Mathematica Pannonica
26/2 (2017–2018), 53–66

Sufficient Conditions for Double Trigonometric Integrals to Belong to a Zygmund Class of Functions

Vanda **Fülöp**

Bolyai Institute, University of Szeged
Aradi vértanúk tere 1
6720 Szeged, Hungary

Ferenc **Móricz**

Bolyai Institute, University of Szeged
Aradi vértanúk tere 1
6720 Szeged, Hungary

Received: May 4, 2018

MSC 2010: 26 A 16, 26 B 35; 42 A 16, 42 A 38, 42 B 10.

Keywords: single and double trigonometric integrals over $\mathbb{R}$ and $\mathbb{R}^2$, respectively; Lipschitz classes $\mathrm{Lip}(\alpha)$ and $\mathrm{lip}(\alpha)$, Zygmund classes $\mathrm{Zyg}(\alpha)$ and $\mathrm{zyg}(\alpha)$, $0 < \alpha \le 2$; $\mathrm{Zyg}(\alpha, \beta)$ and $\mathrm{zyg}(\alpha, \beta)$, $0 < \alpha, \beta \le 2$; initial integral means, tail integral means; single and double Fourier transforms.

Abstract: Let $f : \mathbb{R}^2 \to \mathbb{C}$ be a Lebesgue integrable function on the real plane $\mathbb{R}^2$, and consider the double trigonometric integral defined by

$$F(x, y) := \int\int_{\mathbb{R}^2} f(u, v) e^{i(ux+vy)} du dv, \quad (x, y) \in \mathbb{R}^2 := \mathbb{R} \times \mathbb{R}.$$

We give sufficient conditions in terms of certain integral means of f to ensure that $F(x, y)$ belong to one of the Zygmund classes $\mathrm{Zyg}(\alpha, \beta)$ or $\mathrm{zyg}(\alpha, \beta)$ for some $0 < \alpha, \beta \le 2$. Our present theorems are the extensions of those proved in [4] from single to double trigonometric integrals, and they may also be applied in the case of double Fourier transform. The starting point of our investigation goes back to the monograph [1] by Boas. We also note that in the recent years Tikhonov in [7], [8] and Volosivets in [9], [10] dealt with the same problem. In our auxiliary results we establish interesting interrelations between the order of magnitude of certain initial integral means and the order of magnitude of certain tail integral means of a function $f \in L^1_{\mathrm{loc}}(\mathbb{R}^2)$. They may be useful in the investigation of other two-dimensional problems, as well.

1. Preliminaries

Let $g : \mathbb{R} \to \mathbb{C}$ be a Lebesgue integrable function on the real line $\mathbb{R} := (-\infty, \infty)$, in symbols: $g \in L^1(\mathbb{R})$. In [4] we defined the *trigonometric integral* G of g as follows

$$G(x) := \int_{\mathbb{R}} g(u) e^{iux} du, \quad x \in \mathbb{R}. \tag{1.1}$$

By virtue of the dominated convergence theorem, $G(x)$ is a continuous function on $\mathbb{R}$.

We recall (see, e.g., [2, Ch.2] or [12, Vol. I, Ch.2, §3]) that $G(x)$ is said to belong to the *Lipschitz class* $\mathrm{Lip}(\alpha)$ for some $\alpha > 0$ if for all $x \in \mathbb{R}$ and $h \in \mathbb{R}_+ := (0, \infty)$, we have that

$$|\Delta G(x; h)| := |G(x+h) - G(x)| \le C_\alpha h^\alpha,$$

where C_α is a constant. Furthermore, $G(x)$ is said to belong to the *little Lipschitz class* $\mathrm{lip}(\alpha)$ if

$$\lim_{h \to 0} h^{-\alpha} \Delta G(x; h) = 0 \quad \text{uniformly in} \quad x \in \mathbb{R}.$$

We also recall that a continuous function $G(x)$ is said to belong to the *Zygmund class* $\mathrm{Zyg}(\alpha)$ for some $\alpha > 0$ if for all $x \in \mathbb{R}$ and $h \in \mathbb{R}_+$, we have that

$$|\Delta^2 G(x; h)| := |G(x+h) - 2G(x) + G(x-h)| \le C_\alpha h^\alpha,$$

where C_α is a constant. Furthermore, a continuous function $G(x)$ is said to belong to the *little Zygmund class* $\mathrm{zyg}(\alpha)$ if

$$\lim_{h \to 0} h^{-\alpha} \Delta^2 G(x; h) = 0 \quad \text{uniformly in} \quad x \in \mathbb{R}.$$

Remark 1.1. In the book [12, Vol. I, Ch.2, §3] of Zygmund, the notation Λ_* is used for $\mathrm{Zyg}(1)$ and the notation λ_* is used for $\mathrm{zyg}(1)$.

