Mathematica Pannonica
26/1 (2017–2018), XXX–XXX

Brocard circle of the triangle in an isotropic plane

Z. **Kolar-Begović**

Department of Mathematics,
University of Osijek,
Trg Ljudevita Gaja 6, HR-31 000 Osijek

R. **Kolar-Šuper**

Faculty of Education,
University of Osijek,
Cara Hadrijana 10, HR-31 000 Osijek, Croatia

V. **Volenec**

Department of Mathematics,
University of Zagreb,
Bijenička cesta 30, HR-10 000 Zagreb, Croatia

Received: November 5, 2016

MSC 2000: 51 N 25

Keywords: Isotropic plane, Crelle–Brocard points, Brocard circle

Abstract: The concept of the Brocard circle of a triangle in an isotropic plane is defined in this paper. Some other statements about the introduced concepts and the connection with the concept of complementarity, isogonality, reciprocity, as well as the Brocard diameter, the Euler line, and the Steiner point of an allowable triangle are also considered..

The isotropic (or Galilean) plane is a projective–metric plane, where the absolute consists of one line, i.e., absolute line ω, and one point on that line, i.e., the absolute point Ω. The lines through the point Ω are isotropic lines, and the points on the line ω are isotropic points (the points at infinity). Two lines through the same isotropic point are parallel, and two points on the same isotropic line are parallel points. Each isotropic line is perpendicular to each nonisotropic line. Therefore, an isotropic

E-mail addresses: zkolar@mathos.hr, rkolar@foozos.hr, volenec@math.hr

plane is in fact an affine plane with the pointed direction of isotropic lines where the principle of duality holds.

In [7], it is shown that by a suitable choice of coordinates each allowable triangle in an isotropic plane can be set in the so-called *standard position*, i.e., that its circumscribed circle has the equation $y = x^2$, and its vertices are of the form $A = (a, a^2)$, $B = (b, b^2)$, $C = (c, c^2)$, where $a + b + c = 0$. We shall say then that ABC is a *standard triangle.* To prove geometric facts for each allowable triangle it is sufficient to give a proof for the standard triangle.

With labels

$$p = abc, \qquad q = bc + ca + ab,$$

$$p_1 = \frac{1}{3}(bc^2 + ca^2 + ab^2), \quad p_2 = \frac{1}{3}(b^2c + c^2a + a^2b)$$

and with the *Brocard angle* ω of the standard triangle ABC ([6]) given by the formula

$$\omega = -\frac{1}{3q}(b - c)(c - a)(a - b),$$

a number of useful equalities are valid, as for example

$$p^2 + pp_1 + p_1^2 = -\frac{q^3}{9}.$$

Each circle in the isotropic plane is given by the equation $y = ux^2 + vx + w$, and inversion with respect to this circle is the mapping given by the substitutions $x \to x$, $y \to 2ux^2 + 2vx + 2w - y$ (see [9]).

According to [7], [1], [4], [12], [8] and [10], the centroid G, the symmedian center K, the Steiner point S and Crelle–Brocard points Ω_1 and Ω_2 of the standard triangle are defined by the formulas

$$(1) \quad G = \left(0, -\frac{2}{3}q\right), \qquad K = \left(\frac{3p}{2q}, -\frac{q}{3}\right), \qquad S = \left(-\frac{3p}{q}, -\frac{9p^2}{q^2}\right),$$

$$(2) \quad \Omega_1 = \left(\frac{p - p_1}{q}, \frac{27p_1^2 - 2q^3}{9q^2}\right), \qquad \Omega_2 = \left(\frac{p - p_2}{q}, \frac{27p_2^2 - 2q^3}{9q^2}\right),$$

respectively.

The Euler line $\mathcal{E}$ is defined by the equation $x = 0$ and the Brocard diameter $\mathcal{B}$ by $x = \frac{3p}{2q}$. The orthic line $\mathcal{H}$ and the Lemoine line $\mathcal{L}$ are given by the equations $x = -\frac{q}{3}$ and

$$y = \frac{3p}{q}x + \frac{q}{3}, \tag{3}$$

respectively.

