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INDUCED MAPS ON n-FOLD PSEUDO-HYPERSPACE SUSPENSIONS OF CONTINUA

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Abstract: Let X be a continuum. If n is a positive integer, then $\mathcal{C}_n(X)$ denotes the space of all nonempty closed subsets of X with at most n components and let $\mathcal{F}_1(X)$ denote the space of singletons. The *n-fold pseudo-hyperspace suspension of* X is the quotient space $\mathcal{C}_n(X)/\mathcal{F}_1(X)$, with the quotient topology. For a given map between continua we study the induced maps between n-fold pseudo-hyperspace suspensions.

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1. Introduction

For a continuum X and for a positive integer n we denote by $\mathcal{C}_n(X)$ the hyperspace of all nonempty closed subsets of X with at most n components, and by $\mathcal{F}_1(X)$ the hyperspace of singletons. Denote by $PHS_n(X)$ the quotient space $\mathcal{C}_n(X)/\mathcal{F}_1(X)$ [6]. Given a map $f\colon X \to Y$ between continua X and Y, we let $\mathcal{C}_n(f)\colon \mathcal{C}_n(X) \to \mathcal{C}_n(Y)$ and $PHS_n(f)\colon PHS_n(X) \to PHS_n(Y)$ denote the corresponding induced maps.

Our purpose here is to study the induced maps $PHS_n(f)$ for certain classes of maps f.

The paper consists of eight sections. After the Introduction and Definitions, the third section is devoted to general properties of induced maps the main result of this section (Theorem 3.6) establishes that given a map f, the induced maps $\mathcal{C}_n(f)$ and $PHS_n(f)$ are surjective if f is weakly confluent. In the fourth, we work with atomic maps and homeomorphisms. The fifth section is devoted to monotone maps. In the sixth section we work with refinable and monotonically refinable maps. The seventh section is devoted to light maps. In the eighth section we present an example of an open map f such that neither $\mathcal{C}_n(f)$ nor $PHS_n(f)$ is open. We pose some questions.

Some of our arguments are similar to some that appear in the literature, we include them here for the convenience of the reader.

2. Definitions

If (Z, d) is a metric space, then given $A \subset Z$ and $\varepsilon > 0$, the open ball about A of radius ε is denoted by $\mathcal{V}^d_\varepsilon(A)$.

A *continuum* is a nonempty compact, connected metric space. A *subcontinuum* is a continuum contained in a space Z. A subcontinuum K of a continuum X is *terminal* provided that for each subcontinuum L of X such that $L \cap K \neq \emptyset$, we have that either $L \subset K$ or $K \subset L$.

A *map* means a continuous function. A surjective map $f\colon X \twoheadrightarrow Y$ between continua is said to be:

• *atomic* if for each subcontinuum K of Y such that $f(K)$ is nondegenerate, then $K = f^{-1}(f(K))$.

• *confluent* provided that for each subcontinuum Q of Y and each component K of $f^{-1}(Q)$, we have that $f(K) = Q$;

- an *ε-map* if $\varepsilon > 0$ and $\operatorname{diam}(f^{-1}(y)) < \varepsilon$ for each $y \in Y$.
- *k-to-1* provided that $f^{-1}(y)$ has exactly k points for each $y \in Y$;
- *light* if $f^{-1}(y)$ is totally disconnected for every $y \in Y$;
- *monotone* provided that $f^{-1}(y)$ is connected for each $y \in Y$;
- *monotonically refinable* if for each $\varepsilon > 0$, there exists a monotone ε-map $g\colon X \twoheadrightarrow Y$ such that $d(f(x), g(x)) < \varepsilon$ for all $x \in X$.
- *open* provided that for each open subset U of X, $f(U)$ is an open subset of Y.
- *refinable* if for every $\varepsilon > 0$, there exists an ε-map $g\colon X \twoheadrightarrow Y$ such that $d(f(x), g(x)) < \varepsilon$ for all $x \in X$.
- *weakly confluent* provided that for each subcontinuum Q of Y, there exists a subcontinuum K of X such that $f(K) = Q$.

An *arc* is any space homeomorphic to $[0, 1]$.

Given a continuum X and a positive integer n, $\mathcal{C}_n(X)$ denotes the *n-fold hyperspace* of X; that is:

$$\mathcal{C}_n(X) = \{A \subset X \mid A \text{ is nonempty, closed and has at most } n \text{ components}\},$$

topologized with the Hausdorff metric defined as follows:

$$\mathcal{H}_X(A, B) = \inf\{\varepsilon > 0 \mid A \subset \mathcal{V}^d_\varepsilon(B) \text{ and } B \subset \mathcal{V}^d_\varepsilon(A)\}.$$

The symbol $\mathcal{F}_n(X)$ denotes the *n-fold symmetric product* of X; that is:

$$\mathcal{F}_n(X) = \{A \subset X \mid A \text{ is nonempty and has at most } n \text{ points}\}.$$

Note that $\mathcal{F}_n(X) \subset \mathcal{C}_n(X)$. It is known that $\mathcal{C}_n(X)$ is an arcwise connected continuum (for $n = 1$ see [12, (1.12)], for $n \geq 2$ see [7, 3.1]) and $\mathcal{F}_n(X)$ is a continuum [1, p. 877]. Observe that $\mathcal{F}_1(X)$ is an isometric copy of X.