It is well known (see also in [2, Ch.2] or [12, Vol. I, Ch.2, §3]) that if $G \in \mathrm{lip}(1)$; in particular, if $G \in \mathrm{Lip}(\alpha)$ for some $\alpha > 1$, then $G(x)$ is a constant function. Furthermore, if $G \in \mathrm{zyg}(2)$; in particular, if $G \in \mathrm{Zyg}(\alpha)$ for some $\alpha > 2$, then $G(x)$ is a linear function.

E-mail addresses: fulopv@math.u-szeged.hu, moricz@math.u-szeged.hu

Remark 1.2. We recall (see, e.g., in [12, Vol I, on p. 43]) that a continuous function $G(x)$ is said to be *smooth* at some point $x \in \mathbb{R}$ if

$$\lim_{h\to 0} h^{-1}\Delta^2 G(x;h) = 0.$$

Clearly, zyg(1) is exactly the class of those continuous functions that are uniformly smooth in $x \in \mathbb{R}$.

In our recent paper [4] we proved the following two theorems, where by the symbol $g \in L^1_{\mathrm{loc}}(\mathbb{R})$ we mean that the function $g : \mathbb{R} \to \mathbb{C}$ is Lebesgue integrable on all bounded intervals.

Theorem 1.3. *Let $g \in L^1_{loc}(\mathbb{R})$ and $0 < \alpha \le 2$. If there exists a constant C_α such that for all $U \in \mathbb{R}_+$, we have that*

$$U^{\alpha-2}\int_{|u|<U} u^2|g(u)|du \le C_\alpha, \tag{1.2}$$

then $g \in L^1(\mathbb{R})$ and $G \in \mathrm{Zyg}(\alpha)$, where $G(x)$ is defined in (1.1).

Theorem 1.4. Let $g \in L^1_{\mathrm{loc}}(\mathbb{R})$ and $0 < \alpha < 2$. If

$$\lim_{U\to\infty} U^{\alpha-2}\int_{|u|<U} u^2|g(u)|du = 0,$$

then $g \in L^1(\mathbb{R})$ and $G \in \mathrm{zyg}(\alpha)$.

Remark 1.5. In the special case $\alpha = 1$, our Theorems 1.3 and 1.4 are the nonperiodic versions of the corresponding theorems of Zygmund [11] (see also in [12, Vol I, on p. 320]) in the case of trigonometric series.

Remark 1.6. It is easy to check that condition (1.2) above may be weakened as follows: If there exist constants C_α and $\eta > 0$ such that condition (1.2) holds only for all $U \ge \eta$, then we still have $g \in L^1(\mathbb{R})$ and $G(x) \in \mathrm{Zyg}(\alpha)$. Namely if $0 < U < \eta$, then we have that

$$U^{\alpha-2}\int_{|u|<U} u^2|g(u)|du \le U^\alpha \int_{|u|<U} |g(u)|du \le \eta^\alpha \int_{|u|<\eta} |g(u)|du =: C'_\alpha.$$

So, if (1.2) holds with a constant C_α for all $U > \eta$, then it holds with the constant $\max\{C_\alpha, C'_\alpha\}$ for all $U > 0$.

Remark 1.7. Theorems 1.3 and 1.4 are also valid in the case of the Fourier transform $\hat{g}$ of a function $g \in L^1(\mathbb{R})$ defined by

$$\hat{g}(x) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} g(u) e^{-iux} du, \quad x \in \mathbb{R},$$

(see, e.g., in [12, Vol. II, on p. 254]) in place of the trigonometric integral G defined in (1.1).

2. Definitions of double Lipschitz- and Zygmund classes of continuous functions

Let $f : \mathbb{R}^2 \to \mathbb{C}$ be a Lebesgue integrable function on the real plane $\mathbb{R}^2$, in symbols: $f \in L^1(\mathbb{R}^2)$. Analogously to (1.1), we define the *double trigonometric integral* F of f as follows

$$(2.1) \qquad F(x, y) := \int\int_{\mathbb{R}^2} f(u, v) e^{i(ux+vy)} du dv, \quad (x, y) \in \mathbb{R}^2.$$

By Lebesgue's dominated convergence theorem, we have that the function $F(x, y)$ is continuous on $\mathbb{R}^2$. Thus, the marginal function $F(\cdot, y)$ is continuous in its first variable for every fixed $y \in \mathbb{R}$, and the marginal function $F(x, \cdot)$ is also continuous in its second variable for every fixed $x \in \mathbb{R}$.