The straight line joining CB–points Ω_1 and Ω_2 is determined by the equation

$$y = \frac{3p}{q}x - \frac{54p^2 + 5q^3}{9q^2}, \tag{4}$$

the circumscribed circle $\mathcal{K}_c$ is given by $y = x^2$ and the first Lemoine circle $\mathcal{L}_1$ is determined by

$$y = 2x^2 - \frac{3p}{q}x + \frac{27p^2 - 2q^3}{18q^2}. \tag{5}$$

The tangential triangle of the triangle ABC has the Feuerbach point $\Phi_t = (0, 0)$.

Now we are going to prove some interesting facts about the aforementioned concepts.

Theorem 1. Symmedian center and CB–points Ω_1 and Ω_2 of the triangle lie on the same circle. Ω_1 and Ω_2 are symmetric with respect to the Brocard diameter of the considered triangle. (Figure 1)

Proof. The circle with the equation

$$y = 2x^2 - \frac{3p}{q}x - \frac{q}{3} \tag{6}$$

passes through the points K, Ω_1, Ω_2 from (1) and (2) because for the first two points we get

$$2\left(\frac{3p}{2q}\right)^2 - \frac{3p}{q} \cdot \frac{3p}{2q} - \frac{q}{3} = -\frac{q}{3},$$

$$
\begin{aligned}
2\frac{(p-p_1)^2}{q^2}-\frac{3p}{q}\cdot\frac{p-p_1}{q}-\frac{q}{3}&=\frac{1}{q^2}(2p_1^2-p^2-pp_1)-\frac{q}{3}\\
&=\frac{1}{q^2}(3p_1^2-p^2-pp_1-p_1^2)-\frac{q}{3}\\
&=\frac{1}{q^2}(3p_1^2+\frac{q^3}{9})-\frac{q}{3}=\frac{1}{9q^2}(27p_1^2-2q^3),
\end{aligned}
$$

and in the same way, this also holds for the point Ω_2. ♢

By analogy with the Euclidean case, the circle from Theorem 1 will be called the *Brocard circle* of the considered triangle.

Corollary 1. The Brocard circle $\mathcal{K}_b$ of the standard triangle ABC has equation (6).

The intersection of the Euler line and the orthic line of the triangle ABC is the point $\left(0,-\frac{q}{3}\right)$. Owing to (6), this immediately gives the following statement.

Corollary 2. The Brocard circle $\mathcal{K}_b$ of the triangle passes through the intersection of its Euler line $\mathcal{E}$ with its orthic line $\mathcal{H}$ (Figure 1).

Owing to equations (6) and (5), it follows that the first Lemoine circle of the standard triangle ABC can be obtained from its Brocard circle by the translation in the isotropic direction for the span

$$\frac{27p^2-2q^3}{18q^2}+\frac{q}{3}=\frac{27p^2+4q^3}{18q^2}.$$

If we eliminate x^2 from equation (6) and the equation $y=x^2$ of the circumscribed circle of the triangle ABC, we get the equation of the potential line of these two circles in the form (3), i.e., we have the following statement.

Corollary 3. The Lemoine line $\mathcal{L}$ of the triangle is the potential line of its circumscribed circle $\mathcal{K}_c$ and its Brocard circle $\mathcal{K}_b$.

Theorem 2. The Lemoine line $\mathcal{L}$ and the Brocard circle $\mathcal{K}_b$ of the triangle are mutually inverse curves with respect to its circumscribed circle $\mathcal{K}_c$.

Proof. Inversion with respect to its circumscribed circle is given by the substitution $y \to 2x^2 - y$, and by this substitution equations (3) and (6) can be transformed into each other. ♢

Theorem 3. The Feuerbach point $\Phi_t = (0,0)$ of the tangential triangle of a given triangle is the inverse point of the centroid G of this triangle with respect to its Brocard circle $\mathcal{K}_b$.

Proof. Inversion with respect to the Brocard circle (6) is given by the substitution $y \to 4x^2 - \frac{6p}{q}x - \frac{2q}{3} - y$. The point $\Phi_t = (0,0)$ can obviously be transformed into the point G from (1). ♢

Theorem 4. The Brocard circle of the triangle touches the potential line of its Lemoine circles at its symmedian center (Figure 1).

