Mathematica Pannonica
26/2 (2017–2018), 107–118

Solutions of a higher order difference equation

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Received: 2018. January 30

MSC 2010: 39 A 20.

Keywords: difference equation, forbidden set, periodic solution, unbounded solution.

Abstract:
In this paper, we determine the forbidden set, introduce an explicit formula for the solutions and discuss the global behavior of solutions of the difference equation

$$x_{n+1} = \frac{ax_n x_{n-k}}{-bx_n + cx_{n-k-1}}, \quad n = 0, 1, \ldots,$$

where a, b, c are positive real numbers and the initial conditions $x_{-k-1}, x_{-k}, \ldots, x_{-1}, x_0$ are real numbers.

1. Introduction

The study of nonlinear difference equations that having quadratic terms is not easy and worth to be discussed. Results concerning rational difference equations having quadratic terms are included in some publications such as [1]-[21] and the references cited therein.

In [2], we determined the forbidden set and investigated the global behavior of all solutions of the rational difference equation

$$x_{n+1} = \frac{ax_n x_{n-1}}{bx_n - cx_{n-2}}, \quad n = 0, 1, \ldots,$$

where a, b, c are positive real numbers and the initial conditions x_{-2}, x_{-1}, x_0 are real numbers.

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In this paper, we determine the forbidden set, introduce an explicit formula for the solutions and discuss the global behavior of solutions of the difference equation

$$x_{n+1} = \frac{a x_n x_{n-k}}{-b x_n + c x_{n-k-1}}, \qquad n = 0, 1, \ldots, \tag{1.1}$$

where a, b, c are positive real numbers and the initial conditions $x_{-k-1}, x_{-k}, \ldots, x_{-1}, x_0$ are real numbers.

We have been investigated the global behavior of all possible solutions of equation (1.1) when $k = 1$ in [1].

2. Forbidden set and solutions of equation (1.1)

In this section we derive the forbidden set and introduce an explicit formula for the solutions of the difference equation (1.1).

Proposition 2.1. The forbidden set F of equation (1.1) is
$F = \bigcup_{n=0}^{\infty}\{(v_0, v_{-1}, \ldots, v_{-k-1}) \in \mathbb{R}^{k+2} : v_0 = v_{-k-1}(\frac{c}{b\sum_{l=0}^{n}(\frac{a}{c})^i})\} \cup \cup \bigcup_{i=0}^{k}\{(v_0, v_{-1}, \ldots, v_{-k}, v_{-k-1}) \in \mathbb{R}^{k+2} : v_{-i} = 0\}$.

Proof. Suppose that $\prod_{i=0}^{k+1} x_{-i} = 0$. Then we have the following:

If $x_0 = 0$ and $\prod_{i=1}^{k+1} x_{-i} \neq 0$, then x_{k+2} is undefined.

If $x_{-k} = 0$ and $\prod_{i=0, i\neq k}^{k+1} x_{-i} \neq 0$, then x_2 is undefined.

By induction we can show that, if for a certain $i_0 \in \{0, 1, \ldots, k\}$, such that $x_{-i_0} = 0$ and $\prod_{i=0, i\neq i_0}^{k} x_{-i} \neq 0$, then x_{k-i_0+2} is undefined.

Finally, if $x_{-k-1} = 0$ and $\prod_{i=0}^{k} x_{-i} \neq 0$, then $x_1 = -\frac{a}{b}x_{-k} \neq 0$. It follows that we can start with the nonzero initial point $(x_1, x_0, x_{-1}, \ldots, x_{-k})$, which the case we shall investigate.

Suppose that $x_{-i} \neq 0$ for all $i \in \{0, 1, \ldots, k+1\}$. From equation (1.1), using the substitution $l_n = \frac{x_{n-k-1}}{x_n}$, we can obtain the first order difference equation

$$l_{n+1} = \frac{c}{a} l_n - \frac{b}{a}, \quad l_0 = \frac{x_{-k-1}}{x_0}. \tag{2.1}$$

Consider the function $\phi(x) = \frac{c}{a}x - \frac{b}{a}$ and suppose that we start from an initial point $(x_0, x_{-1}, \ldots, x_{-k-1})$ such that $\frac{x_{-k-1}}{x_0} = \frac{b}{c}$.