We recall that the *difference operators* $\Delta_{x,h}$ and $\Delta_{y,k}$ are defined by

$$\Delta_{x,h} F(x, y) := F(x + h, y) - F(x, y),$$

$$\Delta_{y,k} F(x, y) := F(x, y + k) - F(x, y), \quad (x, y) \in \mathbb{R}^2 \quad \text{and} \quad (h, k) \in \mathbb{R}^2_+.$$

The iterated applications of these operators are defined in the usual way as follows

$$(2.2) \qquad \Delta F(x, y; h, k) := \Delta_{x,h}(\Delta_{y,k} F(x, y))$$

$$= \Delta_{y,k}(\Delta_{x,h} F(x, y)) = \Delta_{y,k}(F(x + h, y) - F(x, y))$$
$$= F(x + h, y + k) - F(x + h, y) - F(x, y + k) + F(x, y).$$

Now, we recall (see, e.g., in [5]) that a continuous function $F(x, y)$ is said to belong to the *double* (called also multiplicative) *Lipschitz class*

$\operatorname{Lip}(\alpha, \beta)$ for some $\alpha, \beta > 0$ if for all $(x, y) \in \mathbb{R}^2$ and $(h, k) \in \mathbb{R}^2_+$, we have that

$$|\Delta F(x, y; h, k)| \le C_{\alpha,\beta} h^\alpha k^\beta,$$

where $C_{\alpha,\beta}$ is a constant. Furthermore, a function $F(x, y) \in \operatorname{Lip}(\alpha, \beta)$ is said to belong to the *little Lipschitz class* $\operatorname{lip}(\alpha, \beta)$ for some $\alpha, \beta > 0$ if

$$\lim_{h,k\to 0} h^{-\alpha} k^{-\beta} \Delta F(x, y; h, k) = 0 \quad \text{uniformly in} \quad (x, y) \in \mathbb{R}^2.$$

It is routine to check that

$$\Delta^2 F(x, y; h, k) := \Delta(\Delta F(x, y; h, k); h, k) \tag{2.3}$$

$$\begin{aligned}= F(x + 2h, y + 2k) + F(x + 2h, y) + F(x, y + 2k) + F(x, y)\\ -2F(x + 2h, y + k) - 2F(x + h, y + 2k) - 2F(x + h, y)\\ -2F(x, y + k) + 4F(x + h, y + k).\end{aligned}$$

Next, we recall (see, e.g., in [3]) that a continuous function $F(x, y)$ belongs to the *double* (also called multiplicative) *Zygmund class* $\operatorname{Zyg}(\alpha, \beta)$ for some $(\alpha, \beta) \in \mathbb{R}^2_+$ if for all $(x, y) \in \mathbb{R}^2$ and $(h, k) \in \mathbb{R}^2_+$, we have that

$$|\Delta^2 F(x, y; h, k)| \le C_{\alpha,\beta} h^\alpha k^\beta, \tag{2.4}$$

where $\Delta^2 F(x, y; h, k)$ is defined in (2.3) and $C_{\alpha,\beta}$ is a constant. Furthermore, a function $F(x, y) \in \operatorname{Zyg}(\alpha, \beta)$ is said to belong to the *little Zygmund class* $\operatorname{zyg}(\alpha, \beta)$ if

$$\lim_{h,k\to 0} h^{-\alpha} k^{-\beta} \Delta^2 F(x, y; h, k) = 0 \quad \text{uniformly in} \quad (x, y) \in \mathbb{R}^2.$$

In the sequel, instead of (2.3) we will use the following equivalent (symmetric) form:

$$\Delta^2 F(x - h, y - k; h, k) \tag{2.5}$$

$$\begin{aligned}= F(x + h, y + k) + F(x + h, y - k) + F(x - h, y + k) + F(x - h, y - k)\\ -2F(x + h, y) - 2F(x, y + k) - 2F(x - h, k) - 2F(x, y - k) + 4F(x, y).\end{aligned}$$

By (2.2) and (2.5) it is easy to check that for all $(x, y) \in \mathbb{R}^2$ and $(h, k) \in \mathbb{R}^2_+$, we have that

$$\Delta^2 F(x - h, y - k; h, k) = \Delta F(x, y; h, k) - \Delta F(x - h, y; h, k)$$

$$-\Delta F(x, y-k; h, k) + \Delta F(x-h, y-k; h, k).$$

Hence it immediately follows that for all $(\alpha, \beta) \in \mathbb{R}^2_+$, we have that

$$\mathrm{Zyg}(\alpha, \beta) \supseteq \mathrm{Lip}(\alpha, \beta) \quad \text{and} \quad \mathrm{zyg}(\alpha, \beta) \supseteq \mathrm{lip}(\alpha, \beta).$$

Analogously to the corresponding one variable classes, in the sequel we will assume that $0 < \alpha, \beta \le 2$.

Remark 2.1. Motivated by Remark 1.2, a continuous function $F(x, y)$ may be called to be *smooth* at some point $(x, y) \in \mathbb{R}^2$ if

$$\lim_{h,k \to 0} h^{-1} k^{-1} \Delta^2 F(x, y; h, k) = 0.$$

Clearly, $\mathrm{zyg}(1, 1)$ is exactly the class of those continuous functions that are uniformly smooth in $(x, y) \in \mathbb{R}^2$.

