INDUCED MAPS ON $n$-FOLD PSEUDO-HYPERSPACE SUSPENSIONS OF CONTINUA

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Abstract: Let $X$ be a continuum. If $n$ is a positive integer, then $C_n(X)$ denotes the space of all nonempty closed subsets of $X$ with at most $n$ components and let $F_1(X)$ denote the space of singletons. The $n$-fold pseudo-hyperspace suspension of $X$ is the quotient space $C_n(X)/F_1(X)$, with the quotient topology. For a given map between continua we study the induced maps between $n$-fold pseudo-hyperspace suspensions.

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1. Introduction

For a continuum $X$ and for a positive integer $n$ we denote by $C_n(X)$ the hyperspace of all nonempty closed subsets of $X$ with at most $n$ components, and by $F_1(X)$ the hyperspace of singletons. Denote by $PHS_n(X)$ the quotient space $C_n(X)/F_1(X)$ [6]. Given a map $f: X \to Y$ between continua $X$ and $Y$, we let $C_n(f): C_n(X) \to C_n(Y)$ and $PHS_n(f): PHS_n(X) \to PHS_n(Y)$ denote the corresponding induced maps.

Our purpose here is to study the induced maps $PHS_n(f)$ for certain classes of maps $f$.

The paper consists of eight sections. After the Introduction and Definitions, the third section is devoted to general properties of induced maps the main result of this section (Theorem 3.6) establishes that given a map $f$, the induced maps $C_n(f)$ and $PHS_n(f)$ are surjective if $f$ is weakly confluent. In the fourth, we work with atomic maps and homeomorphisms. The fifth section is devoted to monotone maps. In the sixth section we work with refinable and monotonically refinable maps. The seventh section is devoted to light maps. In the eighth section we present an example of an open map $f$ such that neither $C_n(f)$ nor $PHS_n(f)$ is open. We pose some questions.

Some of our arguments are similar to some that appear in the literature, we include them here for the convenience of the reader.

2. Definitions

If $(Z,d)$ is a metric space, then given $A \subset Z$ and $\varepsilon > 0$, the open ball about $A$ of radius $\varepsilon$ is denoted by $B^d_\varepsilon(A)$.

A continuum is a nonempty compact, connected metric space. A subcontinuum is a continuum contained in a space $Z$. A subcontinuum $K$ of a continuum $X$ is terminal provided that for each subcontinuum $L$ of $X$ such that $L \cap K \neq \emptyset$, we have that either $L \subset K$ or $K \subset L$.

A map means a continuous function. A surjective map $f: X \to Y$ between continua is said to be:

- **atomic** if for each subcontinuum $K$ of $Y$ such that $f(K)$ is non-degenerate, then $K = f^{-1}(f(K))$.
- **confluent** provided that for each subcontinuum $Q$ of $Y$ and each component $K$ of $f^{-1}(Q)$, we have that $f(K) = Q$;
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- an $\varepsilon$-map if $\varepsilon > 0$ and $\text{diam}(f^{-1}(y)) < \varepsilon$ for each $y \in Y$.
- $k$-to-$1$ provided that $f^{-1}(y)$ has exactly $k$ points for each $y \in Y$;
- light if $f^{-1}(y)$ is totally disconnected for every $y \in Y$;
- monotone provided that $f^{-1}(y)$ is connected for each $y \in Y$;
- monotonically refinable if for each $\varepsilon > 0$, there exists a monotone $\varepsilon$-map $g : X \to Y$ such that $d(f(x), g(x)) < \varepsilon$ for all $x \in X$.
- open provided that for each open subset $U$ of $X$, $f(U)$ is an open subset of $Y$.
- refinable if for every $\varepsilon > 0$, there exists an $\varepsilon$-map $g : X \to Y$ such that $d(f(x), g(x)) < \varepsilon$ for all $x \in X$.
- weakly confluent provided that for each subcontinuum $Q$ of $Y$, there exists a subcontinuum $K$ of $X$ such that $f(K) = Q$.

An arc is any space homeomorphic to $[0, 1]$.