Proof. In [5], it is shown that the potential line of the Lemoine circles of the standard triangle ABC with the equation

$$y = \frac{3p}{q}x - \frac{9p^2}{2q^2} - \frac{q}{3} \tag{7}$$

passes through its symmedian center. It is enough to prove that circle (6) touches line (7). However, these two equations imply the equation

$$2x^2 - \frac{6p}{q}x + \frac{9p^2}{2q^2} = 0, \quad \text{i. e.} \quad \left(x - \frac{3p}{2q}\right)^2 = 0$$

with the double solution $x = \frac{3p}{2q}$. ♢

The assertion of Theorem 4 in Euclidean plane can be found in [3].

Theorem 5. The polar line with respect to the Brocard circle of the point L, isogonal to the reciprocal point K' of the symmedian center K of the triangle ABC, is the straight line joining CB–points Ω_1 and Ω_2 of the triangle ABC. The points K and L are parallel (Figure 1).

Proof. By [13], owing to

$$\begin{aligned} &q^2x^2 - 9pxy - 3qy^2 - 6pqx - 4q^2y + 9p^2 \\ &= \frac{9p^2}{4} + \frac{9p^2}{2} - \frac{q^3}{3} - 9p^2 + \frac{4q^3}{3} + 9p^2 = \frac{27}{4}p^2 + q^3 = \frac{1}{4}(27p^2 + 4q^3), \end{aligned}$$

$$\begin{aligned}
&3pqx^2+4q^2xy-9py^2+(9p^2+4q^3)x-12pqy-4pq^2\\
&\quad=\frac{27p^3}{4q}-2pq^2-pq^2+\frac{27p^3}{2q}+6pq^2+4pq^2-4pq^2\\
&\quad=\frac{81p^3}{4q}+3pq^2=\frac{3p}{4q}(27p^2+4q^3),
\end{aligned}$$

$$\begin{aligned}
&9p^2x^2+12pqxy+4q^2y^2+8pq^2x-(9p^2-4q^3)y-12p^2q\\
&\quad=\frac{81p^4}{4q^2}-6p^2q+\frac{4q^4}{9}+12p^2q+3p^2q-\frac{4q^4}{3}-12p^2q\\
&\quad=\frac{81p^4}{4q^2}-3p^2q-\frac{8q^4}{9}=\frac{1}{36q^2}(729p^4-108p^2q^3-32q^6)\\
&\quad=\frac{1}{36q^2}(27p^2+4q^3)(27p^2-8q^3),
\end{aligned}$$

the reciprocal point of the point $K=\left(\frac{3p}{2q},-\frac{q}{3}\right)$ is the point with the coordinates

$$-\frac{3p}{4q}(27p^2+4q^3):\frac{1}{4}(27p^2+4q^3)=-\frac{3p}{q},$$

$$\frac{1}{36q^2}(27p^2+4q^3)(27p^2-8q^3):\frac{1}{4}(27p^2+4q^3)=\frac{1}{9q^2}(27p^2-8q^3),$$

i.e., the point

(8) $$K'=\left(-\frac{3p}{q},\ \frac{1}{9q^2}(27p^2-8q^3)\right).$$

For this point we get

$$\begin{aligned}
y-x^2&=\frac{1}{9q^2}(27p^2-8q^3)-\frac{9p^2}{q^2}=-\frac{54p^2+8q^3}{9q^2}\\
&=-\frac{2}{9q^2}(27p^2+4q^3),\\
xy+qx-p&=-\frac{p}{3q^3}(27p^2-8q^3)-3p-p=-\frac{p}{3q^3}(27p^2+4q^3),\\
px-qy-y^2&=-\frac{3p^2}{q}-\frac{1}{9q}(27p^2-8q^3)-\frac{(27p^2-8q^3)^2}{81q^4}\\
&=-\frac{1}{81q^4}(729p^4+54p^2q^3-8q^6)\\
&=-\frac{1}{81q^4}(27p^2+4q^3)(27p^2-2q^3),
\end{aligned}$$

and owing to [9], its isogonal point is the point with the coordinates

$$-\frac{p}{3q^3}(27p^2+4q^3) : -\frac{2}{9q^2}(27p^2+4q^3) = \frac{3p}{2q},$$

$$-\frac{1}{81q^4}(27p^2+4q^3)(27p^2-2q^3) : -\frac{2}{9q^2}(27p^2+4q^3) = \frac{1}{18q^2}(27p^2-2q^3),$$

i.e., the point

$$L = \left(\frac{3p}{2q}, \frac{1}{18q^2}(27p^2-2q^3)\right). \tag{9}$$

The polar line of the point (x_o, y_o) with respect to circle (6) has the equation

