Second order parallel tensors and Ricci solitons on Lorentzian Para $r$-Sasakian manifold with a coefficient $\alpha$

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Abstract: The geometry of Lorentzian para $r$-Sasakian manifold is developed by Takahashi [15] and Matsumoto [10]. The present paper deals with the study of second order parallel tensor in an LP $r$-Sasakian manifold with a coefficient $\alpha$. It is proved that a second order parallel symmetric tensor on an LP $r$-Sasakian manifold with a coefficient $\alpha$, is a constant multiple of the metric tensor, where as the second order parallel skew-symmetric tensor on an LP $r$-Sasakian manifold with a coefficient $\alpha$ does not exist.

1. Introduction

In 1923, Eisenhart [5] showed that a Riemannian manifold admitting a second order symmetric parallel tensor other than a constant multiple of metric tensor is reducible. Then Levy [8] had obtained the necessary and sufficient conditions for the existence of such tensors. Sharma

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[13] has generalized Levy’s result by showing that a second order parallel (not necessarily symmetric and non singular) tensor of an n-dimensional \((n > 2)\) space of constant curvature is a constant metric tensor. Then Li [9] studied second order parallel tensors on \(P\)-Sasakian manifold with a coefficient \(k\). Also in [14], Singh et al. studied second order parallel tensors on LP-Sasakian manifolds. Recently Das ([2],[3]) has proved that on a Para \(r\)-Sasakian manifold with a coefficient \(\alpha\), a second order symmetric parallel tensor is a constant multiple of the associated positive definite Riemannian metric tensor. In this paper we have extended these ideas further and we have defined Lorentzian Para \(r\)-Sasakian manifold with a coefficient \(\alpha\) (non-zero scalar function) and it is proved that a second order parallel symmetric tensor on an LP \(r\)-Sasakian manifold with a coefficient \(\alpha\), is a constant multiple of the metric tensor. However, it is proved that there do not exist second order parallel skew-symmetric tensors on an LP \(r\)-Sasakian manifold with a coefficient \(\alpha\).

In 1982, Hamilton [6] introduced the notion of Ricci flow to find a canonical metric on a smooth manifold. Then Ricci flow has become a powerful tool for the study of Riemannian manifolds, especially for those manifolds with positive curvature. Perelman [11] used Ricci flow and its surgery to prove Poincare conjecture. The Ricci flow is an evolution equation for metrics on a Riemannian manifold defined as follows:

\[
\frac{\partial}{\partial t} g_{ij}(t) = -2R_{ij}.
\]

A Ricci soliton emerges as the limit of the solutions of the Ricci flow. A solution to the Ricci flow is called Ricci soliton if it moves only by a one parameter group of diffeomorphism and scaling. A Ricci soliton \((g, V, \lambda)\) on a Riemannian manifold \((M, g)\) is a generalization of Einstein metric such that

\begin{equation}
\mathcal{L}_V g + 2S + 2\lambda g = 0,
\end{equation}

where \(S\) is the Ricci tensor and \(\mathcal{L}_V\) is the Lie derivative along the vector field \(V\) on \(M\) and \(\lambda\) is a real number. The Ricci soliton is said to be shrinking, steady and expanding according as \(\lambda\) is negative, zero and positive respectively. Recently Chandra et al. [1] studied second order parallel tensors and Ricci solitons on \((LCS)_n\)-manifolds.

In this paper we prove that if the tensor field \(\mathcal{L}_V g + 2S\) on an LP \(r\)-Sasakian manifold with a coefficient \(\alpha\) is parallel then \((g, V, \lambda)\) is a Ricci soliton.
2. Preliminaries

Let $M^{2n+r}$ be an $(2n+r)$-dimensional smooth manifold equipped with the ring of real valued differentiable functions $C^\infty(M)$ and the module of derivation $\chi(M)$ and an $(1,1)$tensor field $\phi$ as a linear map such that such that $\phi : \chi(M) \to \chi(M)$. Let there be a $r$ $C^\infty$-contravariant vector fields $\xi_1, \xi_2, \cdots, \xi_r$ satisfying the following condition:

\begin{align*}
(2.1) \quad & \eta_p(\xi^p) = \epsilon \delta^p_q, \quad p, q = 1, 2, \cdots, r \\
(2.2) \quad & \phi(\xi^p) = 0, \quad p = 1, 2, \cdots, r \\
(2.3) \quad & \eta_p(\phi X) = 0, \quad p = 1, 2, \cdots, r \\
(2.4) \quad & \phi^2 X = X - \epsilon \eta_p(X)\xi^p, \quad p = 1, 2, \cdots, r
\end{align*}

for any vector field $X \in \chi(M)$. Here the summation convention is employed on repeated indices for $p = 1, 2, \cdots, r$ and

\[\delta^p_p = 1, \quad p \neq q = 0, \quad p \neq q.\]