An *order arc* in $\mathcal{C}_n(X)$ is an arc $\alpha\colon [0, 1] \to \mathcal{C}_n(X)$ such that if $0 \leq s < t \leq 1$ then $\alpha(s) \subsetneq \alpha(t)$.

By the *n-fold pseudo-hyperspace suspension* of a continuum X, denoted by $PHS_n(X)$, we mean the quotient space:

$$PHS_n(X) = \mathcal{C}_n(X)/\mathcal{F}_1(X)$$

with the quotient topology [6]. The fact that $PHS_n(X)$ is a continuum follows from [9, 1.7.3]. Notice that $PHS_1(X)$ corresponds to the hyperspace suspension $HS(X)$ defined by Professor Sam B. Nadler, Jr. in [13].

Notation 2.1. Given a continuum X and a positive integer n, $q_X^n : \mathcal{C}_n(X) \to PHS_n(X)$ denotes the quotient map. Let $T_X^n = \mathcal{C}_n(f)(X)$. Also, let F_X^n denote the point $q_X^n(\mathcal{F}_1(X))$.

Remark 2.2. Let X be a continuum and let n be a positive integer. Note that $PHS_n(X) \setminus \{F_X^n\}$ is homeomorphic to $\mathcal{C}_n(X) \setminus \mathcal{F}_1(X)$ using the restriction of q_X^n to $\mathcal{C}_n(X) \setminus \mathcal{F}_1(X)$.

Given a map $f\colon X \to Y$ between continua and a positive integer n, the function $\mathcal{C}_n(f)\colon \mathcal{C}_n(X) \to \mathcal{C}_n(Y)$ given by $\mathcal{C}_n(f)(A) = f(A)$ is the *induced map by f between the n-fold hyperspaces of X and Y*. Note that $\mathcal{C}_n(f)$ is continuous ([9, 1.8.23]). Also, we have an induced map $PHS_n(f)\colon PHS_n(X) \to PHS_n(Y)$, given by

$$PHS_n(f)(\chi) = \begin{cases} q_Y^n(\mathcal{C}_n(f)((q_X^n)^{-1}(\chi))), & \text{if } \chi \neq F_X^n; \\ F_Y^n, & \text{if } \chi = F_X^n; \end{cases}$$

called the *induced map by f between the n-fold pseudo-hyperspace suspensions of X and Y*. Note that, by [4, 4.3, p. 126], $PHS_n(f)$ is continuous. In addition, the following diagram

$$\begin{array}{ccc} \mathcal{C}_n(X) & \xrightarrow{\mathcal{C}_n(f)} & \mathcal{C}_n(Y) \\ {\scriptstyle q_X^n}\big\downarrow & & \big\downarrow{\scriptstyle q_Y^n} \\ PHS_n(X) & \xrightarrow[PHS_n(f)]{} & PHS_n(Y) \end{array} \qquad (*)$$

is commutative.

3. Induced Maps

We begin with the following observation:

Remark 3.1. Let X and Y be continua, let n be a positive integer and let $f\colon X \to Y$ be a map. If $\alpha\colon [0,1] \to \mathcal{C}_n(X)$ is an order arc and $\mathcal{C}_n(f)(\alpha(0)) = \mathcal{C}_n(f)(\alpha(1))$, then for each $t \in [0,1]$ $\mathcal{C}_n(f)(\alpha(0)) = \mathcal{C}_n(f)(\alpha(t)) = \mathcal{C}_n(f)(\alpha(1))$.

Lemma 3.2. *Let X and Y be continua, let n be a positive integer and let $f\colon X \to Y$ be a map. Then $\mathcal{C}_n(f)^{-1}(\mathcal{C}_n(f)(X))$ is pathwise connected.*

Proof. If $\mathcal{C}_n(f)^{-1}(\mathcal{C}_n(f)(X)) = \{X\}$, there is nothing to prove. Suppose $\mathcal{C}_n(f)^{-1}(\mathcal{C}_n(f)(X)) \neq \{X\}$, and let $A \in \mathcal{C}_n(f)^{-1}(\mathcal{C}_n(f)(X)) \setminus \{X\}$. Let $\alpha\colon [0,1] \to \mathcal{C}_n(X)$ be an order arc such that $\alpha(0) = A$ and $\alpha(1) = X$ [12, (1.8)]. Since $\mathcal{C}_n(f)(A) = \mathcal{C}_n(f)(X)$, by Remark 3.1, we have that for each $t \in [0,1]$, $\mathcal{C}_n(f)(\alpha(t)) = \mathcal{C}_n(f)(X)$. Hence, $\alpha([0,1]) \subset \mathcal{C}_n(f)^{-1}(\mathcal{C}_n(f)(X))$. Therefore, $\mathcal{C}_n(f)^{-1}(\mathcal{C}_n(f)(X))$ is pathwise connected. $\Diamond$

Lemma 3.3. *Let X and Y be continua, let n be a positive integer and let $f\colon X \to Y$ be a map. If $\xi = PHS_n(f)(T^n_X)$, then $PHS_n(f)^{-1}(\xi) = q^n_X(\mathcal{C}_n(f)^{-1}(\mathcal{C}_n(f)(X)))$. In particular, $PHS_n(f)^{-1}(\xi)$ is pathwise connected.*

Proof. We consider two cases. Suppose $\xi = F^n_Y$. Since $(*)$ commutes, we have that
$$F^n_Y = PHS_n(T^n_X) = q^n_Y(\mathcal{C}_n(f)(X)).$$
Hence, $\mathcal{C}_n(f)(X) \in \mathcal{F}_1(Y)$. Thus, $f(X)$ is a singleton. This implies that f is a constant map. In particular both $\mathcal{C}_n(f)$ and $PHS_n(f)$ are constant maps too. As a consequence of this, we obtain that $PHS_n(f)^{-1}(\xi) = PSH_n(X)$ and $q^n_X(\mathcal{C}_n(f)^{-1}(\mathcal{C}_n(f)(X))) = q^n_X(\mathcal{C}_n(X)) = PHS_n(X)$. Therefore, $PHS_n(f)^{-1}(\xi) = q^n_X(\mathcal{C}_n(f)^{-1}(\mathcal{C}_n(f)(X)))$.