3. Main results

In a recent paper [4] we proved Theorems 1.3 and 1.4 for trigonometric integrals, which are the nonperiodic versions of the classical theorems by Zygmund [11] (see also in [12, Vol II, on pp. 320-321]) on the smoothness of the sum of a trigonometric series. In the present paper, our goal is to prove analogous theorems for the double trigonometric integral $F(x, y)$ defined in (2.1).

In the case, where a function $f : \mathbb{R}^2 \to \mathbb{C}$ is Lebesgue integrable on any bounded rectangle of $\mathbb{R}^2$, it will be indicated in symbols: $f \in L^1_{\mathrm{loc}}(\mathbb{R}^2)$. In the sequel, we will also assume that there exists some constant $\eta \in \mathbb{R}_+$ such that

$$f \in L^1((\mathbb{R} \times [-\eta, \eta]) \cup ([-\eta, \eta] \times \mathbb{R})) \tag{3.1}$$

Now, our main results are formulated in the following two theorems.

Theorem 3.1. Let $f \in L^1_{\mathrm{loc}}(\mathbb{R}^2)$, $\eta \in \mathbb{R}_+$ and $0 < \alpha, \beta \le 2$. If conditions (3.1) and

$$U^{\alpha-2} V^{\beta-2} \int_{|u|<U} \int_{|v|<V} u^2 v^2 |f(u, v)| du dv \le C_{\alpha,\beta} \tag{3.2}$$

are satisfied, the latter one for all $U, V \ge \eta$, where $C_{\alpha,\beta}$ is a constant, then $f \in L^1(\mathbb{R}^2)$ and the double trigonometric integral $F(x, y)$ defined in (2.1) belongs to the class $\mathrm{Zyg}(\alpha, \beta)$.

Theorem 3.2. Suppose $0 < \alpha, \beta < 2$ and that the conditions in Theorem 3.1 are satisfied. If, in addition, we have that

$$\lim_{U,V\to\infty} U^{\alpha-2}V^{\beta-2}\int_{|u|<U}\int_{|v|<V} u^2v^2|f(u,v)|dudv = 0, \tag{3.3}$$

then the double trigonometric integral $F(x,y)$ defined in (2.1) belongs to the class $\mathrm{zyg}(\alpha,\beta)$.

Remark 3.3. A real-valued function $G(U,V)$ defined for all $U,V > \eta$ may converge to 0 as $U,V \to \infty$, and yet it is not bounded. This explains the fact that we required the fulfillment of both (3.2) and (3.3) in Theorem 3.2.

4. Auxiliary results

The next Lemmas 4.1-4.4 will be the basic tools in the proofs of our Theorems 3.1 and 3.2. They are also of some interest in themselves, since they exhibit useful interrelations between the order of magnitude of certain initial integral means and the order of magnitude of certain tail integral means of any function $f \in L^1_{\mathrm{loc}}(\mathbb{R}^2)$.

Lemma 4.1. Let $f \in L^1_{\mathrm{loc}}(\mathbb{R}^2)$, $\eta > 0$ and $0 < \alpha, \beta \le 2$. If condition (3.2) is satisfied for all $U,V \ge \eta$, then there exists another constant $C^{(1)}_{\alpha,\beta}$ such that for all $U,V \ge \eta$, we also have that

$$U^{\alpha}V^{\beta-2}\int_{|u|\ge U}\int_{|v|<V} v^2|f(u,v)|dudv \le C^{(1)}_{\alpha,\beta}. \tag{4.1}$$

Proof. By (3.2), for any $p \in \mathbb{N}_0 := \{0,1,2,\dots\}$ and $V \ge \eta$, we have that

$$\begin{aligned}(2^p\eta)^2 &\int_{2^p\eta\le|u|<2^{p+1}\eta}\int_{|v|<V} v^2|f(u,v)|dudv\\ &\le \int_{2^p\eta\le|u|<2^{p+1}\eta}\int_{|v|<V} u^2v^2|f(u,v)|dudv\\ &\le C_{\alpha,\beta}(2^{p+1}\eta)^{2-\alpha}V^{2-\beta} = 2^{2-\alpha}C_{\alpha,\beta}(2^p\eta)^{2-\alpha}V^{2-\beta}.\end{aligned}$$

Clearly, it follows that for all $p \in \mathbb{N}_0$ and $V \ge \eta$, we also have that

$$\int_{2^p\eta\le|u|<2^{p+1}\eta}\int_{|v|<V} v^2|f(u,v)|dudv \le 2^{2-\alpha}C_{\alpha,\beta}(2^p\eta)^{-\alpha}V^{2-\beta}, \tag{4.2}$$

whence we conclude that for any $r \in \mathbb{N}_0$ and $V \geq \eta$, we have that