Given a continuum $X$ and a positive integer $n$, $C_n(X)$ denotes the $n$-fold hyperspace of $X$; that is:

$$C_n(X) = \{ A \subset X \mid A \text{ is nonempty, closed and has at most } n \text{ components} \},$$

topologized with the Hausdorff metric defined as follows:

$$H_X(A, B) = \inf \{\varepsilon > 0 \mid A \subset V^d_\varepsilon(B) \text{ and } B \subset V^d_\varepsilon(A)\}.$$

The symbol $F_n(X)$ denotes the $n$-fold symmetric product of $X$; that is:

$$F_n(X) = \{ A \subset X \mid A \text{ is nonempty and has at most } n \text{ points} \}.$$

Note that $F_n(X) \subset C_n(X)$. It is known that $C_n(X)$ is an arcwise connected continuum (for $n = 1$ see [12, (1.12)], for $n \geq 2$ see [7, 3.1]) and $F_n(X)$ is a continuum [1, p. 877]. Observe that $F_1(X)$ is an isometric copy of $X$.

An order arc in $C_n(X)$ is an arc $\alpha : [0, 1] \to C_n(X)$ such that if $0 \leq s < t \leq 1$ then $\alpha(s) \ess \alpha(t)$.

By the $n$-fold pseudo-hyperspace suspension of a continuum $X$, denoted by $PHS_n(X)$, we mean the quotient space:

$$PHS_n(X) = C_n(X) / F_1(X)$$
with the quotient topology [6]. The fact that \( PHS_n(X) \) is a continuum follows from [9, 1.7.3]. Notice that \( PHS_1(X) \) corresponds to the hyper-space suspension \( HS(X) \) defined by Professor Sam B. Nadler, Jr. in [13].

**Notation 2.1.** Given a continuum \( X \) and a positive integer \( n \),
\[ q^n_X : \mathcal{C}_n(X) \rightarrow PHS_n(X) \]
denotes the quotient map. Let \( T^n_X = \mathcal{C}_n(f)(X) \).
Also, let \( F^n_X \) denote the point \( q^n_X(\mathcal{F}_1(X)) \).

**Remark 2.2.** Let \( X \) be a continuum and let \( n \) be a positive integer.
Note that \( PHS_n(X) \setminus \{ F^n_X \} \) is homeomorphic to \( \mathcal{C}_n(X) \setminus \mathcal{F}_1(X) \) using the restriction of \( q^n_X \) to \( \mathcal{C}_n(X) \setminus \mathcal{F}_1(X) \).

Given a map \( f : X \rightarrow Y \) between continua and a positive integer \( n \), the function \( \mathcal{C}_n(f) : \mathcal{C}_n(X) \rightarrow \mathcal{C}_n(Y) \) given by \( \mathcal{C}_n(f)(A) = f(A) \) is the induced map by \( f \) between the \( n \)-fold hyperspaces of \( X \) and \( Y \). Note that \( \mathcal{C}_n(f) \) is continuous ([9, 1.8.23]). Also, we have an induced map \( PHS_n(f) : PHS_n(X) \rightarrow PHS_n(Y) \), given by
\[
PHS_n(f)(\chi) = \begin{cases} 
q^n_Y(C_n(f)((q^n_X)^{-1}(\chi))), & \text{if } \chi \neq F^n_X; \\
F^n_Y, & \text{if } \chi = F^n_X;
\end{cases}
\]
called the induced map by \( f \) between the \( n \)-fold pseudo-hyperspace suspensions of \( X \) and \( Y \). Note that, by [4, 4.3, p. 126], \( PHS_n(f) \) is continuous. In addition, the following diagram
\[
\begin{array}{ccc}
\mathcal{C}_n(X) & \xrightarrow{\mathcal{C}_n(f)} & \mathcal{C}_n(Y) \\
q^n_X \downarrow & & \downarrow q^n_Y \\
PHS_n(X) & \xrightarrow{PHS_n(f)} & PHS_n(Y) \\
\end{array}
\]
is commutative.

**3. Induced Maps**

We begin with the following observation:

**Remark 3.1.** Let \( X \) and \( Y \) be continua, let \( n \) be a positive integer and let \( f : X \rightarrow Y \) be a map. If \( \alpha : [0, 1] \rightarrow \mathcal{C}_n(X) \) is an order arc and \( \mathcal{C}_n(f)(\alpha(0)) = \mathcal{C}_n(f)(\alpha(1)) \), then for each \( t \in [0, 1] \) \( \mathcal{C}_n(f)(\alpha(0)) = \mathcal{C}_n(f)(\alpha(t)) = \mathcal{C}_n(f)(\alpha(1)) \).
Lemma 3.2. Let $X$ and $Y$ be continua, let $n$ be a positive integer and let $f: X \to Y$ be a map. Then $C_n(f)^{-1}(C_n(f)(X))$ is pathwise connected.

