each positive integer $n \geq 1$;

(v) The set of *-ideals of R is linearly ordered and every proper ideal of R is completely semiprime.

Proof. (i) implies (ii). Let $a \in R$ and suppose that $\langle a^2 \rangle_* \neq R$. Then $a(aa^*) \in \langle a^2 \rangle_*$ implies that $a \in \langle a^2 \rangle_*$, since $\langle a^2 \rangle_*$ is completely *-prime. To prove that the set of *-ideals of R is linearly ordered, let I and J be *-ideals of R and suppose that $I \nsubseteq J$ and $J \nsubseteq I$. Then there exist $a \in I$ such that $a \notin J$ and $b \in J$ such that $b \notin I$. Therefore $ab, ab^* \in I \cap J$

and $a, b \notin I \cap J$, which means that the *-ideal $I \cap J$ is not completely *-prime.

(ii) implies (i). Let P be a proper *-ideal of R and a and b be symmetric or skew-symmetric elements in R such that $ab \in P$. Then, for any $x \in R$, $bxa \in \langle bxa \rangle_* = \left\langle (bxa)^2 \right\rangle_* \subseteq P$ and hence $\langle b \rangle_* \langle a \rangle_* \subseteq P$. Since the set of *-ideals of R is linearly ordered, we may assume, without loss of generality, that $\langle b \rangle_* \subseteq \langle a \rangle_*$. Then $\langle b \rangle_* = \langle b^2 \rangle_* \subseteq \langle b \rangle_* \langle b^2 \rangle_* \subseteq \langle b \rangle_* \langle a \rangle_*$ and so $b \in P$.

(ii) implies (iii). If (ii) holds, then $a \in \langle a^2 \rangle_*$ and $a^2 \in \langle a^4 \rangle *$, for each $a \in R$. Hence $a \in \langle a^2 \rangle_* \subseteq \langle a^4 \rangle_* \subseteq Ra^2R + R(a^*)^2R$.

It can be easily seen that statements (iii) and (iv) are equivalent and that (iii) implies (ii).

(i) implies (v). By [15, Proposition 5], if every proper *-ideal of R is completely *-prime, then every proper *-ideal of R is completely semiprime. Hence, from the previous proposition, every ideal of R is completely semiprime.

It follows from [9, Proposition 1.2] that (v) implies (iii). $\Diamond$

For any ring R, let R^{op} denote the opposite ring of R. As is well known, $R \oplus R^{op}$ is a ring with involution given by $(a, b)^* = (b, a)$, called the exchange involution. From [5, Theorem 2.7], every ideal of $R \oplus R^{op}$ is idempotent if and only if every ideal of R is idempotent, and the *-ideals of $R \oplus R^{op}$ are of the form $I \oplus I^{op}$, where I is an ideal of R. By [12, Theorem 2], there exist strong domains with exactly n proper ideals, for each positive integer n. So, the next proposition yields examples of strong *-domains which are not strong domains.

Proposition 2.5. Let R be a ring. Then $R \oplus R^{op}$, with the exchange involution, is a strong *-domain if and only if R is a strong domain.

Proof. Let J be a proper *-ideal of $R \oplus R^{op}$, where R is a strong domain. Then $J = I \oplus I^{op}$ where I is an ideal of R. Let $(a, b)\,(c, d) \in J$ and $(a, b)\,(d, c) \in J$; that is, $(ac, db) \in J$ and $(ad, cb) \in J$, where ac, db, ad, $cb \in I$. As I is a completely prime ideal of R, we can deduce that $(a, b) \in I \oplus I^{op}$ or $(c, d) \in I \oplus I^{op}$. The converse is clear. $\Diamond$

Proposition 2.6. Let R be a ring. Then $R \oplus R^{op}$, equipped with the exchange involution, is fully *-prime if and only if R is fully prime.