(4.3)
$$\int_{|u|\geq 2^r\eta} \int_{|v|<V} v^2|f(u,v)|dudv = \sum_{p=r}^{\infty} \int_{2^p\eta\leq|u|<2^{p+1}\eta} \int_{|v|<V} v^2|f(u,v)|dudv$$
$$\leq 2^{2-\alpha} C_{\alpha,\beta} V^{2-\beta} \sum_{p=r}^{\infty} (2^p\eta)^{-\alpha} = \frac{4}{2^\alpha - 1} C_{\alpha,\beta} (2^r\eta)^{-\alpha} V^{2-\beta}.$$

This proves (4.1) in the special case $U := 2^r\eta$ with $r \in \mathbb{N}_0$ and $V \geq \eta$. Hence the fulfillment of (4.1) in the general case $U, V \geq \eta$ clearly follows. ♢

Lemma 4.2. Let $f \in L^1_{\text{loc}}(\mathbb{R}^2)$, $\eta > 0$ and $0 < \alpha, \beta \leq 2$. If condition (3.3) is satisfied, then

$$\lim_{U,V\to\infty} U^\alpha V^{\beta-2} \int_{|u|\geq U} \int_{|v|<V} v^2|f(u,v)|dudv = 0. \tag{4.4}$$

Proof. By (3.3), for every $\varepsilon > 0$ there exists some $p_0 = p_0(\varepsilon) \in \mathbb{N}_0$ such that for all $U \geq 2^{p_0}\eta$ and $V \geq 2^{p_0}$, we have that

$$U^{\alpha-2} V^{\beta-2} \int_{|u|<U} \int_{|v|<V} u^2 v^2 |f(u,v)|dudv \leq \varepsilon.$$

Let $r(\geq p_0)$ be an arbitrary integer. Analogously to (4.2) and (4.3), this time we obtain that

$$\int_{|u|\geq 2^r\eta} \int_{|v|<V} v^2|f(u,v)|dudv \leq \frac{4\varepsilon}{2^\alpha - 1} (2^r\eta)^{-\alpha} V^{2-\beta},$$

whence we conclude that for any integer $r \geq p_0$ and $V \geq 2^{p_0}$, we also have that

$$(2^r\eta)^\alpha V^{\beta-2} \int_{|u|\geq 2^r\eta} \int_{|v|<V} v^2|f(u,v)|dudv \leq \frac{4\varepsilon}{2^\alpha - 1}.$$

Since $\varepsilon > 0$ is arbitrary, the inequality just received proves (4.4). ♢

Lemma 4.3. Let $f \in L^1_{\text{loc}}(\mathbb{R}^2)$, $\eta > 0$ and $0 < \alpha, \beta \leq 2$. If condition (3.2) is satisfied for all $U, V \geq \eta$, then there exists another constant $C^{(2)}_{\alpha,\beta}$ such that for all $U, V \geq \eta$, we have that

$$U^2 V^2 \int_{|u|\geq U} \int_{|v|\geq V} |f(u,v)|dudv \leq C^{(2)}_{\alpha,\beta}. \tag{4.5}$$

In particular, from (4.5) it follows that

$$f \in L^1(\{u \in \mathbb{R} : |u| \geq \eta\} \times \{v \in \mathbb{R} : |v| \geq \eta\}). \tag{4.6}$$

Proof. By (3.2), for any $p, q \in \mathbb{N}_0$, we have that

$$(2^p\eta)^2(2^q\eta)^2 \int_{2^p\eta \leq |u| < 2^{p+1}\eta} \int_{2^q\eta \leq |v| < 2^{q+1}} |f(u,v)|dudv$$

$$\leq \int_{2^p\eta \leq |u| < 2^{p+1}\eta} \int_{2^q\eta \leq |v| < 2^{q+1}\eta} u^2v^2|f(u,v)|dudv$$

$$\leq C_{\alpha,\beta}(2^{p+1}\eta)^{2-\alpha}(2^{q+1}\eta)^{2-\beta},$$

whence it follows that

$$\int_{2^p\eta \leq |u| < 2^{p+1}\eta} \int_{2^q\eta \leq |v| < 2^{q+1}\eta} |f(u,v)|dudv \leq 2^{4-\alpha-\beta}C_{\alpha,\beta}(2^p\eta)^{-\alpha}(2^q\eta)^{-\beta}.$$