The backward orbits $u_n = \frac{x_{n-k-1}}{x_n}$ satisfy

$$u_n = \phi^{-1}(u_{n-1}) = \frac{a}{c}u_{n-1} + \frac{b}{c} \quad \text{with} \quad u_0 = \frac{x_{-k-1}}{x_0} = \frac{b}{c}.$$

It follows that $u_n = \frac{x_{n-k-1}}{x_n} = \phi^{-n}(u_0) = \frac{b}{c}\sum_{i=0}^{n}(\frac{a}{c})^i$.

Therefore, $x_n = x_{n-k-1}(\frac{c}{b\sum_{i=0}^{n}(\frac{a}{c})^i})$.

On the other hand, we can observe that if we start from an initial point $(x_0, x_{-1}, \dots, x_{-k-1})$ such that $l_0 = \frac{x_{-k-1}}{x_0} = \frac{b}{c}\sum_{i=0}^{n_0}(\frac{a}{c})^i$ for a certain $n_0 \in \mathbb{N}$, then according to equation (2.1) we obtain

$$l_{n_0} = \frac{x_{n_0-k-1}}{x_{n_0}} = \frac{b}{c}.$$

This implies that $-bx_{n_0} + cx_{n_0-k-1} = 0$. Therefore, x_{n_0+1} is undefined.

This completes the proof. $\Diamond$

Let $\theta = \frac{a-c+b\alpha}{\alpha}$ where $\alpha = \frac{x_0}{x_{-k-1}}$.

Lemma 2.2. Let $x_{-k-1}, \dots, x_{-1}$ and x_0 be real numbers such that $(x_0, x_{-1}, \dots, x_{-k-1}) \notin F$. If $a \neq c$, then

$$x_i = x_{-k-1+i}\frac{a-c}{\theta(\frac{c}{a})^i - b}, \quad i = 1, 2, \dots, k+1.$$

Proof. The proof is by induction on i, where $i \in \{1, 2, \dots, k+1\}$.

When $i = 1$,

$$x_1 = \frac{ax_0x_{-k}}{-bx_0 + cx_{-k-1}} = \frac{a\frac{x_0}{x_{-k-1}}x_{-k}}{-b\frac{x_0}{x_{-k-1}} + c} = \frac{a\alpha x_{-k}}{-b\alpha + c}.$$

But as $\theta = \frac{a-c+b\alpha}{\alpha}$, we get $\alpha = \frac{a-c}{\theta-b}$.

It follows that

$$x_1 = \frac{a\alpha x_{-k}}{-b\alpha + c} = \frac{a-c}{(\frac{c}{a})\theta - b}x_{-k}.$$

Suppose for $1 \leq i \leq k$ that $x_i = x_{-k-1+i}\frac{a-c}{\theta(\frac{c}{a})^i - b}$. Then

$$x_{i+1} = \frac{ax_ix_{i-k}}{-bx_i + cx_{i-k-1}} = \frac{a\frac{x_i}{x_{i-k-1}}}{-b\frac{x_i}{x_{i-k-1}} + c}x_{i-k}$$

$$= \frac{a(a-c)}{-b(a-c)+c(\theta(\frac{c}{a})^i-b)}x_{i-k} = \frac{a(a-c)}{-ba+bc+c\theta(\frac{c}{a})^i-cb}x_{i-k}$$
$$= \frac{(a-c)}{(\frac{c}{a})^{i+1}\theta-b}x_{-k+i}.$$

This completes the proof. ♢

Theorem 2.3. Let $x_{-k-1}, x_{-k}, \ldots, x_{-1}$ and x_0 be real numbers such that $(x_0, x_{-1}, \ldots, x_{-k-1}) \notin F$. If $a \neq c$, then the solution $\{x_n\}_{n=-k-1}^{\infty}$ of equation (1.1) is
(2.2)
$$x_n = \begin{cases} x_{-k}\prod_{j=0}^{\frac{n-1}{k+1}} \frac{a-c}{\theta(\frac{c}{a})^{(k+1)j+1}-b}, & n = 1, k+2, 2k+3, \ldots, \\ x_{-k+1}\prod_{j=0}^{\frac{n-2}{k+1}} \frac{a-c}{\theta(\frac{c}{a})^{(k+1)j+2}-b}, & n = 2, k+3, 2k+4, \ldots, \\ \vdots & \\ x_{-1}\prod_{j=0}^{\frac{n-k}{k+1}} \frac{a-c}{\theta(\frac{c}{a})^{(k+1)j+k}-b}, & n = k, 2k+1, 3k+2, \ldots, \\ x_0\prod_{j=0}^{\frac{n-(k+1)}{k+1}} \frac{a-c}{\theta(\frac{c}{a})^{(k+1)j+k+1}-b}, & n = k+1, 2k+2, 3k+3, \ldots, \end{cases}$$