If moreover $M^{2n+r}$ admits an indefinite metric $g$ such that

\begin{align*}
(2.5) \quad & g(\xi^p, \xi^p) = \epsilon \\
(2.6) \quad & \eta_p(X) = g(X, \xi^p) \\
(2.7) \quad & g(\phi X, \phi Y) = g(X, Y) - \epsilon \eta_p(X)\eta_p(Y)
\end{align*}

for any vector field $X$ and $Y \in \chi(M)$, where $\epsilon$ is 1 or -1 according as $\xi$ is a spacelike or timelike vector field, then a manifold satisfying condition (2.1)-(2.7) is called a Lorentzian Para $r$-Sasakian manifold (briefly LP $r$-Sasakian manifold).

In an LP $r$-Sasakian manifold $M^{2n+r}$, the following relations hold:

\begin{align*}
(2.8) \quad & \Phi(X, Y) = g(X, \phi Y) = g(Y, \phi X) = \Phi(Y, X), \\
(2.9) \quad & \Phi(X, \xi^p) = 0.
\end{align*}
Definition 2.1. If in an LP $r$-Sasakian manifold $M^{2n+r}$, the following relations
\begin{equation}
\phi X = \frac{1}{\alpha} \nabla_X \xi_p, \quad \Phi(X, Y) = \frac{1}{\alpha} (\nabla_X \eta_p)(Y),
\end{equation}
\begin{equation}
(\nabla_X \phi)(Y, Z) = \alpha \left[ \{g(X, Y) - \epsilon \eta_p(X) \eta_p(Y)\} \eta_p(Z) + \{g(X, Z) - \epsilon \eta_p(X) \eta_p(Z)\} \eta_p(Y) \right]
\end{equation}
hold for arbitrary smooth vector fields $X, Y, Z \in \chi(M)$, where $\nabla$ denotes the Riemannian coefficient of the metric tensor $g$, then $M^{2n+r}$ is called an $\epsilon$-Lorentzian Para $r$-Sasakian manifold with a coefficient $\alpha$.

In an LP $r$-Sasakian manifold with a coefficient $\alpha$, the following relations hold:
\begin{equation}
\eta_p(\nabla(X, Y)Z) = \alpha^2 [g(Y, Z) \eta_p(X) + g(X, Z) \eta_p(Y)] - \alpha \eta_p(\Phi(Y, Z) - \alpha \Phi(X, Z)],
\end{equation}
\begin{equation}
R(\xi_p, X)Y = \alpha^2 [\epsilon g(X, Y) \xi_p - \eta_p(Y) X] + \alpha \eta_p(\phi X - \sigma \phi(X, Y),
\end{equation}
\begin{equation}
R(\xi_p, X)\xi_p = \beta \phi X + \alpha^2 [X - \epsilon \eta_p(X) \xi_p]
\end{equation}
for all vector fields $X, Y, Z \in \chi(M), p = 1, 2, \cdots, r$, where $\alpha(\xi_p) = \beta$.

3. second order parallel symmetric tensors and Ricci solitons

Let $J$ be a symmetric $(0,2)$ tensor field on an LP $r$-Sasakian manifold $M^{2n+r}$ with a coefficient $\alpha$ such that $\nabla J = 0$. Then we have
\begin{equation}
J(R(W, X)Y, Z) + J(Y, R(W, X)Z) = 0
\end{equation}
for arbitrary vector fields $X, Y, Z, W$ on $M^{2n+r}$.

Putting $W = Y = Z = \xi_p$ in (3.1), we get
\begin{equation}
J(\xi_p, R(\xi_p, X)\xi_p) = 0.
\end{equation}
In view of (2.15) and (2.9) it follows from (3.2) that
\[(3.3) \quad \alpha^2 [J(X, \xi^p) - \epsilon \eta_p(X)]J(\xi^p, \xi^p)] = 0.\]

Since \(\alpha^2 \neq 0\) and \(\epsilon\) is either 1 or -1, we have from (3.3) that
\[(3.4) \quad J(X, \xi^p) - \epsilon \eta_p(X)J(\xi^p, \xi^p) = 0.\]

Differentiating (3.4) covariantly along \(Y\), we get
\[(3.5) \quad -\epsilon g(\nabla_Y X, \xi^p)J(\xi^p, \xi^p) - 2\epsilon g(X, \xi^p)J(\nabla_Y \xi^p, \xi^p) + J(X, \nabla_Y \xi^p) = 0.\]

Putting \(X = \nabla_Y X\) in (3.4) we obtain
\[(3.6) \quad J(\nabla_Y X, \xi^p) - \epsilon g(\nabla_Y X, \xi^p)J(\xi^p, \xi^p) = 0.\]