Now, similarly to (4.3), this time for all $r, s \in \mathbb{N}_0$ we conclude that

$$\int_{|u| \geq 2^r\eta} \int_{|v| \geq 2^s v} |f(u,v)|dudv$$

$$= \sum_{p=r}^{\infty} \sum_{q=s}^{\infty} \int_{2^p\eta \leq |u| < 2^{p+1}\eta} \int_{2^q\eta \leq |v| < 2^{q+1}\eta} |f(u,v)|dudv$$

$$\leq 2^{4-\alpha-\beta}C_{\alpha,\beta} \sum_{p=r}^{\infty}(2^p\eta)^{-\alpha} \sum_{q=s}^{\infty}(2^q\eta)^{-\beta}$$

$$= \frac{16}{(2^\alpha - 1)(2^\beta - 1)} C_{\alpha,\beta}(2^r\eta)^{-\alpha}(2^s\eta)^{-\beta}.$$

The inequality just received proves (4.5) in the special case $U := 2^r\eta$ and $V := 2^s\eta$ with $r, s \in \mathbb{N}_0$. Hence the fulfillment of (4.5) in the general case $U, V \geq \eta$ clearly follows. $\Diamond$

Lemma 4.4. Let $f \in L^1_{\text{loc}}(\mathbb{R}^2)$, $\eta > 0$ and $0 < \alpha, \beta \leq 2$. If condition (3.3) is satisfied, then we have that

$$\lim_{U,V \to \infty} U^\alpha V^\beta \int_{|u| \geq U} \int_{|v| \geq V} |f(u,v)|dudv = 0.$$

Proof. It goes along analogous lines as Lemma 4.2 was proved above. $\Diamond$

5. Proofs of our main results

Proof of Theorem 3.1. By condition (3.2), we may apply Lemma 4.3 with $U, V \geq \eta$, where $\eta \in \mathbb{R}_+$ occurs in condition (3.1). As a result, we obtain (4.5), whence (4.6) follows. Combining (3.1) and (4.6) yields that $f(x,y) \in L^1(\mathbb{R}^2)$ as it is stated in Theorem 3.1. Thus, the double trigonometric integral defined in (2.1) exists and it is continuous on $\mathbb{R}^2$.

It remains to prove that inequality (2.4) is satisfied. For the sake of brevity in writing, instead of (2.3) we make use of (2.5) with $2h$ and $2k$ in place of h and k, respectively. By (2.1) and (2.5), we get the following representations:

$$\Delta^2 F(x-2h, y-2k; 2h, 2k) \tag{5.1}$$

$$= \int\int_{\mathbb{R}^2} f(u,v)(e^{i(u(x+2h)+v(y+2k))} - e^{i(u(x+2h)+v(u-2k))}$$
$$+e^{i(u(x-2h)+v(y+2k))} + e^{i(u(x-2h))+v(y-2k))}$$
$$-2e^{i(u(x+2h)+vy)} - 2e^{i(ux+v(y+2k))}$$
$$-2e^{i(u(x-2h)+vy)} - 2e^{i(ux+v(y-2k))} + 4e^{i(ux+vy)})dudv$$
$$= \int\int_{\mathbb{R}^2} f(u,v)e^{i(ux+vy)}(e^{iu2h} + e^{-iu2h} - 2)(e^{iv2k} + e^{-iv2k} - 2)dudv$$
$$= 16\int\int_{\mathbb{R}^2} f(u,v)e^{i(ux+vy)} \sin^2 uh \sin^2 vkdudv.$$

Without loss of generality, we may assume that $0 < h, k \leq 1/\eta$, where η occurs in (3.1). We set

$$U := \frac{1}{h} \quad \text{and} \quad V := \frac{1}{k}. \tag{5.2}$$

By virtue of (5.1), we estimate as follows

$$Q(F; 2h, 2k) := \sup_{(x,y)\in\mathbb{R}^2} \frac{|\Delta^2 F(x-2h, y-2k; 2h, 2k)|}{(2h)^\alpha (2k)^\beta} \tag{5.3}$$

$$\leq 2^{4-\alpha-\beta}\Big(h^{2-\alpha}k^{2-\beta}\int_{|u|<U}\int_{|v|<V} u^2v^2|(u,v)|dudv$$
$$+h^{-\alpha}k^{2-\beta}\int_{|u|\geq U}\int_{|v|<V} v^2|f(u,v)|dudv$$

$$+h^{2-\alpha}k^{-\beta}\int_{|u|<U}\int_{|v|\geq V} u^2|f(u,v)|dudv$$
$$+h^{-\alpha}k^{-\beta}\int_{|u|\geq U}\int_{|v|\geq V} |f(u,v)|dudv\Big)$$
$$=: 2^{4-\alpha-\beta}(I_1+I_2+I_3+I_4),$$

say, where U and V are defined in (5.2).