Assume that $\xi \neq F^n_Y$. Let $\chi \in PHS_n(f)^{-1}(\xi)$, we prove that $\xi \in q^n_X(\mathcal{C}_n(f)^{-1}(\mathcal{C}_n(f)(X)))$. Note that, since $\xi \neq F^n_Y$,
$$\xi = PHS_n(f)(\chi) = q^n_Y(\mathcal{C}_n(f)((q^n_X)^{-1}(\chi)))$$
and
$$\xi = PHS_n(f)(T^n_X) = q^n_Y(\mathcal{C}_n(f)((q^n_X)^{-1}(T^n_X))) = q^n_Y(\mathcal{C}_n(f)(X)).$$
Hence, $PHS_n(f)(\chi) = q^n_Y(\mathcal{C}_n(f)(X))$. Since $\xi \neq F^n_X$, this implies that $\mathcal{C}_n(f)((q^n_X)^{-1}(\chi)) = \mathcal{C}_n(f)(X)$. Thus, $(q^n_X)^{-1}(\chi) \in \mathcal{C}_n(f)^{-1}(\mathcal{C}_n(f)(X))$, and $\chi \in q^n_X(\mathcal{C}_n(f)^{-1}(\mathcal{C}_n(f)(X)))$. Therefore,
$$PHS_n(f)^{-1}(\xi) \subset q^n_X(\mathcal{C}_n(f)^{-1}(\mathcal{C}_n(f)(X))).$$

Now, let $\chi \in q_X^n(\mathcal{C}_n(f)^{-1}(\mathcal{C}_n(f)(X)))$, we show that $\chi \in PHS_n(f)^{-1}(\xi)$. Note that $\chi \neq F_X^n$, otherwise, $\mathcal{C}_n(f)^{-1}(\mathcal{C}_n(f)(X)) \subset \mathcal{F}_1(X)$ and $\mathcal{C}_n(f)(X) \in \mathcal{F}_1(Y)$. Hence,

$$\xi = PHS_n(f)(T_X^n) = q_Y^n(\mathcal{C}_n(f)((q_X^n)^{-1}(T_X^n))) = q_Y^n(\mathcal{C}_n(f)(X)) = F_Y^n,$$

a contradiction to our assumption.

Since $\chi \in q_X^n(\mathcal{C}_n(f)^{-1}(\mathcal{C}_n(f)(X)))$, $\mathcal{C}_n(f)((q_X^n)^{-1}(\chi)) = \mathcal{C}_n(f)(X)$. Thus, $q_Y^n(\mathcal{C}_n(f)((q_X^n)^{-1}(\chi))) = q_Y^n(\mathcal{C}_n(f)(X))$. Hence,

$$PHS_n(f)(\chi) = q_Y^n(\mathcal{C}_n(f)((q_X^n)^{-1}(\chi))) = q_Y^n(\mathcal{C}_n(f)(X)) =$$

$$q_Y^n(\mathcal{C}_n(f)((q_X^n)^{-1}(T_X^n))) = PHS_n(f)(T_X^n) = \xi.$$

Therefore, $q_X^n(\mathcal{C}_n(f)^{-1}(\mathcal{C}_n(f)(X))) \subset PHS_n(f)^{-1}(\xi)$, and $PHS_n(f)^{-1}(\xi) = q_X^n(\mathcal{C}_n(f)^{-1}(\mathcal{C}_n(f)(X)))$.

The fact that $PHS_n(f)^{-1}(\xi)$ is pathwise connected now follows from Lemma 3.2. $\Diamond$

Corollary 3.4. Let X and Y be continua and let n be a positive integer. If $f\colon X \twoheadrightarrow Y$ is a surjective map, then $PHS_n(f)^{-1}(T_Y^n)$ is a pathwise connected subcontinuum of $PHS_n(X)$.

Proof. Since f is surjective, we have that $f(X) = Y$; i.e., $\mathcal{C}_n(f)(X) = Y$. Since $(*)$ commutes, this implies that $PHS_n(f)(T_X^n) = T_Y^n$. Hence, $PHS_n(f)^{-1}(T_Y^n)$ is nonempty.

Since $PHS_n(f)$ is continuous, $PHS_n(f)^{-1}(T_Y^n)$ is a compact subset of $PHS_n(X)$. By Lemma 3.3, $PHS_n(f)^{-1}(T_Y^n)$ is pathwise connected. $\Diamond$

Lemma 3.5. Let X and Y be continua, let n be a positive integer and let $f\colon X \to Y$ be a map. If $PHS_n(f)$ is surjective, then f is surjective.