where $\theta = \frac{a-c+b\alpha}{\alpha}$ and $\alpha = \frac{x_0}{x_{-k-1}}$.

Proof. We can write the given solution (2.2) as

$$(2.3)\quad x_{(k+1)m+i} = x_{-k-1+i}\prod_{j=0}^{m}\gamma_i(j),\ i = 1, 2, \ldots, k+1 \text{ and } m = 0, 1, \ldots,$$

where
$$\gamma_i(j) = \frac{a-c}{\theta(\frac{c}{a})^{(k+1)j+i}-b}, \quad i = 1, 2, .., k+1.$$
When $m = 0$, we have

$$x_i = x_{-k-1+i}\frac{a-c}{\theta(\frac{c}{a})^i-b}, \quad i = 1, 2, \ldots, k+1,$$

which is true by Lemma (2.2).

Now for $m \geq 0$, we can see that

$$\frac{ax_{(k+1)(m+1)+i}x_{(k+1)m+i+1}}{-bx_{(k+1)(m+1)+i}+cx_{(k+1)m+i}} = \frac{ax_{-k-1+i}\prod_{j=0}^{m+1}\gamma_i(j)x_{-k+i}\prod_{j=0}^{m}\gamma_{i+1}(j)}{-bx_{-k-1+i}\prod_{j=0}^{m+1}\gamma_i(j)+cx_{-k-1+i}\prod_{j=0}^{m}\gamma_i(j)}$$
$$=\frac{ax_{-k-1+i}\prod_{j=0}^{m+1}\gamma_i(j)x_{-k+i}\prod_{j=0}^{m}\gamma_{i+1}(j)}{x_{-k-i+1}\prod_{j=0}^{m}\gamma_i(j)(-b\gamma_i(m+1)+c)} = \frac{a\gamma_i(m+1)x_{-k+i}\prod_{j=0}^{m}\gamma_{i+1}(j)}{-b\gamma_i(m+1)+c}$$
$$=\frac{a\frac{a-c}{\theta(\frac{c}{a})^{(k+1)(m+1)+i}-b}x_{-k+i}\prod_{j=0}^{m}\gamma_{i+1}(j)}{-b\frac{a-c}{\theta(\frac{c}{a})^{(k+1)(m+1)+i}-b}+c} = \frac{a(a-c)x_{-k+i}\prod_{j=0}^{m}\gamma_{i+1}(j)}{-b(a-c)+c(\theta(\frac{c}{a})^{(k+1)(m+1)+i}-b)}$$
$$=\frac{a(a-c)x_{-k+i}\prod_{j=0}^{m}\gamma_{i+1}(j)}{c\theta(\frac{c}{a})^{(k+1)(m+1)+i}-ab} = x_{-k+i}\frac{a-c}{\theta(\frac{c}{a})^{(k+1)(m+1)+i+1}-b}\prod_{j=0}^{m}\gamma_{i+1}(j)$$
$$=x_{-k+i}\gamma_{i+1}(m+1)\prod_{j=0}^{m}\gamma_{i+1}(j) = x_{-k+i}\prod_{j=0}^{m+1}\gamma_{i+1}(j) = x_{(k+1)(m+1)+i+1}.$$

This completes the proof.

◊

3. Global behavior of equation (1.1)

In this section, we investigate the global behavior of equation (1.1) with $a \neq c$, using the explicit formula of its solution.

Theorem 3.1. Let $\{x_n\}_{n=-k-1}^{\infty}$ be a solution of equation (1.1) such that $(x_0, x_{-1}, \ldots, x_{-k-1}) \notin F$. Then the following statements are true.