Proof. Every ideal of $R \oplus R^{op}$ is idempotent if and only if every ideal of R is idempotent. The set of ideals of R is linearly ordered if and only if the set of *-ideals of $R \oplus R^{op}$ is linearly ordered. Hence, taking into account Corollary 1, we have the desired result. $\Diamond$

In [13], Leavitt and van Leeuwen studied the structure of rings R with the property that $R/I \cong R$ for each proper ideal I of R. In particular, they proved that such a ring R is either fully prime or is a zero ring (that is, $R^2 = 0$), and they also constructed a non-simple, non-nilpotent ring with this property. We shall now consider the analoguous condition for rings with involution:

($\Diamond$) $R/I \cong_* R$ for each proper *-ideal I of R.

If R is an idempotent ring such that $R/I \cong R$ for each proper ideal I of R, then the ring $R \oplus R^{op}$, endowed with the exchange involution, satisfies property ($\Diamond$). In fact, a proper *-ideal of $R \oplus R^{op}$ is of the form $I \oplus I^{op}$, where I is a proper ideal of R, and $(R \oplus R^{op}) / (I \oplus I^{op}) \cong_* (R/I) \oplus (R/I)^{op} \cong_* R \oplus R^{op}$. In order to show that idempotent involution rings satisfying condition ($\Diamond$) are fully *-prime, we require the following:

Lemma 2.7. Let R be a ring with involution and $Ann_*(R) = \{x \in R : RxR = 0\}$. If R is idempotent and satisfies ($\Diamond$), then $Ann_*(R) = 0$.

Proof. Clearly, $Ann_*(R)$ is a proper *-ideal of R. Let $\overline{R} = R/Ann_*(R)$ and $\overline{x} = x + Ann_*(R) \in Ann_*(\overline{R})$. Then $\overline{R}\overline{x}\overline{R} = \overline{0}$, whence $RxR \subseteq Ann_*(R)$. This means that $R(RxR)R = 0$. Since R is idempotent, it follows that $RxR = 0$ or, equivalently, $x \in Ann_*(R)$. Hence $\overline{x} = \overline{0}$ and, since $R/Ann_*(R) \cong_* R$, it follows that $Ann_*(R) = 0$. $\Diamond$

Let us recall that an involution ring R is called *-subdirectly irreducible if the intersection of all its nonzero *-ideals (called the *-heart of R) is nonzero. A routine application of Birkhoff's theorem yields that every involution ring R is a subdirect product of *-subdirectly irreducible rings.

Proposition 2.8. Let R be an idempotent ring with involution satisfying property ($\Diamond$). Then R is fully *-prime.

Proof. By our assumption, it is clear that R is *-subdirectly irreducible. Let H be the *-heart of R. It is well known that $\beta(R)$, the prime radical of R, is a *-ideal of R. If $R \neq \beta(R)$, then $R/\beta(R) \cong R$ has no nonzero nilpotent ideals. This impiles that $H^2 \neq 0$ and therefore R is *-prime.

Now suppose that $R = \beta(R)$. By the previous lemma, $RxR \neq 0$ for any $0 \neq x \in H$. Clearly, $RxR \subseteq H \subseteq RxR$ and so $H = RxR$. Hence $x = \sum_{i=1}^{n} a_i x b_i$, for some positive integer n and $a_i, b_i \in R$. Every subring generated by a finite subset of a β-radical ring is nilpotent, since the β-radical is contained in the Levitzki radical (see [6]). This implies that $x = 0$, which is a contradiction. $\Diamond$

In what follows, let $Z(R)$ denote the centre of a ring R. In [4], Blair and Tsutsui showed that a fully prime ring with nonzero centre has identity and that its centre is a field. We now present the involutive version of this result.

Proposition 2.9. Let R be a fully *-prime involution ring with centre $Z(R) \neq 0$. Then R has identity, and either $Z(R)$ is a field or $Z(R) = F \oplus F^*$ where F and F^* are fields.