Proof. Since $PHS_n(f)$ is surjective, there exists $\chi \in PHS_n(X) \setminus \{F_X^n\}$ such that $PHS_n(f)(\chi) = T_Y^n$. Since $(*)$ commutes and $\chi \neq F_X^n$, we have that $\mathcal{C}_n(f)((q_X^n)^{-1}(\chi)) = Y$. Since $(q_X^n)^{-1}(\chi) \subset X$, $Y = \mathcal{C}_n(f)((q_X^n)^{-1}(\chi)) \subset \mathcal{C}_n(f)(X)$. Hence, $\mathcal{C}_n(f)(X) = Y$; i.e., $f(X) = Y$. Therefore, f is surjective. $\Diamond$

Theorem 3.6. Let X and Y be continua and let n be a positive integer. If $f\colon X \to Y$ is a map, then the following are equivalent:

(1) f is weakly confluent;

(2) $\mathcal{C}_n(f)$ is surjective;

(3) $PHS_n(f)$ is surjective.

Proof. By [3, Proposition 1], we have that f is weakly confluent if and only if $\mathcal{C}_n(f)$ is surjective. Since $(*)$ commutes and q^n_Y is surjective, if $\mathcal{C}_n(f)$ is surjective, then $PHS_n(f)$ is surjective.

Suppose $PHS_n(f)$ is surjective, we see that $\mathcal{C}_n(f)$ is surjective. By Lemma 3.5, f is surjective. Thus, if B is in $\mathcal{F}_1(Y)$, then there exists $A \in \mathcal{F}_1(X)$ such that $\mathcal{C}_n(f)(A) = B$. Assume $B \in \mathcal{C}_n(Y) \setminus \mathcal{F}_1(Y)$. Then $q^n_Y(B) \in PHS_n(Y) \setminus \{F^n_Y\}$. Since $PHS_n(f)$ is surjective, there exists $\chi \in PHS_n(X) \setminus \{F^n_X\}$ such that $PHS_n(f)(\chi) = q^n_Y(B)$. Hence, $(q^n_X)^{-1}(\chi) \in \mathcal{C}_n(X) \setminus \mathcal{F}_1(X)$. Since $(*)$ commutes, we have that $\mathcal{C}_n(f)((q^n_X)^{-1}(\chi)) = B$. Therefore, $\mathcal{C}_n(f)$ is surjective. $\Diamond$

4. Homeomorphisms

Theorem 4.1. Let X and Y be continua and let n be a positive integer. If $f\colon X \to Y$ is a map, then the following are equivalent:

(1) f is a homeomorphism;

(2) $\mathcal{C}_n(f)$ is a homeomorphism;

(3) $\mathcal{C}_n(f)$ is atomic;

(4) $PHS_n(f)$ is a homeomorphism;

(5) $PHS_n(f)$ is atomic.

Proof. By [3, Theorem 46], we have that (1), (2) and (3) are equivalent.

Suppose $\mathcal{C}_n(f)$ is a homeomorphism, we prove that $PHS_n(f)$ is a homeomorphism. Note that, by Theorem 3.6, $PHS_n(f)$ is surjective. To show that $PHS_n(f)$ is one-to-one, let $\chi \in PHS_n(Y)$. Suppose that $\chi \neq F^n_Y$. Since $(*)$ commutes, by Remark 2.2, we have that there exists a unique element $A \in \mathcal{C}_n(X)$ such that $PHS_n(f) \circ q^n_X(A) = \chi$. Assume that $\chi = F^n_Y$ and that there exists $\chi' \in PHS_n(X) \setminus \{F^n_X\}$ such that $PHS_n(f)(\chi') = \chi$. Note that $(q^n_X)^{-1}(\chi') \in \mathcal{C}_n(X) \setminus \mathcal{F}_1(X)$. Since $\mathcal{C}_n(f)$ is a homeomorphism, we have that $\mathcal{C}_n(f)((q^n_X)^{-1}(\chi')) \in \mathcal{C}_n(Y) \setminus \mathcal{F}_1(Y)$. Thus, by Remark 2.2, $q^n_Y(\mathcal{C}_n(f)((q^n_X)^{-1}(\chi'))) \in PHS_n(Y) \setminus \{F^n_Y\}$. Since $(*)$ commutes, we obtain that $PHS_n(f)(\chi') \in PHS_n(Y) \setminus \{F^n_Y\}$, a contradiction to our election of χ'. Therefore, $PHS_n(f)$ is one-to-one and a homeomorphism.

Suppose $PHS_n(f)$ is a homeomorphism, we show that $\mathcal{C}_n(f)$ is a homeomorphism. By [8, Theorem 3.4], this is true for $n = 1$. Assume $n \geq 2$. Observe that, by Theorem 3.6, $\mathcal{C}_n(f)$ is surjective. To prove $\mathcal{C}_n(f)$ is one-to-one, note that, by Remark 2.2, the commutativity of $(*)$ and the fact that $PHS_n(f)$ is a homeomorphism, we have that $\mathcal{C}_n(f)|_{\mathcal{C}_n(X)\setminus\mathcal{F}_1(X)}\colon \mathcal{C}_n(X) \setminus \mathcal{F}_1(X) \twoheadrightarrow \mathcal{C}_n(Y) \setminus \mathcal{F}_1(Y)$ is a homeomorphism. We only need to prove that $\mathcal{C}_n(f)|_{\mathcal{F}_1(X)}$ is one-to-one. Suppose there exist $\{x_1\}, \{x_2\} \in \mathcal{F}_1(X)$, with $\{x_1\} \neq \{x_2\}$, such that $\mathcal{C}_n(f)(\{x_1\}) = \mathcal{C}_n(f)(\{x_2\})$. This implies that $\mathcal{C}_n(f)(\{x_1, x_2\}) \in \mathcal{F}_1(Y)$, a contradiction to the fact that $\mathcal{C}_n(f)|_{\mathcal{C}_n(X)\setminus\mathcal{F}_1(X)}$ is a homeomorphism. Therefore, $\mathcal{C}_n(f)$ is a homeomorphism.