1. If $a < c$, then $\{x_n\}_{n=-k-1}^{\infty}$ converges to 0.
2. If $a > c$, then we have the following:
   (a) If $\frac{a-c}{b} < 1$, then $\{x_n\}_{n=-k-1}^{\infty}$ converges to 0.
   (b) If $\frac{a-c}{b} > 1$, then $\{x_n\}_{n=-k-1}^{\infty}$ is unbounded.

Proof. 1. If $a < c$, then $\gamma_i(j)$ converges to 0 as $j \to \infty$, $i = 1, 2, \ldots, k+1$. It follows that, for a given $0 < \epsilon < 1$, there exists $j_0 \in \mathbb{N}$ such that $\mid \gamma_i(j) \mid< \epsilon$ for all $j \geq j_0$ and $i = 1, 2, \ldots, k+1$. Therefore, for each $i \in \{1, 2, \ldots, k+1\}$, we have

$$\begin{aligned}
\mid x_{(k+1)m+i} \mid &=\mid x_{-k-1+i} \mid\mid \prod_{j=0}^{m} \gamma_i(j) \mid \\
&=\mid x_{-k-1+i} \mid\mid \prod_{j=0}^{j_0-1} \gamma_i(j) \mid\mid \prod_{j=j_0}^{m} \gamma_i(j) \mid \\
&<\mid x_{-k-1+i} \mid\mid \prod_{j=0}^{j_0-1} \gamma_i(j) \mid \epsilon^{m-j_0+1}.
\end{aligned}$$

As m tends to infinity, the solution $\{x_n\}_{n=-k-1}^{\infty}$ converges to 0.

2. Suppose that $a > c$. Then we have the following:

   (a) If $\frac{a-c}{b} < 1$, then $\gamma_i(j)$ converges to $-\frac{a-c}{b} > 1 \in (-1, 0)$ as $j \to \infty$, $i = 1, 2, \ldots, k+1$. This implies that, there exists $j_1 \in \mathbb{N}$ such that $\gamma_i(j) \in (\mu_1, 0)$, with some $0 > -\frac{a-c}{b} > \mu_1 > -1$ for all $j \geq j_1$ and $i = 1, 2, \ldots, k+1$. This implies that, $|\gamma_i(j)| < |\mu_1|$ for all $j \geq j_1$ and $i = 1, 2, \ldots, k+1$. Therefore, the solution $\{x_n\}_{n=-k-1}^{\infty}$ converges to 0 as in (1).

   (b) If $\frac{a-c}{b} > 1$, then $\gamma_i(j)$ converges to $-\frac{a-c}{b} < -1$ as $j \to \infty$, $i = 1, 2, \ldots, k+1$. Then for a given $-\frac{a-c}{b} < \mu_2 < -1$ there exists $j_2 \in \mathbb{N}$ such that $\gamma_i(j) < \mu_2 < -1$, for all $j \geq j_2$ and $i = 1, 2, \ldots, k+1$.

   For large values of m we have for each $i \in \{1, 2, \ldots, k+1\}$

$$\begin{aligned}
\mid x_{(k+1)m+i} \mid &=\mid x_{-k-1+i} \mid\mid \prod_{j=0}^{m} \gamma_i(j) \mid \\
&=\mid x_{-k-1+i} \mid\mid \prod_{j=0}^{j_2-1} \gamma_i(j) \mid\mid \prod_{j=j_2}^{m} \gamma_i(j) \mid \\
&>\mid x_{-k-1+i} \mid\mid \prod_{j=0}^{j_2-1} \gamma_i(j) \mid\mid \mu_2 \mid^{m-j_2+1} .
\end{aligned}$$

   Therefore, the subsequences $\{x_{(k+1)m+i}\}_{m=-1}^{\infty}$, $i = 1, 2, \ldots, k+1$ are unbounded and the result follows.

$\Diamond$

Example (1) Figure 1 shows that if $k = 2$, $a = 2.3$, $b = 1.3$ and $c = 1.5$ $(a - c < b)$, then the solution $\{x_n\}_{n=-3}^{\infty}$ with initial conditions $x_{-3} = 8$, $x_{-2} = 6$, $x_{-1} = -0.2$ and $x_0 = 1.2$ converges to 0.