Proof. We first show that R has identity. Let a be a nonzero symmetric or skew-symmetric element in $Z(R)$. Then $aRa = a^2R$ is a nonzero *-ideal of R. Since the *-ideal a^2R is *-prime, $aRa \subseteq a^2R$ implies that $a \in a^2R$. Therefore $a = a^2b$ for some $b \in R$. If $e = ab$ and $r \in R$, then $a\,(er - re) = a\,(r^*e^* - e^*r^*) = 0$. Therefore $aR\,(er - re) = aR\,(r^*e^* - e^*r^*) = 0$ and hence $er = re$. Similarly, $a\,(r - re) = a\,(r^* - e^*r^*) = 0$, whence $r = re$. Thus $e = 1$, the identity of R, and a is invertible in R. Now, $a\,(br - rb) = a\,(r^*b^* - b^*r^*) = 0$ implies that $aR\,(br - rb) = aR\,(r^*b^* - b^*r^*) = 0$ and so $br - rb = 0$. Thus $b \in Z(R)$. Moreover, b is either symmetric or skew-symmetric. Indeed, since $ab = 1$, we have $abb^* = b^*$. Thus $(abb^*)^* = (b^*)^*$, that is, $bb^*a^* = b$. Now it is clear that if a is symmetric (skew-symmetric), then b is symmetric (skew-symmetric). Since all nonzero symmetric and skew-symmetric elements in $Z(R)$ are invertible, these are not zero divisors. Hence $Z(R)$ is a *-domain. Therefore, by [15, Theorem 10] , $Z(R)$ is either a domain or has a nonzero ideal F such that F and F^* are domains, $F \cap F^* = 0$ and $F \oplus F^*$ is a *-essential *-ideal in $Z(R)$. If $Z(R)$ is a domain, then, for any nonzero element $a \in Z(R)$, $aa^* \neq 0$. Hence aa^* is invertible in $Z(R)$ and therefore a is also invertible in $Z(R)$. Finally, we consider the case when $Z(R)$ is not a domain and show that F is a field. If c is a nonzero element in F, then c is neither symmetric nor skew-symmetric, and thus $c + c^*$ is a nonzero symmetric element in $Z(R)$. Therefore, as seen above, $c + c^*$ is invertible, that is, there exists $d \in Z(R)$ such that $(c + c^*)\,d = 1$, where d is symmetric.

Therefore $c\,(c+c^*)\,d = c$, that is, $c^2 d = c$. Now $f = cd$ is a nonzero idempotent in F and, for any $x \in F$, $(xf - x)\,f = 0$. This implies that F has identity element f. Hence the *-essential *-ideal $F \oplus F^*$ has identity $f + f^*$. From [14, Lemma 8], it follows that $F \oplus F^* = Z(R)$. It remains to show that every element in F is invertible. Since $f = cd$, where $d = d_1 + d_2$ for some $d_1 \in F$ and $d_2 \in F^*$, $f = c\,(d_1 + d_2)$. Hence $f - cd_1 = cd_2 = 0$ and so $f = cd_1$. Therefore c is invertible in F. $\Diamond$

We may therefore conclude that a commutative fully *-prime involution ring is either a field or a direct sum of the form $F \oplus F$ (where F is a field), endowed with the exchange involution. More generally, the structure of fully *-prime involution rings which satisfy a polynomial identity is given below. The non-involutive analogue may be found in [4, Theorem 3.3].

Corollary 2.10. Let R be an involution ring. Then R is fully *-prime satisfying a polynomial identity if and only if either $R \cong_* M_n(D)$ or $R \cong_* M_n(D) \oplus M_n(D)^{op}$ (endowed with the exchange involution), where $M_n(D)$ is a full matrix ring over a division ring which is finite dimensional over its centre.