Assume that $PHS_n(f)$ is atomic, we see that f is a homeomorphism. Since $PHS_n(f)$ is an atomic map, for each $\chi \in PHS_n(Y)$, we have that $PHS_n(f)^{-1}(\chi)$ is a terminal subcontinuum of $PHS_n(X)$ [11, (2.1)]. Since $\mathcal{C}_n(X)$ is arcwise connected [9, 1.8.12], $PHS_n(X)$ is arcwise connected. It is easy to see that the only terminal subcontinua of an arcwise connected continuum are the continuum itself and singletons. Hence, if $\chi \in PHS_n(Y)$, then either $PHS_n(f)^{-1}(\chi) = PHS_n(X)$ or $PHS_n(f)^{-1}(\chi)$ is a singleton of $PHS_n(X)$. Note that $PHS_n(f)^{-1}(\chi) = PHS_n(X)$ is impossible, otherwise $PHS_n(f)$ would be a constant map, in particular, $PHS_n(f)$ would not be surjective. Thus, for each $\chi \in PHS_n(Y)$, $PHS_n(f)^{-1}(\chi)$ is a singleton. This implies that $PHS_n(f)$ is a homeomorphism, and, by the above argument, f is a homeomorphism. $\diamondsuit$

5. Monotone maps

Theorem 5.1. Let X and Y be continua and let n be a positive integer. If $f\colon X \to Y$ is a map, then the following are equivalent:

(1) f is monotone;

(2) $\mathcal{C}_n(f)$ is monotone;

(3) $PHS_n(f)$ is monotone.

Proof. By [3, Theorem 4], we have that f is monotone if and only if $\mathcal{C}_n(f)$ is monotone.

Suppose $\mathcal{C}_n(f)$ is monotone. Since $(*)$ commutes and q^n_Y is monotone, $q^n_Y \circ \mathcal{C}_n(f)$ is monotone [10, (5.1)]. Hence, by [10, (5.15)], we obtain that $PHS_n(f)$ is monotone.

Assume $PHS_n(f)$ is monotone, we show that f is monotone. Let $B \in \mathcal{C}_1(Y) \setminus \mathcal{F}_1(Y)$. By definition, $PHS_n(f)$ is surjective. Thus, by Theorem 3.6, f is weakly confluent. Hence, there exists $A \in \mathcal{C}_1(X)$ such that

$f(A) = B$. Since $(*)$ commutes and $q_X^n|_{\mathcal{C}_n(X)\setminus\mathcal{F}_1(X)}$ is a homeomorphism, we have that

$$\mathcal{C}_n(f)^{-1}(B) = (q_X^n)^{-1}(PHS_n(f)^{-1}(q_Y^n(B))).$$

Since $PHS_n(f)$ and q_X^n are monotone, by [9, 2.1.12], $\mathcal{C}_n(f)^{-1}(B)$ is connected. Since $A \in \mathcal{C}_1(X) \cap \mathcal{C}_n(f)^{-1}(B)$, by [12, (1.49)], $\bigcup \mathcal{C}_n(f)^{-1}(B) \in \mathcal{C}_1(X)$. Note that $\bigcup \mathcal{C}_n(f)^{-1}(B) \subset f^{-1}(B)$. To prove the other inclusion, let $x \in f^{-1}(B)$. Then $A \cup \{x\} \in \mathcal{C}_n(X)$ and $\mathcal{C}_n(f)(A \cup \{x\}) = B$. Thus, $A \cup \{x\} \in \mathcal{C}_n(f)^{-1}(B)$. In particular, $x \in \bigcup \mathcal{C}_n(f)^{-1}(B)$. Hence, $\bigcup \mathcal{C}_n(f)^{-1}(B) = f^{-1}(B)$. In particular, $f^{-1}(B)$ is connected.

Now, let $y \in Y$ and let $\{K_m\}_{m=1}^{\infty}$ be a decreasing sequence of subcontinua of Y such that $\bigcap_{m=1}^{\infty} K_m = \{y\}$.

This implies that $\bigcap_{m=1}^{\infty} f^{-1}(K_m) = f^{-1}(y)$. Hence, by the previous paragraph, we obtain that $\{f^{-1}(K_m)\}_{m=1}^{\infty}$ is a decreasing sequence of subcontinua. Therefore, $f^{-1}(y)$ is connected [9, 1.7.2], and f is monotone. $\Diamond$