Example (2) Figure 2 shows that if $k = 2$, $a = 2.1$, $b = 1.2$ and $c = 0.6$ $(a - c > b)$, then for the solution $\{x_n\}_{n=-3}^{\infty}$ with initial conditions $x_{-3} = -8$, $x_{-2} = 6$, $x_{-1} = -0.2$ and $x_0 = -1.2$, we have that the subsequences $\{x_{4n+i}\}_{n=-1}^{\infty}$, $i = 1, 2, 3, 4$ are unbounded.

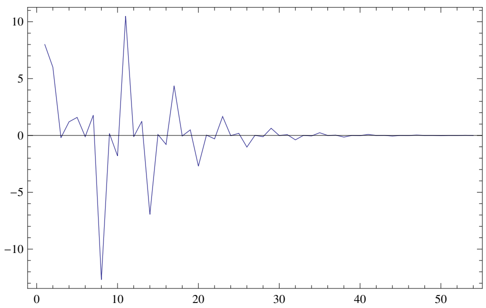

Figure 1: $x_{n+1} = \frac{2.3x_nx_{n-2}}{-1.3x_n+1.5x_{n-3}}$

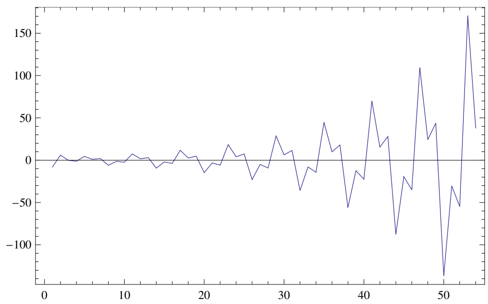

Figure 2: $x_{n+1} = \frac{2.1x_nx_{n-2}}{-1.2x_n+0.6x_{n-3}}$

4. Case $a - c = b$

In this section, we study the case when $a - c = b$.

Theorem 4.1. Assume that $\{x_n\}_{n=-k-1}^{\infty}$ is a solution of equation (1.1) such that $(x_0, x_{-1}, \ldots, x_{-k-1}) \notin F$ and let $a - c = b$. If $\alpha = -1$, then $\{x_n\}_{n=-k-1}^{\infty}$ is periodic solution with period $2(k+1)$.

Proof. Assume that $a - c = b$. If $\alpha = -1$, then $\theta = 0$. Therefore,

$$\begin{aligned} x_{(k+1)m+i} &= x_{-k-1+i} \prod_{j=0}^{m} \frac{a-c}{\theta(\frac{c}{a})^{(k+1)j+i} - b} \\ &= (-1)^{m+1} x_{-k-1+i}, \;\; i = 1, 2, \ldots, k+1 \text{ and } m = 0, 1, \ldots. \end{aligned}$$

It follows that

$$x_{(k+1)(m+2)+i} = x_{-k-1+i} \prod_{j=0}^{m+2} \frac{a-c}{\theta(\frac{c}{a})^{(k+1)j+i} - b}$$

$$\begin{aligned} &= (-1)^{m+3}x_{-k-1+i} = (-1)^{m+1}x_{-k-1+i} \\ &= x_{(k+1)m+i},\ i = 1,2,\dots,k+1 \text{ and } m = 0,1,\dots. \end{aligned}$$

This completes the proof. ◊

Theorem 4.2. Assume that $\{x_n\}_{n=-k-1}^{\infty}$ is a solution of equation (1.1) such that $(x_0, x_{-1}, \dots, x_{-k-1}) \notin F$ and let $a - c = b$. If $\alpha \neq -1$, then $\{x_n\}_{n=-k-1}^{\infty}$ converges to a period-$2(k+1)$ solution.