Proof. Let R be fully *-prime. Since R is a semiprime ring satisfying a polynomial identity, we have, by [11, Theorem 1.4.2], that $Z(R) \neq 0$. Hence, from the previous proposition, R has identity. Since R is *-prime, by [2, Theorem 4.2], R is either a prime ring or R has a nonzero ideal P such that P and P^* are prime rings, $P \cap P^* = 0$ and $P \oplus P^*$ is a *-essential *-ideal of R. If R is prime, then $Z(R)$ is a domain and hence, by the previous proposition, $Z(R)$ is a field. Therefore, by [16, Corollary 1.6.28], R is a simple Artinian involution ring. Thus R is a square matrix ring over a division ring which is finite dimensional over its centre. If R is not prime, then $Z(P \oplus P^*) = Z(P) \oplus Z(P^*)$, where $Z(P)$ and $Z(P^*) = Z(P)^*$ are fields. Since P and P^* are prime rings satisfying a polynomial identity, they are simple Artinian rings. As $P \oplus P^*$ has identity and is a *-essential *-ideal of R, it follows that $P \oplus P^* = R$ and thus the result holds.

Conversely, if $R \cong_* M_n(D)$ or $R \cong_* M_n(D) \oplus M_n(D)^{op}$, then R is *-prime and has no nonzero proper *-ideals. Since D is finite dimensional over its centre, R satisfies a polynomial identity. $\Diamond$

The next corollary is the involutive version of [12, Proposition 1].

Corollary 2.11. Let R be an involution ring. Then R is a strong *-domain satisfying a polynomial identity if and only if R is either a division ring which is finite dimensional over its centre, or R is the direct sum of such a division ring and its opposite, endowed with the exchange involution.

Proof. If R is a *-domain, then it is either a domain or it has nonzero ideal P such that P and P^* are domains, $P \cap P^* = 0$ and $P \oplus P^*$ is a *-essential *-ideal of R [15, Theorem 10]. The result is now clear from the proof of the previous corollary. $\Diamond$

As a result of the next lemma, we can describe the structure of the additive groups of fully *-prime involution rings. In the subsequent, the characteristic of a ring R shall be shortly written as $char(R)$ and the additive group of R will be denoted by R^+.

Lemma 2.12. If R is a *-prime involution ring, then either $char(R) = p$, where p is a prime, or R^+ is torsion-free.

Proof. Suppose that R^+ is not torsion-free. Then there exists $0 \neq a \in R$ and a least positive integer n such that $na = 0$. For any $b \in R$, $0 = (na)\,Rb = aR\,(nb) = aR\,(nb^*)$ implies that $nb = 0$ and so R has finite characteristic n. Moreover, n is prime. In fact, if $n = n_1 n_2$ for integers n_1, n_2 such that $1 < n_1 < n$ and $1 < n_2 < n$, then $0 = (na)\,Ra = (n_1 a)\,R\,(n_2 a) = (n_1 a)\,R\,(n_2 a^*)$. This implies that $n_1 a = 0$ or $n_2 a = 0$, which is a contradiction. $\Diamond$

The additive groups of strong domains were determined in [12, Corollary 1]. The structure of the additive groups of fully *-prime involution rings is given below. The cyclic additive group of order p is denoted by $\mathbb{Z}(p)$.

Proposition 2.13. Let G be an abelian group. Then G is the additive group of a fully *-prime involution ring if and only if either $G \cong \oplus_\alpha \mathbb{Q}$ or $G \cong \oplus_\alpha \mathbb{Z}(p)$ for some prime p and cardinal α.

Proof. Let G be the additive group of a fully *-prime involution ring R. Suppose that $G = R^+$ is torsion-free. Then, for every integer n, $nR = (nR)^2 = n^2 R$, which implies that $R = nR$. Thus R^+ is an abelian torsion-free divisible group. By [7, Theorem 23.1], $R^+ \cong \oplus_\alpha \mathbb{Q}$ for some cardinal α. On the other hand, if $char(R) = p$, then R^+ is an elementary abelian p-group. From [7, Theorem 17.2], $R^+ \cong \oplus_\alpha \mathbb{Z}(p)$ for some prime p and cardinal α.