In view of (3.6) it follows from (3.5) that
\[(3.7) \quad -\epsilon g(X, \nabla_Y \xi^p)J(\xi^p, \xi^p) - 2\epsilon g(X, \xi^p)J(\nabla_Y \xi^p, \xi^p) + J(X, \nabla_Y \xi^p) = 0.\]

Using (2.10) in (3.7) we get
\[(3.8) \quad -\epsilon g(X, \phi Y)J(\xi^p, \xi^p) - 2\epsilon \eta_p(X)J(\phi Y, \xi^p) + J(X, \phi Y) = 0, \quad \text{since} \quad \alpha \neq 0.\]

Replacing \(Y\) by \(\phi Y\) in (3.8) and then using (2.4) and (3.4) we obtain
\[(3.9) \quad J(X, Y) = \epsilon J(\xi^p, \xi^p)g(X, Y).\]

Differentiating (3.9) covariantly along any vector field on \(M^{2n+r}\), it can be easily shown that \(J(\xi^p, \xi^p)\) is constant. This leads to the following:

**Theorem 3.1.** A second order parallel symmetric tensor on an LP \(r\)-Sasakian manifold with a coefficient \(\alpha\), is a constant multiple of the associated metric tensor.

**Corollary 3.2.** [14] A second order parallel symmetric tensor on an LP-Sasakian manifold is a constant multiple of the associated metric tensor.

Suppose that the (0, 2) type symmetric tensor field \(\mathcal{L}_V g + 2S\) is parallel for any vector field \(V\) on an LP \(r\)-Sasakian manifold \(M^{2n+r}\). Then by Theorem 3.1, it follows that \(\mathcal{L}_V g + 2S\) is a constant multiple of the metric tensor \(g\), i.e. \(\mathcal{L}_V g + 2S = -2\lambda g\) for all \(X, Y\) on \(M^{2n+r}\), where \(\lambda\) is a constant. Hence the relation (1.1) holds. This implies that \((g, V, \lambda)\) yields a Ricci soliton. Thus we can state the following:
**Theorem 3.3.** If the tensor field $\mathcal{L}_V g + 2S$ on an LP $r$-Sasakian manifold with a coefficient $\alpha$, is parallel for any vector field $V$, then $(g, V, \lambda)$ is a Ricci soliton.

**Corollary 3.4.** If the tensor field $\mathcal{L}_V g + 2S$ on an LP-Sasakian manifold is parallel for any vector field then $(g, V, \lambda)$ is a Ricci solution.

Let $(g, \xi^p, \lambda)$ be a Ricci soliton on a LP $r$-Sasakian manifold $M^{2n+r}$ with a coefficient $\alpha$. Then we have

$$(3.10) \quad (\mathcal{L}_{\xi^p} g)(Y, Z) + 2S(Y, Z) + 2\lambda g(Y, Z) = 0$$

where $\mathcal{L}_{\xi^p}$ is the Lie derivative along the vector field $\xi^p$ on $M^{2n+r}$.

From (2.10), we have

$$(3.11) \quad (\mathcal{L}_{\xi} g)(Y, Z) = g(\nabla_Y \xi^p, Z) + g(Y, \nabla_Z \xi^p)$$

$$= \alpha [g(\phi Y, Z) + g(Y, \phi Z)]$$

$$= 2\alpha \Phi(Y, Z).$$

Using (3.11) in (3.10) we get

$$S(Y, Z) = -\lambda g(Y, Z) - \alpha \Phi(Y, Z),$$

which implies that the manifold under consideration is nearly quasi-Einstein manifold [4]. This leads the following:

**Theorem 3.5.** If $(g, \xi^p, \lambda)$ is a Ricci soliton on an LP $r$-Sasakian manifold $M^{2n+r}$ with a coefficient $\alpha$, then $M^{2n+r}$ is nearly quasi-Einstein manifold.

**Corollary 3.6.** If $(g, \xi, \lambda)$ is a Ricci soliton on an LP-Sasakian manifold $M$ then $M$ is nearly quasi-Einstein manifold.

If possible, let $J$ be a second order skew symmetric parallel tensor field on an LP $r$-Sasakian manifold $M^{2n+r}$ with a coefficient $\alpha$. Then we have the relation (3.1). Putting $W = Y = \xi^p$ in (3.1) we get

$$(3.12) \quad J(R(\xi^p, X)\xi^p, Z) + J(\xi^p, R(\xi^p, X)Z) = 0.$$
Theorem 3.7. There do not exist second order parallel skew-symmetric tensor on an LP $r$-Sasakian manifold with a coefficient $\alpha$.

Corollary 3.8. There do not exist second order parallel skew-symmetric tensor on an LP-Sasakian manifold.

References