Proof. Suppose that $\{x_n\}_{n=-k-1}^{\infty}$ is a solution of equation (1.1) such that $(x_0, x_{-1}, \dots, x_{-k-1}) \notin F$ and let $a - c = b$. As

$$\lim_{j\to\infty} \gamma_i(j) = \lim_{j\to\infty} \frac{a-c}{\theta(\frac{c}{a})^{kj+i} - b} = -1,\ i = 1,2,\dots,k+1,$$

there exists $j_3 \in \mathbb{N}$ such that, $\gamma_i(j) < 0$, for all $i = 1,2,\dots,k+1$ and $j \geq j_3$.
It follows that

$$\begin{aligned} |x_{(k+1)m+i}| &= |x_{-k-1+i}||\prod_{j=0}^{m}\gamma_i(j)| = |x_{-k-1+i}||\prod_{j=0}^{j_3-1}\gamma_i(j)|\prod_{j=j_3}^{m}|\gamma_i(j)| \\ &= |x_{-k-1+i}||\prod_{j=0}^{j_3-1}\gamma_i(j)|\exp(\sum_{j=j_3}^{m}\ln(|\gamma_i(j)|). \end{aligned}$$

For all $j \geq j_3$, we can write

$$\ln(|\gamma_i(j)|) = \ln(-\gamma_i(j)) = \ln(-\frac{a-c}{\theta(\frac{c}{a})^{(k+1)j+i} - b}) = -\ln(1-(\frac{\theta}{b})(\frac{c}{a})^{(k+1)j+i}).$$

We shall test the convergence of the series $\sum_{j=j_3}^{\infty} |\ln(\gamma_i(j))|$.
Let $a_j = \ln(|\gamma_i(j)|) = -\ln(1 - (\frac{\theta}{b})(\frac{c}{a})^{(k+1)j+i})$ and $b_j = (\frac{c}{a})^{(k+1)j}$. Then for each $i \in \{1,2,\dots,k+1\}$ we get

$$\lim_{j\to\infty}\frac{a_j}{b_j} = \lim_{j\to\infty}\frac{-\ln(1-(\frac{\theta}{b})(\frac{c}{a})^{(k+1)j+i})}{(\frac{c}{a})^{(k+1)j}} = \frac{0}{0}.$$

Using L'Hospital's rule we obtain

$$\lim_{j\to\infty}\frac{a_j}{b_j} = -\lim_{j\to\infty}\frac{(\frac{-\theta}{b})(\frac{c}{a})^{(k+1)j+i}\ln(\frac{c}{a})(k+1)}{(1-(\frac{\theta}{b})(\frac{c}{a})^{(k+1)j+i})}/((\frac{c}{a})^{(k+1)j}\ln(\frac{c}{a})(k+1)) = (\frac{\theta}{b})(\frac{c}{a})^i.$$

Therefore, the series $\sum_{j=j_3}^{\infty} \mid \ln(\gamma_i(j)) \mid$ is convergent.
It follows that there are $k+1$ real numbers ρ_1, ρ_2, $\ldots$,ρ_{k+1} such that

$$\lim_{j\to\infty} |x_{(k+1)m+i}| = \rho_i \quad , i = 1, 2, \ldots, k+1.$$

Now set $\lim_{j\to\infty} x_{2(k+1)m+i} = \mu_i$, $i = 1, 2, \ldots, k+1$.
Then we get

$$x_{2(k+1)m+k+1+i} = x_{-k-1+i} \prod_{j=0}^{2m+1} \gamma_i(j) = x_{2(k+1)m+i}\gamma_i(2m+1).$$

It follows that $\mu_{k+1+i} = -\mu_i$, $i = 1, 2, \ldots, k+1$.
But for each $1 \le i \le k+1$, $\{x_{2(k+1)m+i}\}_{m=0}^{\infty}$ and $\{x_{2(k+1)m+k+1++i}\}_{m=0}^{\infty}$ are subsequences of $\{x_{(k+1)m+i}\}_{m=0}^{\infty}$, from which we get

$$|\mu_i| = \rho_i, \;\; 1 \le i \le k+1.$$

That is

$$\mu_i = \rho_i \text{ or } (-\rho_i), \;\; 1 \le i \le k+1.$$

Without loss of generality, we can take

$$\mu_i = \rho_i, \;\; 1 \le i \le k+1.$$

Then the solution $\{x_n\}_{n=-k-1}^{\infty}$ converges to the period-$2(k+1)$ solution

$$\{\ldots, \rho_1, \rho_2, \ldots, \rho_{k+1}, -\rho_1, -\rho_2, \ldots, -\rho_{k+1}, \ldots\}.$$

This completes the proof.