Conversely, if either $G \cong \oplus_\alpha \mathbb{Q}$ or $G \cong \oplus_\alpha \mathbb{Z}(p)$ for some prime p and cardinal α, then G is the additive group of a field, which is obviously a fully *-prime involution ring. $\Diamond$

3. Further properties of fully *-prime rings

We recall that an involution ring is said to be *-simple if it has no nonzero proper *-ideals. The next proposition shows that any *-domain, which is neither a division ring nor the direct sum of a division ring and its opposite, contains a *-subring which is fully *-prime and is not *-simple. The following results are the involutive versions of those in [4, 4]. Let us first note that every nonzero *-biideal B of a *-domain R is such that $B^2 \neq 0$. Indeed, B cannot contain symmetric or skew-symmetric zero divisors. Hence, if R is *-simple with identity, then $B^2 \subseteq BRB = B\left(RB^2RB^2R\right)B = (BRB)^3 \subseteq B^3$, and so $B^2 = BRB$ is an idempotent *-biideal of R contained in B.

Proposition 3.1. Let R be a *-domain with identity. The conditions below are equivalent.

(i) R is *-simple;

(ii) $S = B + Z(R)$ is a fully *-prime involution ring for every idempotent *-biideal B of R;

(iii) $S = aRa^* + Z(R)$ is a fully *-prime involution ring for every $a \in R$.

Proof. (i) implies (ii). Suppose that R is *-simple and that B is a proper idempotent *-biideal of R. If $B = 0$, then S is either a field or $S = F \oplus F^*$, where F is a field. If $B \neq 0$, then B is a *-ideal of S and $B \cap Z(R) = 0$. Indeed, if $B \cap Z(R) \neq 0$, then there exists a nonzero symmetric or skew-symmetric element $a \in B \cap Z(R)$ which is invertible in R. Hence, for any $r \in R$, $r = aa^{-1}ra^{-1}a \in B$ so that $B = R$; a contradiction. Thus $S/B \cong Z(R)$ so that B is a maximal *-ideal of S. Furthermore, B is the only nonzero proper *-ideal of S. In fact, if J is a nonzero *-ideal of S, then $B = B^2 \subseteq BRB = B\left(RBJBR\right)B = (BRB)\,J\,(BRB) \subseteq BJB \subseteq J$. The maximality of B implies that $B = J$. Since $S/B \cong Z(R)$, S/B is *-prime. Clearly, S is also *-prime.

(ii) implies (iii). Clearly, for any $a \in R$, aRa^*, is an idempotent *-biideal.

(iii) implies (i). Let I be a nonzero *-ideal of R and let $0 \neq a \in I$ be symmetric or skew–symmetric. If $S = aRa + Z(R)$ is fully *-prime, then $aRa = aRa^2Ra$, where aRa is a *-ideal of S. Therefore, for any $r \in R$, $ara = a\left(\sum_{i=1}^n s_i a^2 t_i\right) a$ for some positive integer n and $s_i, t_i \in R$. Hence $a\left(r - \sum_{i=1}^n s_i a^2 t_i\right) a = 0$, which implies that $r = \sum_{i=1}^n s_i a^2 t_i$, since R is a *-domain. Thus $r \in I$ and so $R = I$. ◊

Proposition 3.2. Let R be a fully *-prime involution ring R with identity and P a proper *-ideal of R. If $Z(R)$ is a field, let $S = P + K$ where K is a *-subfield of $Z(R)$, and if $Z(R) = F \oplus F^*$ (F a field), let $S = P + K \oplus K^*$ where K is a subfield of F. Then S is a fully *-prime involution ring whose proper *-ideals are precisely the *-ideals of R contained in P.