6. Refinable Maps

Let X be a continuum and let n be a positive integer. We follow Professor Sam B. Nadler, Jr. [13] (compare with [8, p. 149]) to define a metric on $PHS_n(X)$. Let

$$\Im_n(X) = \{\mathcal{F}_1(X) \cup \{A\} \mid A \in \mathcal{C}_n(X)\}.$$

Note that $\Im_n(X) \subset \mathcal{C}_2(\mathcal{C}_n(X))$. Let $G_n \colon PHS_n(X) \twoheadrightarrow \Im_n(X)$ be given by

$$G_n(\chi) = \mathcal{F}_1(X) \cup (q_X^n)^{-1}(\chi).$$

Then G_n is a homeomorphism. Next, define

$$\rho_X^n \colon PHS_n(X) \times PHS_n(X) \to [0, \infty)$$

by

$$\rho_X^n(\chi_1, \chi_2) = \mathcal{H}_X^2(G_n(\chi_1), G_n(\chi_2)),$$

where $\mathcal{H}_X^2$ is the Hausdorff metric on $\mathcal{C}_2(\mathcal{C}_n(X))$ induced by the Hausdorff metric $\mathcal{H}_X$ on $\mathcal{C}_n(X)$. Then ρ_X^n is a metric. We rewrite [13, (2.3)], with our terminology, to obtain:

Theorem 6.1. Let X and Y be continua, let n be a positive integer, let $f\colon X \to Y$ be a map, and let $\varepsilon > 0$. If $\mathcal{C}_n(f)$ is an ε-map, then $PHS_n(f)$ is an ε-map.

Lemma 6.2. Let X and Y be continua, let n be a positive integer, let $g, f\colon X \to Y$ be two maps, and let $\varepsilon > 0$. If $d(f(x), g(x)) < \varepsilon$ for all $x \in X$, then $\rho^n_X(PHS_n(f)(\chi), PHS_n(g)(\chi)) < \varepsilon$ for each $\chi \in PHS_n(X)$.

Proof. Let $\chi \in PHS_n(X)$. If $\chi = F^n_X$, then

$$\rho^n_Y(PHS_n(f)(\chi), PHS_n(g)(\chi)) = \rho^n_Y(PHS_n(f)(F^n_X), PHS_n(g)(F^n_X)) =$$

$$\rho^n_Y(F^n_Y, F^n_Y) = 0 < \varepsilon.$$

Suppose that $\chi \neq F^n_X$. Then $(q^n_X)^{-1}(\chi) \in \mathcal{C}_n(X) \setminus \mathcal{F}_1(X)$, and

$$\rho^n_Y(PHS_n(f)(\chi), PHS_n(g)(\chi)) =$$

$$\mathcal{H}^2_Y(\mathcal{F}_1(Y) \cup \mathcal{C}_n(f)((q^n_X)^{-1}(\chi)), \mathcal{F}_1(Y) \cup \mathcal{C}_n(g)((q^n_X)^{-1}(\chi))) =$$
$$\mathcal{H}_Y(\mathcal{C}_n(f)((q^n_X)^{-1}(\chi)), \mathcal{C}_n(g)((q^n_X)^{-1}(\chi))) < \varepsilon.$$

The last inequality follows from [3, Lemma 37]. ◊

A continuum Y is in $Class(W)$ provided that for each continuum X, each surjective map $f\colon X \twoheadrightarrow Y$ is weakly confluent.

Theorem 6.3. Let X and Y be continua, where Y is in $Class(W)$, and let n be a positive integer. If $f\colon X \twoheadrightarrow Y$ is a refinable map, then $PHS_n(f)$ is a refinable map.

Proof. Let $\varepsilon > 0$. Since f is refinable, there exists an ε-map $g\colon X \twoheadrightarrow Y$ such that $d(f(x), g(x)) < \varepsilon$ for all $x \in X$. Since g is surjective and Y is in $Class(W)$, we have that g is weakly confluent. Hence, $\mathcal{C}_n(g)$ and $PHS_n(g)$ are both surjective (Theorem 3.6). By [3, Lemma 36] and Theorem 6.1, $PHS_n(g)$ is an ε-map.

By Lemma 6.2, $\rho^n_Y(PHS_n(f)(\chi), PHS_n(g)(\chi)) < \varepsilon$, for each $\chi \in PHS_n(X)$. Therefore, $PHS_n(f)$ is a refinable map. ◊

Theorem 6.4. *Let X and Y be continua and let n be a positive integer. If $f\colon X \twoheadrightarrow Y$ is a monotonically refinable map, then $PHS_n(f)$ is a monotonely refinable map.*

Proof. Let $\varepsilon > 0$. Since f is monotonely refinable, there exists a monotone ε-map $g\colon X \twoheadrightarrow Y$ such that $d(f(x), g(x)) < \varepsilon$ for every $x \in X$. By Theorem 5.1 and the argument given in the proof of Theorem 6.3, we obtain that $PHS_n(g)$ is a monotone ε-map such that

$$\rho_Y^n(PHS_n(f)(\chi), PHS_n(g)(\chi)) < \varepsilon$$

for each $\chi \in PHS_n(X)$. Therefore, $PHS_n(f)$ is a monotonely refinable map. $\Diamond$

In [3, Theorems 38 and 40] it is shown that if f is a (monotonically) refinable map, then $\mathcal{C}_n(f)$ is a (monotonically) refinable map.

Questions 6.5. Let X and Y be continua, let n be a positive integer and let $f\colon X \to Y$.

(*i*) If $\mathcal{C}_n(f)$ is (monotonically) refinable, then is f (monotonically) refinable?

(*ii*) If $\mathcal{C}_n(f)$ is (monotonically) refinable, then is $PHS_n(f)$ (monotonically) refinable?

(*iii*) If $PHS_n(f)$ is (monotonically) refinable, then f is (monotonically) refinable?

(*iv*) If $PHS_n(f)$ is (monotonically) refinable, then $\mathcal{C}_n(f)$ is (monotonically) refinable?