By (5.2) and (5.3), we get

$$I_1 \leq U^{\alpha-2}V^{\beta-2}\int_{|u|<U}\int_{|v|<V} u^2v^2|f(u,v)|dudv \leq C_{\alpha,\beta}.$$

By Lemma 4.1, we get

$$I_2 \leq U^{\alpha}V^{\beta-2}\int_{|u|\geq U}\int_{|v|<V} v^2|f(u,v)|dudv \leq C^{(1)}_{\alpha,\beta}.$$

By the symmetric counterpart of Lemma 4.1, where the roles of U and V are interchanged, we get that

$$I_3 \leq U^{\alpha-2}V^{\beta}\int_{|u|<U}\int_{|v|\geq V} u^2|f(u,v)|dudv \leq C^{(1)}_{\beta,\alpha},$$

where the constant $C^{(1)}_{\beta,\alpha}$ is the symmetric counterpart the constant of $C^{(1)}_{\alpha,\beta}$ when the roles of α and β are interchanged. Finally, by Lemma 4.3, we get that

$$I_4 \leq U^{\alpha}V^{\beta}\int_{|u|\geq U}\int_{|v|\geq V} |f(u,v)|dudv \leq C^{(2)}_{\alpha,\beta}.$$

Combining (5.3) and the last four inequalities gives that for all $(x,y)\in\mathbb{R}^2$ we have that

$$Q(F;2h,2k) \leq 2^{4-\alpha-\beta}(C_{\alpha,\beta}+C^{(1)}_{\alpha,\beta}+C^{(1)}_{\beta,\alpha}+C^{(2)}_{\alpha,\beta}). \tag{5.4}$$

Keeping in mind the notation in (5.3), the inequality (5.4) proves the fulfillment of (2.4) with $2h$ and $2k$ in place of h and k, respectively, for all $(x,y)\in\mathbb{R}^2$. Thus, we conclude that $F(x,y)\in \mathrm{Zyg}(\alpha,\beta)$ and the proof of Theorem 3.1 is complete.

◊

Proof of Theorem 3.2. It runs along analogous lines as the proof of Theorem 3.1, except that this time we use Lemmas 4.2 and 4.4 instead of Lemmas 4.1 and 4.3, respectively. We emphasize that we also have that $f \in L^1(\mathbb{R}^2)$, due to the conditions (3.1) and (3.2). We do not enter into further details. $\Diamond$

6. Concluding remarks

Remark 6.1. Analysing the proof of [4, Lemma 3.1] it turns out that one can even prove the following more general auxiliary result in the one-dimensional case (cf. the proof of Lemma 4.1 of the present paper).

Lemma 6.2. Let $g \in L^1_{\mathrm{loc}}(\mathbb{R})$, $\eta > 0$ and $0 < \alpha \leq 2$. If for every $U \geq \eta$ we have that

$$U^{\alpha-2} \int_{|u|<U} u^2 |g(u)| du \leq C_\alpha,$$

where C_α is a constant, then there exists another constant $C_\alpha^{(1)}$ such that for every $U \geq \eta$ we have that

$$U^\alpha \int_{|u|\geq U} |g(u)| du \leq C_\alpha^{(1)}.$$

In particular, we also have that $g \in L^1(\mathbb{R})$.

Now, let $f \in L^1_{\mathrm{loc}}(\mathbb{R}^2)$ and consider the function

$$g(u) := \int_{|v|\leq\eta} |f(u,v)| dv, \quad u \in \mathbb{R}. \tag{6.1}$$

It is clear that $g \in L^1_{\mathrm{loc}}(\mathbb{R})$. Applying Lemma 6.2 for the function g defined in (6.1) results in the following auxiliary result in the two-dimensional case.

Lemma 6.3. Let $f \in L^1_{\mathrm{loc}}(\mathbb{R}^2)$, $\eta > 0$ and $0 < \alpha \leq 2$. If there exists a constant C_α such that for every $U \geq \eta$, we have that

$$U^{2-\alpha} \int_{|u|<U} \int_{|v|\leq\eta} u^2 |f(u,v)| du dv \leq C_\alpha, \tag{6.2}$$

then the function $g(u)$ defined in (6.1) belongs to $L^1(\mathbb{R})$, and consequently, we have that

$$f \in L^1(\mathbb{R} \times [-\eta, \eta]).$$

We note that the symmetric counterpart of (6.2) says that if $f \in L^1_{\mathrm{loc}}(\mathbb{R}^2)$ and there exists another constant C_β such that for every $V \geq \eta$, we have that

$$V^{2-\beta} \int_{|u|\leq\eta} \int_{|v|<V} v^2 |f(u,v)| du dv \leq C_\beta, \tag{6.3}$$

then we also have that $f \in L^1([-\eta,\eta] \times \mathbb{R})$.

After these preliminaries, the next version of Theorem 3.1 also holds true.