Proof. Let $P \neq 0$ and suppose that I is a proper *-ideal of S. Then $I \subseteq P$. In fact, if $I \nsubseteq P$, then $S/I = (I + P)/I \cong P/(I \cap P)$, since P is a maximal *-ideal of S. Since $P/(I \cap P)$ is a *-essential *-ideal of $R/(I \cap P)$ and $P/(I \cap P)$ has identity, it follows that $P/(I \cap P) = R/(I \cap P)$, whence $P = R$; a contradiction. Therefore, every proper *-ideal of S is a *-ideal of R and so the set of *-ideals of S is linearly ordered and every *-ideal of S is idempotent. ◊

In what follows, let $\mathcal{R}$ be a hereditary radical property of associative rings (in the sense of rings without involution) such that the $\mathcal{R}$-radical $\mathcal{R}(R)$ of R is a *-ideal, for every ring R with involution. As usual, a ring is $\mathcal{R}$-semisimple if it has no nonzero ideals with the property $\mathcal{R}$. For details regarding radical theory of rings, we refer the reader to [6] and [8].

Proposition 3.3. Let R be a fully *-prime involution ring with identity, P be a proper *-ideal of R and $S = P + Z(R)$. If $\mathcal{R}(S)$ is a proper *-ideal of S, then either $\mathcal{R}(S) = \mathcal{R}(R)$ or $\mathcal{R}(S) = P$.

Proof. Taking into account the previous proposition and the fact that $\mathcal{R}$ is hereditary, we have $\mathcal{R}(P) = \mathcal{R}(S) \cap P \subseteq \mathcal{R}(R) \cap P = \mathcal{R}(P)$. The result follows from the fact that the *-ideals of R are linearly ordered. ◊

Corollary 3.4. Let R be a fully *-prime involution ring with identity and P be a nonzero proper *-ideal of R. Then R is $\mathcal{R}$-semisimple if and only if $S = P + Z(R)$ is $\mathcal{R}$-semisimple.

4. Almost fully *-prime rings

In this section, we consider almost fully *-prime involution rings and almost strong *-domains, which are the *-analogues of almost fully prime rings and almost strong domains, respectively. An almost strong *-domain is an involution ring in which every nonzero proper *-ideal is completely *-prime. The proofs are analogous to those of [17, Theorem 2.1 and Theorem 2.2] and [12, Theorem 4], but we outline the proof for ease of reading.

Proposition 4.1. An involution ring R is almost fully *-prime if and only if one of the following holds:

(i) R is fully *-prime;

(ii) R is *-simple and $R^2 = 0$;

(iii) R is *subdirectly irreducible with *-heart H such that $H^2 = 0$ and R/H is fully *-prime;

(iv) R is the direct sum of two *-simple *-prime involution rings;

(v) There are two fully *-prime involution rings A and B that are *-subdirectly irreducible with *-hearts P and Q, respectively, such that there is a *-isomorphism $\varphi : A/P \to B/Q$ and $R \cong_* \{(a,b) \in A \times B : \varphi(a+P) = b+Q\}$.

Proof. Let R be almost fully *-prime and suppose that R is not *-prime. If R has no nonzero proper *-ideals, then R satisfies (ii). Now assume that R has a nonzero proper *-ideal. If the set of *-ideals of R is linearly ordered, then R satisfies (iii). If, on the other hand, the set of *-ideals of R is not linearly ordered, then there exist *-ideals P and Q such that $P \nsubseteq Q$ and $Q \nsubseteq P$. Then the *-ideal $P \cap Q$ is not a *-prime *-ideal and so $P \cap Q = 0$. Moreover, these are minimal *-ideals of R and they are the only minimal *-ideals of R. If $R = P + Q$, then (iv) holds. Finally, if $R \neq P + Q$, let $A = R/Q$ and $B = R/P$, which are fully *-prime. Since $P \cap Q = 0$, P and Q can be viewed as *-ideals of A and B, respectively. Moreover, $A/P \cong_* R/(P+Q) \cong_* B/Q$. Let $\phi : A/P \to B/Q$ be a *-isomorphism. Consider the injective*-homomorphism $f : R \to A \times B$ defined by $f(a) = (a+Q, a+P)$. Then $f(R) = \{(a,b) \in A \times B : \varphi(a+P) = b+Q\}$ and hence (v) holds.