7. Light Maps

We begin with the following easy lemma.

Lemma 7.1. Let X and Y be continua, let $f\colon X \to Y$ be a map, and let A be a nonempty subset of X. If f is light, then $f|_A\colon A \twoheadrightarrow f(A)$ is light.

Theorem 7.2. Let X and Y be continua, let n be a positive integer and let $f\colon X \twoheadrightarrow Y$ be a surjective map. Consider the following statements:

(1) f is light;

(2) $\mathcal{C}_n(f)$ is light;

(3) $PHS_n(f)$ is light.

Then (3) implies (2) and (2) implies (1). Hence, (3) implies (1).

Proof. Assume $PHS_n(f)$ is light, we show that $\mathcal{C}_n(f)$ is light. Let $B \in \mathcal{C}_n(Y)$. Suppose first that $B \notin \mathcal{F}_1(Y)$. Since $(*)$ commutes,

$$PHS_n(f)^{-1}(q_Y^n(B)) = q_X^n(\mathcal{C}_n(f)^{-1}(B)).$$

Since $PHS_n(f)$ is light and $q_X^n|_{\mathcal{C}_n(X)\setminus\mathcal{F}_1(X)}$ is a homeomorphism, we have that $\mathcal{C}_n(f)^{-1}(B)$ is totally disconnected.

Assume that $B \in \mathcal{F}_1(Y)$. Then there exists $y \in Y$ such that $B = \{y\}$. Suppose that $\mathcal{C}_n(f)^{-1}(\{y\})$ is not totally disconnected. Let $\mathcal{A}$ be a nondegenerate component of $\mathcal{C}_n(f)^{-1}(\{y\})$. Then $\bigcup \mathcal{A} \in \mathcal{C}_n(X)$ [9, 6.1.2]. Since $\mathcal{A}$ is nondegenerate, at least one component, say A, of $\bigcup \mathcal{A}$ is nondegenerate. Let $a \in A$ and let $\alpha\colon [0,1] \to \mathcal{C}_n(X)$ be an order arc such that $\alpha(0) = \{a\}$ and $\alpha(1) = A$. Note that for each $t \in [0,1]$, $\mathcal{C}_n(f)(\alpha(t)) = \{y\}$. Then $q_X^n(\alpha([0,1]))$ is an arc in $PHS_n(X)$ such that $q_X^n(\alpha(0)) = F_X^n$. Since $(*)$ commutes, we have that

$$PHS_n(f)(q_X^n(\alpha([0,1]))) = q_Y^n(\mathcal{C}_n(f)(\alpha([0,1]))) = q_Y^n(\{\{y\}\}) = \{F_Y^n\}.$$

A contradiction to the fact that $PHS_n(f)$ is light. Therefore, $\mathcal{C}_n(f)^{-1}(\{y\})$ is totally disconnected, and $\mathcal{C}_n(f)$ is light.

The fact that $\mathcal{C}_n(f)$ implies f is light follows easily from Lemma 7.1. $\diamondsuit$

The following example shows a light map f such that neither $\mathcal{C}_n(f)$ nor $PHS_n(f)$ are light for any positive integer n.

Example 7.3. Let $f\colon [-1,1] \twoheadrightarrow [0,1]$ be given by $f(t) = |t|$. Clearly, f is a light map. Note that if $\mathcal{A} = \{A \in \mathcal{C}([-1,1]) \mid [0,1] \subset A\}$, then $\mathcal{A} \subset \mathcal{C}_1(f)^{-1}([0,1])$. Hence, $\mathcal{C}_1(f)$ is not light. Thus, by Lemma 7.1, $\mathcal{C}_n(f)$ is not light for any positive integer n. Also, by Theorem 7.2, it follows that $PHS_n(f)$ is not light for any positive integer n.

Theorem 7.4. Let X and Y be continua, let n be a positive integer, and let $f\colon X \to Y$ be a map. If k is a positive integer such that $PHS_n(f)$ is a k-to-1 map, then $k = 1$ and all f, $\mathcal{C}_n(f)$ and $PHS_n(f)$ are homeomorphisms.

Proof. Suppose $k \geq 2$. Let $\chi \in PHS_n(f)^{-1}(T_Y^n) \setminus \{T_X^n\}$. Then $(q_X^n)^{-1}(\chi) \in \mathcal{C}_n(X) \setminus \{X\}$. Since $(*)$ commutes, $\mathcal{C}_n(f)((q_X^n)^{-1}(\chi)) = Y$. Let $\alpha\colon [0,1] \to \mathcal{C}_n(X)$ be an order arc such that $\alpha(0) = (q_X^n)^{-1}(\chi)$ and $\alpha(1) = X$. Note that, for each $t \in [0,1]$, $\mathcal{C}_n(f)(\alpha(t)) = Y$. Thus,

$\mathcal{C}_n(f)(\alpha([0,1])) = \{Y\}$. Then $q_X^n \circ \alpha\colon [0,1] \to PHS_n(X)$ is an arc such that $q_X^n \circ \alpha(0) = \chi$, $q_X^n \circ \alpha(1) = T_X^n$, and $PHS_n(f)(q_X^n \circ \alpha([0,1])) = \{T_Y^n\}$. Hence, $q_X^n \circ \alpha([0,1]) \subset PHS_n(f)^{-1}(T_Y^n)$, a contradiction to the fact that $PHS_n(f)$ is k-to-1. Therefore, $k = 1$. Since $PHS_n(f)$ is 1-to-1, $PHS_n(f)$ is a homeomorphism. Thus, by Theorem 4.1, f and $\mathcal{C}_n(f)$ are both homeomorphism. $\lozenge$

Questions 7.5. Let X and Y be continua, let n be a positive integer and let $f\colon X \to Y$. If $\mathcal{C}_n(f)$ is light, then is $PHS_n(f)$ light?