$\diamondsuit$

Example (3) Figure 3 shows that if $k = 3$, $a = 2.5$, $b = 1.8$ and $c = 0.7$ $(a - c = b)$, then the solution $\{x_n\}_{n=-4}^{\infty}$ with initial conditions $x_{-4} = 2$, $x_{-3} = -1.8$, $x_{-2} = -0.1$, $x_{-1} = -3.1$ and $x_0 = 3.5$ converges to a period-8 solution.

5. Case $a = b = c$

We end this work by introducing the main results when $a = b = c$.

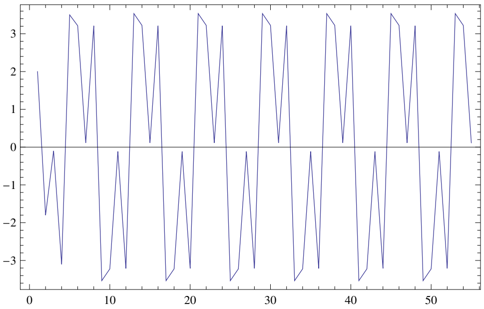

Figure 3: $x_{n+1} = \frac{2.5x_n x_{n-3}}{-1.8x_n + 0.7x_{n-4}}$

Proposition 5.1. Assume that $a = b = c$. Then the forbidden set G of equation (1.1) is
$G = \bigcup_{n=0}^{\infty}\{(u_0, u_{-1}, \dots, u_{-k-1}) \in \mathbb{R}^{k+2} : u_0 = u_{-k-1}(\frac{1}{n+1})\} \cup$
$\cup \bigcup_{i=0}^{k}\{(u_0, u_{-1}, \dots, u_{-k-1}) \in \mathbb{R}^{k+2} : u_{-i} = 0\}$.

Theorem 5.2. Let $x_{-k-1}, \dots, x_{-1}$ and x_0 be real numbers such that $(x_0, x_{-1}, \dots, x_{-k-1}) \notin G$. If $a = c$, then the solution $\{x_n\}_{n=-k-1}^{\infty}$ of equation (1.1) is

(5.1)
$$x_n = \begin{cases} x_{-k}\prod_{j=0}^{\frac{n-1}{k+1}} \frac{\alpha}{1-\alpha((k+1)j+1)}, & n = 1, k+2, 2k+3, \dots, \\ x_{-k+1}\prod_{j=0}^{\frac{n-2}{k+1}} \frac{\alpha}{1-\alpha((k+1)j+2)}, & n = 2, k+3, 2k+4, \dots, \\ \vdots & \\ x_{-1}\prod_{j=0}^{\frac{n-k}{k+1}} \frac{\alpha}{1-\alpha((k+1)j+k)}, & n = k, 2k+1, 3k+2, \dots, \\ x_0\prod_{j=0}^{\frac{n-(k+1)}{k+1}} \frac{\alpha}{1-\alpha((k+1)j+k+1)}, & n = k+1, 2k+2, 3k+3, \dots, \end{cases}$$

where $\alpha = \frac{x_0}{x_{-k-1}}$.

Theorem 5.3. Let $\{x_n\}_{n=-k-1}^{\infty}$ be a solution of equation (1.1) such that $(x_0, x_{-1}, \dots, x_{-k-1}) \notin G$. If $a = b = c$, then $\{x_n\}_{n=-k-1}^{\infty}$ converges to 0.

Example (4) Figure 4 shows that if $k = 3$ and $a = b = c$, then the solution $\{x_n\}_{n=-4}^{\infty}$ with initial conditions $x_{-4} = 2$, $x_{-3} = -1$, $x_{-2} = -0.5$, $x_{-1} = -5.1$ and $x_0 = 3.5$ converges to 0.

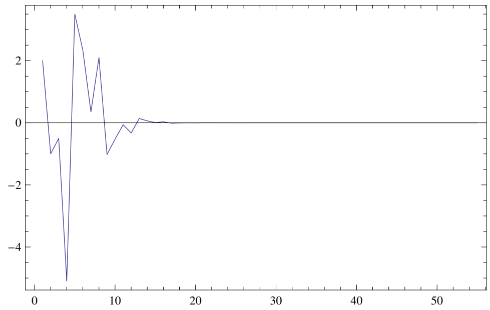

Figure 4: $x_{n+1} = \frac{x_n x_{n-3}}{-x_n + x_{n-4}}$

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