Conversely, if R satisfies any of the conditions (i)-(iv), then it is evident that R is almost fully *-prime. Finally, if R satisfies (v), then its nonzero proper *-ideals are

$$(P,0) = \{(p,0) \in A \times B : p \in P\},$$

$$(0,Q) = \{(0,q) \in A \times B : q \in Q\},$$

and

$$H(I) = \{(a,b) \in A \times B : a \in I, \varphi(a+P) = b+Q\},$$

where I runs over all proper *-ideals of A containing P. Since $R/(P,0) \cong_* B$, $R/(0,Q) \cong_* A$ and $R/H(I) \cong_* A/I$, these nonzero *-ideals are all *-prime. $\Diamond$

The proof of the next proposition is analogous to the proofs of Proposition 2.6 and Proposition 2.5.

Proposition 4.2. Let R be an almost fully prime ring (almost strong domain) with identity. Then $R \oplus R^{op}$, equipped with the exchange involution, is almost fully *-prime (an almost strong *-domain).

Reasoning as in Proposition 4.1, we obtain the following characterization of almost strong *-domains.

Proposition 4.3. Let R be an involution ring. Then R is an almost strong *-domain if and only if one of the following holds:

(i) R is a strong *-domain;

(ii) R is *-simple with symmetric or skew-symmetric zero-divisors;

(iii) R is not a *-domain and R is *-subdirectly irreducible with *-heart H and R/H is a strong *-domain;

(iv) R is the direct sum of two *-simple *-domains;

(v) There are two strong *-domains A and B that are *-subdirectly irreducible with *-hearts P and Q, respectively, such that there is a *-isomorphism $\varphi : A/P \to B/Q$ and $R \cong_* \{(a,b) \in A \times B : \varphi(a+P) = b+Q\}$.

Next, we provide examples of involution rings satisfying (iii) of the previous proposition.

Example 4.4. If R is a commutative ring, then

$$S=\left\{\begin{bmatrix} a & b \\ 0 & a \end{bmatrix} : a, b \in R\right\},$$

equipped with the usual addition and multiplication of matrices and with involution defined by $\begin{bmatrix} a & b \\ 0 & a \end{bmatrix}^* = \begin{bmatrix} a & -b \\ 0 & a \end{bmatrix}$, has a unique nonzero proper *-ideal

$$H=\left\{\begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix} : b \in R\right\},$$

$H^2 = 0$ and S is an almost strong *-domain.

An n-chain ring is a ring with exactly n proper ideals which are linearly ordered. Hirano, in [12, Theorem 2], showed the existence of n-chain strong domains, for each positive integer n. Now, if R is an n-chain strong domain, then, $R \oplus R^{op}$, is a *-n-chain *-domain (the number of *-ideals of R is n and the *-ideals of R are linearly ordered).

Example 4.5. [12, Example 2] Let R be a *-n-chain *-domain with unique minimal *-ideal H. Then the ring

$$S=\left\{\begin{bmatrix} a & h \\ 0 & a \end{bmatrix} : a \in R, h \in H\right\},$$

with the usual addition and multiplication of matrices and with involution defined by $\begin{bmatrix} a & h \\ 0 & a \end{bmatrix}^* = \begin{bmatrix} a^* & h^* \\ 0 & a^* \end{bmatrix}$, has a unique nonzero *-ideal

$$K=\left\{\begin{bmatrix} 0 & h \\ 0 & 0 \end{bmatrix} : h \in H\right\},$$

$K^2 = 0$ and S/K is a *-domain. Hence, S is an almost strong *-domain which is not *-prime.