8. Open maps

In contrast to equivalences in previous sections, we give an example that shows that f being open does not imply that either $\mathcal{C}_n(f)$ or $PHS_n(f)$ is open.

Example 8.1. Let n be a positive integer. For each $j \in \{1, \ldots, n+1\}$, let $L_j = \{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} \mid x_j \in [0,1]$ and $x_k = 0$ if $k \in \{1, \ldots, n+1\} \setminus \{j\}\}$. Let $X = \bigcup_{j=1}^{n+1} L_j$, and let $f\colon X \twoheadrightarrow [0,1]$ be given by

$$f((x_1, \ldots, x_{n+1})) = \begin{cases} 0, & \text{if } (x_1, \ldots, x_{n+1}) = (0, \ldots, 0); \\ x_j, & \text{if } (x_1, \ldots, x_{n+1}) \in L_j \setminus \{(0, \ldots, 0)\}, \\ & \text{for some } j \in \{1, \ldots, n+1\}. \end{cases}$$

Then f is an open map. We see that $\mathcal{C}_n(f)$ is not an open map. To this end, for each $j \in \{1, \ldots, n+1\}$, let $K_j = \{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} \mid x_j \in [0, \frac{1}{2}]$ and $x_k = 0$ if $k \in \{1, \ldots, n+1\} \setminus \{j\}\}$. Let $K = \bigcup_{j=1}^{n+1} K_j$. Also, for every $j \in \{1, \ldots, n+1\}$, let $U_j = \{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} \mid x_j \in (0, \frac{3}{4})$ and $x_k = 0$ if $k \in \{1, \ldots, n+1\} \setminus \{j\}\}$. Let $\mathcal{U} = \{A \in \mathcal{C}_n(X) \mid A \subset \bigcup_{j=1}^{n+1}(U_j \cup \{(0, \ldots, 0)\})$ and $A \cap U_j \neq \emptyset$ for all $j \in \{1, \ldots, n+1\}\}$. It is easy to see that $\mathcal{U}$ is an open subset of $\mathcal{C}_n(X)$ and $K \in \mathcal{U}$. We prove that $\mathcal{C}_n(f)$ is not open. Suppose $\mathcal{C}_n(f)$ is open and consider the sequence $\{[\frac{1}{m}, \frac{1}{2}]\}_{m=3}^{\infty}$, which converges to $[0, \frac{1}{2}]$. Since $\mathcal{C}_n(f)(K) = [0, \frac{1}{2}]$, we obtain that $[0, \frac{1}{2}] \in \mathcal{C}_n(f)(\mathcal{U})$. Since we are assuming that $\mathcal{C}_n(f)(\mathcal{U})$ is open, there exists a positive integer M such that for every $m \geq M$, $[\frac{1}{m}, \frac{1}{2}] \in \mathcal{C}_n(f)(\mathcal{U})$. Let $m \geq M$. Then there exists $B_m \in \mathcal{U}$ such that $\mathcal{C}_n(f)(B_m) = [\frac{1}{m}, \frac{1}{2}]$. This implies that $f(B_m) = [\frac{1}{m}, \frac{1}{2}]$. Observe that $f^{-1}([\frac{1}{m}, \frac{1}{2}]) = \bigcup_{j=1}^{n+1}[\frac{1}{m}, \frac{1}{2}]_j$, where $[\frac{1}{m}, \frac{1}{2}]_j = \{(x_1, \ldots, x_{n+1}) \in L_j \mid x_j \in [\frac{1}{m}, \frac{1}{2}]\}$. Since $B_m \in \mathcal{C}_n(X)$ and $B_m \subset f^{-1}([\frac{1}{m}, \frac{1}{2}])$, without loss of generality, we assume that $B_m \subset \bigcup_{j=1}^{n}[\frac{1}{m}, \frac{1}{2}]_j$. Hence, $B_m \cap U_{m+1} = \emptyset$, a contradiction to the fact that $B_m \in \mathcal{U}$. Therefore, $\mathcal{C}_n(f)$ is not open. To show that $PHS_n(f)$ is not open, note that $\mathcal{U} \cap \mathcal{F}_1(X) = \emptyset$ and $\mathcal{C}_n(f)(\mathcal{U}) \cap \mathcal{F}_1(Y) = \emptyset$. This implies that $q_X^n(\mathcal{U})$ is an open subset of $PHS_n(X)$ and $q_Y^n(\mathcal{C}_n(f)(\mathcal{U}))$ is not an open subset of $PHS_n(Y)$ (Remark 2.2). Since $(*)$ commutes, we obtain that $PHS_n(f)(q_X^n(\mathcal{U})) = q_Y^n(\mathcal{C}_n(f)(\mathcal{U}))$. Therefore, $PHS_n(f)$ is not open.

Questions 8.2. Let X and Y be continua, let n be a positive integer and let $f\colon X \to Y$ be a map. Is it true that $\mathcal{C}_n(f)$ is open if and only if $PHS_n(f)$ is open?

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