$$y + y_o = 4x_o x - \frac{3p}{q}(x+x_o) - \frac{2q}{3}. \tag{10}$$

For the point $L = (x_o, y_o)$, from (9) we obtain

$$4x_o - \frac{3p}{q} = \frac{3p}{q},$$

$$-y_o - \frac{3p}{q}x_o - \frac{2q}{3} = \frac{2q^3-27p^2}{18q^2} - \frac{9p^2}{2q^2} - \frac{2q}{3} = -\frac{54p^2+5q^3}{9q^2},$$

and the polar line of the point L with respect to circle (6) is given by equation (4), and it is the line $\Omega_1\Omega_2$. The points K and L are obviously parallel. $\diamondsuit$

If we consider the points K' and L from (8) and (9) and the points Ω_1' and Ω_2' from Theorem 7 in [10], we obtain that $K' = \Omega_1'$ and $L = \Omega_2'$.

With $x_o = 0$ and $y_o = -\frac{2}{3}q$, equation (10) gets the form $y = -\frac{3p}{q}x$ and the Steiner point S from (1) satisfies it.

We have proved the following statement.

Theorem 6. The polar line of the centroid of a triangle with respect to its Brocard circle passes through its Steiner point.

The mentioned polar line obviously passes through the point $\Phi_t = (0,0)$, the Feuerbach point of the tangential triangle $A_tB_tC_t$ of the triangle ABC. According to [12], Th. 5, this polar line is a tangent line of its circumscribed Steiner ellipse of the triangle ABC at its Steiner point.

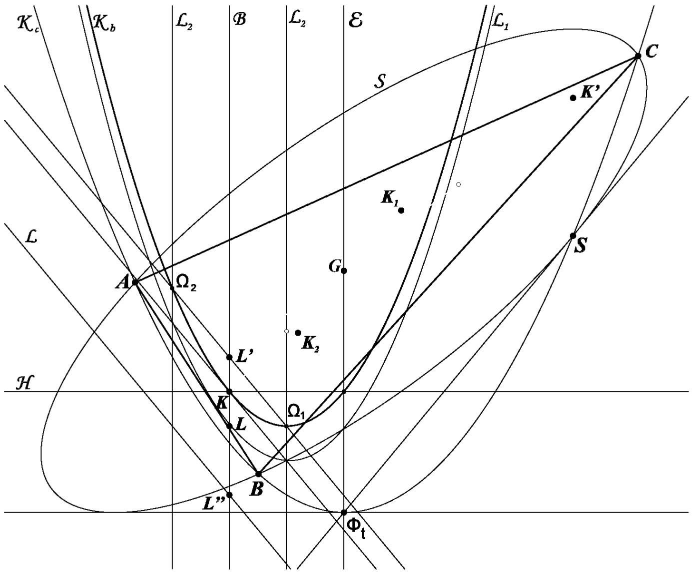

Figure 1.

Theorem 7. Let K' be the reciprocal point of the symmedian center K of the triangle ABC and L and L' the isogonal and complementary point of the point K', respectively. If the point L'' is the intersection of the Lemoine line and the Brocard diameter of the triangle ABC, then the point L is the midpoint of the points L' and L'' and all three points lie on the Brocard diameter of the triangle ABC (Figure 1). (see [11]).

Proof. From the equalities $2L' = 3G - K'$, (1) and (8) we get the coordinates x, y of the point L' $2x = \frac{3p}{q}$ and

$$2y = -2q - \frac{1}{9q^2}(27p^2 - 8q^3) = -\frac{27p^2 + 10q^3}{9q^2},$$

wherefrom

$$L' = \left(\frac{3p}{2q},\ -\frac{1}{18q^2}(27p^2 + 10q^3)\right). \tag{11}$$

With $x = \frac{3p}{2q}$ from equation (3) of the Lemoine line it follows

$$y = \frac{1}{6q^2}(27p^2 + 2q^3),$$

and therefore

$$L'' = \left(\frac{3p}{2q},\ \frac{1}{6q^2}(27p^2 + 2q^3)\right).$$

As

$$\frac{1}{2}\left[\frac{1}{6q^2}(27p^2 + 2q^3) - \frac{1}{18q^2}(27p^2 + 10q^3)\right] = \frac{1}{18q^2}(27p^2 - 2q^3),$$