Theorem 6.4. Let $f \in L^1_{\mathrm{loc}}(\mathbb{R}^2)$, $\eta \in \mathbb{R}_+$ and $0 < \alpha, \beta \leq 2$. If conditions (6.2) and (6.3) are satisfied for all $U, V \geq \eta$, respectively, and condition (3.2) is also satisfied for all $U, V \geq \eta$, where C_α, C_β and $C_{\alpha,\beta}$ are constants, then $f \in L^1(\mathbb{R}^2)$ and the double trigonometric integral $F(x,y)$ defined in (2.1) belongs to the class $\mathrm{Zyg}(\alpha,\beta)$.

Likewise, the next version of Theorem 3.2 also holds true.

Theorem 6.5. Suppose $0 < \alpha, \beta < 2$ and that the conditions of Theorem 6.4 are satisfied. If, in addition, condition (3.3) is also satisfied, then the double trigonometric integral $F(x,y)$ defined in (2.1) belongs to the class $\mathrm{zyg}(\alpha,\beta)$.

Remark 6.6. In the particular case, where $f(u,v) \in L^1(\mathbb{R}^2)$ as well as $u^2 v^2 f(u,v) \in L^1(\mathbb{R}^2)$, the proof of the statement that the trigonometric integral $F(x,y)$ defined in (2.1) belongs to the class $\mathrm{Zyg}(2,2)$ is very simple. Indeed, by (5.1) we may estimate as follows

$$\frac{\Delta^2 F(x-2h, y-2k; 2h, 2k)}{(2h)^2(2k)^2} = \\ = \frac{1}{h^2k^2} \int\int_{\mathbb{R}^2} f(u,v) e^{i(ux+vy)} \sin^2 uh \sin^2 vk du dv,$$

whence we get that

$$Q(F; 2h, 2k) \leq \int\int_{\mathbb{R}^2} |f(u,v)| \left(\frac{\sin uh}{h}\right)^2 \left(\frac{\sin vk}{k}\right)^2 du dv,$$

where $Q(F; 2h, 2k)$ is defined in (5.3). Then because of $\sin^2 uh \leq u^2h^2$ and $\sin^2 vk \leq v^2k^2$, we have that

$$Q(F; 2h, 2k) \leq \int\int_{\mathbb{R}^2} u^2v^2 |f(u,v)| du dv < \infty,$$

due to the assumption that $u^2v^2f(u,v) \in L^1(\mathbb{R}^2)$. This completes the proof of our above statement that $F(x,y) \in \mathrm{Zyg}(2,2)$.

Remark 6.7. Our Theorems 3.1 and 3.2 as well as our statement in Remark 6.6 are also valid for the *double Fourier transform* $\hat{f}$ of $f \in L^1(\mathbb{R}^2)$ defined by (see, e.g., in [6, on p.2])

$$\hat{f}(x,y) := \int\int_{\mathbb{R}^2} f(u,v)e^{-2\pi i(ux+vy)}dudv, \quad (x,y) \in \mathbb{R}^2,$$

in place of the double trigonometric integral $F(x,y)$ defined in (2.1).

ACKNOWLEDGEMENT.
The authors express their gratitude to the referees for careful reading of the manuscript and their valuable suggestions to improve its presentation.

References

[1] R. P. BOAS, *Integrability theorems for trigonometric transforms*, Springer–Verlag, 1967.

[2] R. DEVORE and G.G. LORENTZ, *Constructive Approximation*, Springer, Berlin, 1993.

[3] V. FÜLÖP and F. MÓRICZ, *Absolutely convergent multiple Fourier series and multiplicative Zygmund classes of functions*, Analysis, **28** (2008), 345-354.

[4] V. FÜLÖP and F. MÓRICZ, *Sufficient conditions for trigonometric integrals to belong to a Zygmund class of functions*, Acta Sci. Math. (Szeged), **83** (2017), 433-439.

[5] F. MÓRICZ, *Absolutely convergent multiple Fourier series and multiplicative Lipschitz classes of functions*, Acta Math. Hungar., **121** (2008), 1-19.

[6] E.M. STEIN and G. WEISS, *Introduction to Fourier Analysis on Euclidean Spaces*, Princeton Univ. Press, 1971.

[7] S. TIKHONOV, *Smoothness conditions and Fourier series*, Math. Inequal. Appl., **10** (2007) 229-242.

[8] S. TIKHONOV, *Trigonometric series of Nikol'skii classes* , Acta Math. Hungar., **114** (2007), 61-78.

[9] S. S. VOLOSIVETS *Fourier transforms and generalized Lipschitz classes in uniform metric* J. Math. Anal. Appl. **383** (2011), 344-352.

[10] S. S. VOLOSIVETS, *Multiple Fourier coefficients and generalized Lipschitz classes in uniform metric* J. Math. Anal. Appl., **427** (2015), 1070-1083.

[11] A. ZYGMUND, *Smooth functions*, Duke Math. J., **12** (1945), 47-76.

[12] A. ZYGMUND, *Trigonometric Series,* Cambridge Univ. Press, 1959.