Finally, we describe the structure of the additive groups of almost fully *-prime involution rings. Note that the structure of the additive groups of almost strong domains (rings in which every nonzero proper ideal is completely prime) was described in [12, Corollary 3].

Proposition 4.6. If R is an almost fully *-prime involution ring, then one of the following holds:

(a) $R^+ \cong \oplus_\alpha \mathbb{Q}^+$ where α is a cardinal;

(b) $R^+ \cong \oplus_\alpha \mathbb{Z}(p)$ where p is a prime and α is a cardinal;

(c) $R^+ \cong (\oplus_\alpha \mathbb{Q}^+) \oplus (\oplus_\beta \mathbb{Z}(p))$ where p is a prime and α and β are cardinals;

(d) $R^+ \cong (\oplus_\alpha \mathbb{Z}(p)) \oplus (\oplus_\beta \mathbb{Z}(q))$ where p and q are distinct primes and α and β are cardinals;

(e) $R^+ \cong \oplus_\alpha \mathbb{Z}(p^2)$ where p is a prime and α is a cardinal;

(f) $R^+ \cong (\oplus_\alpha \mathbb{Z}(p)) \oplus (\oplus_\beta \mathbb{Z}(p^2))$ where p is a prime and α and β are cardinals;

Proof. If R satisfies condition (i) of Proposition 4.1, then either (a) or (b) holds, according to Proposition 2.13.

Now suppose that R satisfies condition (ii) of Proposition 4.1. Then, by [3, Corollary 2.3], R is simple and $R^+ \cong \mathbb{Z}\,(p)$ where p is a prime.

Next, assume that R satisfies condition (iii) of Proposition 4.1. Then R is *-subdirectly irreducible with *-heart H such that $H^2 = 0$. Note that since R/H is fully *-prime, $R/H = (R/H)^2$ and so $R = R^2$. Moreover, every *-ideal properly containing H is idempotent. Suppose first that $char(R) = 0$. Then $char(R/H) = 0$. In fact, if $char(R/H) = p \neq 0$ for some prime p, then $0 \neq pR \subseteq H \subseteq pR$. So, $0 = (pR)^2 = p^2R$, which is a contradiction. Therefore $(R/H)^+$ is torsion-free. Since, for any nonzero integer n, $H \subset nR$, it follows that $nR = (nR)^2 = n^2R$. Consequently, $R = nR$ which means that the additive abelian group R^+ is divisible. Hence H^+ is also a divisible abelian group. Moreover, R^+ is torsion-free. Indeed, if not, there exists $0 \neq a \in R$ and a positive integer n such that $na = 0$. Since $n(a + H) = H$ and $(R/H)^+$ is torsion-free, $a \in H$. Hence, by [10, Proposition 6.2], H^+ is an elementary abelian p-group for some prime p. Then $0 = pH = H$, which is a contradiction. Thus we have that R^+ is a torsion-free divisible group and so (a) holds. Next, suppose that $char(R) = n \neq 0$. Then $char(R/H) = p \neq 0$ for some prime p, which yields $pR \subseteq H$. If $pR = 0$, then R^+ is an elementary abelian p-group. If $pR \neq 0$, then $H \subseteq pR$. Consequently, $H = pR$ and

$0 = H^2 = p^2R$. From [7, Theorem 17.2], R^+ satisfies either (b), (e) or (f) .

If R satisfies (iv) of Proposition 4.1, then one of (a), (b), (c), or (d) holds, according to [3, Corollary 2.3].

Finally, suppose that (v) of Proposition 4.1 holds. Since $A/P \cong_* B/Q$, $char(A) = char(B)$. If the fully *-prime rings A and B are torsion-free, then (a) holds, since R is a *-subring of the involution ring $A \times B$. On the other hand, if $char(A) = char(B) = p$ for some prime p, then (b) holds. $\Diamond$

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