the point L from (9) is the midpoint of the points L' and L''. ♢

The complementary point K_1 of the symmedian center K satisfies the equality $2K_1 = 3G - K$, wherefrom

$$K_1 = \left(-\frac{3p}{4q}, -\frac{5}{6}q\right).$$

With $x = -\frac{3p}{4q}$ and $y = \frac{5}{6}q$, for this point we get

$$\begin{aligned} &q^2x^2 - 9pxy - 3qy^2 - 6pqx - 4q^2y + 9p^2 \\ &\qquad = \frac{9}{16}p^2 - \frac{45}{8}p^2 - \frac{25}{12}q^3 + \frac{9}{2}p^2 + \frac{10}{3}q^3 + 9p^2 \\ &\qquad = \frac{135}{16}p^2 + \frac{5}{4}q^3 = \frac{5}{16}(27p^2 + 4q^3), \end{aligned}$$

$$\begin{aligned} &3pqx^2 + 4q^2xy - 9py^2 + (9p^2 + 4q^3)x - 12pqy - 4pq^2 \\ &\quad = \frac{27}{16}\cdot\frac{p^3}{q} + \frac{5}{2}pq^2 - \frac{25}{4}pq^2 - \frac{27}{4}\cdot\frac{p^3}{q} - 3pq^2 + 10pq^2 - 4pq^2 \\ &\quad = -\frac{81}{16}\cdot\frac{p^3}{q} - \frac{3}{4}pq^2 = -\frac{3p}{16q}(27p^2 + 4q^3), \end{aligned}$$

$$\begin{aligned}
&9p^2x^2+12pqxy+4q^2y^2+8pq^2x-(9p^2-4q^3)y-12p^2q\\
&\quad=\frac{81}{16}\cdot\frac{p^4}{q^2}+\frac{15}{2}p^2q+\frac{25}{9}q^4-6p^2q+\frac{15}{2}p^2q-\frac{10}{3}q^4-12p^2q\\
&\quad=\frac{81}{16}\cdot\frac{p^4}{q^2}-3p^2q-\frac{5}{9}q^4=\frac{1}{144q^2}(729p^4-432p^2q^3-80q^6)\\
&\quad=\frac{1}{144q^2}(27p^2+4q^3)(27p^2-20q^3),
\end{aligned}$$

and according to [13], its reciprocal point K_2 has the coordinates

$$\frac{3p}{16q}:\frac{5}{16}=\frac{3p}{5q},\qquad \frac{1}{144q^2}(27p^2-20q^3):\frac{5}{16}=\frac{1}{45q^2}(27p^2-20q^3),$$

i.e., we get

$$K_2=\left(\frac{3p}{5q},\ \frac{1}{45q^2}(27p^2-20q^3)\right). \tag{12}$$

For the point L' from (11) with $x=\frac{3p}{2q},\ -\frac{1}{18q^2}(27p^2+10q^3)$ we obtain

$$\begin{aligned}
y-x^2&=-\frac{1}{18q^2}(27p^2+10q^3)-\frac{9p^2}{4q^2}=-\frac{5}{36q^2}(27p^2+4q^3),\\
xy+qx-p&=-\frac{p}{12q^3}(27p^2+10q^3)+\frac{3}{2}p-p=-\frac{p}{12q^3}(27p^2+4q^3),\\
px-qy-y^2&=\frac{3p^2}{2q}+\frac{1}{18q}(27p^2+10q^3)-\frac{1}{324q^4}(27p^2+10q^3)^2\\
&=-\frac{1}{324q^4}(729p^4-432p^2q^3-80q^6)\\
&=-\frac{1}{324q^4}(27p^2+4q^3)(27p^2-20q^3),
\end{aligned}$$

and according to [9], its isogonal point has the coordinates

$$\frac{p}{12q^3}:\frac{5}{36q^2}=\frac{3p}{5q},\qquad \frac{1}{324q^4}(27p^2-20q^3):\frac{5}{36q^2}=\frac{1}{45q^2}(27p^2-20q^3),$$

and in fact it is the point K_2 from [13]. We have proved

Theorem 8. The isogonal point of the point L' from Theorem 7 is also the reciprocal point of the complementary point of the symmedian center K of the triangle ABC.

The point isogonal to the reciprocal point of the complementary point of the symmedian center of the triangle can be found in Euclidean geometry (see [2]).

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