Proof. If $C_n(f)^{-1}(C_n(f)(X)) = \{X\}$, there is nothing to prove. Suppose $C_n(f)^{-1}(C_n(f)(X)) \neq \{X\}$, and let $A \in C_n(f)^{-1}(C_n(f)(X)) \setminus \{X\}$. Let $\alpha: [0, 1] \to C_n(X)$ be an order arc such that $\alpha(0) = A$ and $\alpha(1) = X$ [12, (1.8)]. Since $C_n(f)(A) = C_n(f)(X)$, by Remark 3.1, we have that for each $t \in [0, 1]$, $C_n(f)(\alpha(t)) = C_n(f)(X)$. Hence, $\alpha([0, 1]) \subset C_n(f)^{-1}(C_n(f)(X))$. Therefore, $C_n(f)^{-1}(C_n(f)(X))$ is pathwise connected. \hfill\Box

Lemma 3.3. Let $X$ and $Y$ be continua, let $n$ be a positive integer and let $f: X \to Y$ be a map. If $\xi = \text{PHS}_n(f)(T_X^n)$, then $\text{PHS}_n(f)^{-1}(\xi) = q_X^n(C_n(f)^{-1}(C_n(f)(X)))$. In particular, $\text{PHS}_n(f)^{-1}(\xi)$ is pathwise connected.

Proof. We consider two cases. Suppose $\xi = F_Y^n$. Since (\ast) commutes, we have that

$$F_Y^n = \text{PHS}_n(T_X^n) = q_X^n(C_n(f)(X)).$$

Hence, $C_n(f)(X) \in F_Y^1(Y)$. Thus, $f(X)$ is a singleton. This implies that $f$ is a constant map. In particular both $C_n(f)$ and $\text{PHS}_n(f)$ are constant maps too. As a consequence of this, we obtain that $\text{PHS}_n(f)^{-1}(\xi) = \text{PSH}_n(X)$ and $q_X^n(C_n(f)^{-1}(C_n(f)(X))) = q_X^n(C_n(X)) = \text{PHS}_n(X)$. Therefore, $\text{PHS}_n(f)^{-1}(\xi) = q_X^n(C_n(f)^{-1}(C_n(f)(X)))$.

Assume that $\xi \neq F_Y^n$. Let $\chi \in \text{PHS}_n(f)^{-1}(\xi)$, we prove that $\xi \in q_X^n(C_n(f)^{-1}(C_n(f)(X)))$. Note that, since $\xi \neq F_Y^n$,

$$\xi = \text{PHS}_n(f)(\chi) = q_X^n(C_n(f)((q_X^n)^{-1}(\chi)))$$

and

$$\xi = \text{PHS}_n(f)(T_X^n) = q_X^n(C_n(f)((q_X^n)^{-1}(T_X^n))) = q_X^n(C_n(f)(X)).$$

Hence, $\text{PHS}_n(f)(\chi) = q_X^n(C_n(f)(X))$. Since $\xi \neq F_Y^n$, this implies that $C_n(f)((q_X^n)^{-1}(\chi)) = C_n(f)(X)$. Thus, $(q_X^n)^{-1}(\chi) \in C_n(f)^{-1}(C_n(f)(X))$, and $\chi \in q_X^n(C_n(f)^{-1}(C_n(f)(X)))$. Therefore,

$$\text{PHS}_n(f)^{-1}(\xi) \subset q_X^n(C_n(f)^{-1}(C_n(f)(X))).$$
Now, let $\chi \in q^n_X(\mathcal{C}_n(f)^{-1}(\mathcal{C}_n(f)(X)))$, we show that $\chi \in \text{PHS}_n(f)^{-1}(\xi)$. Note that $\chi \neq F^n_X$, otherwise, $\mathcal{C}_n(f)^{-1}(\mathcal{C}_n(f)(X)) \subset \mathcal{F}_1(X)$ and $\mathcal{C}_n(f)(X) \in \mathcal{F}_1(Y)$. Hence,

$$\xi = \text{PHS}_n(f)(T^n_X) = q_Y^n(\mathcal{C}_n(f)((q^n_X)^{-1}(T^n_X))) = q_Y^n(\mathcal{C}_n(f)(X)) = F^n_Y,$$

a contradiction to our assumption.

Since $\chi \in q^n_X(\mathcal{C}_n(f)^{-1}(\mathcal{C}_n(f)(X)))$, $\mathcal{C}_n(f)((q^n_X)^{-1}(\chi)) = \mathcal{C}_n(f)(X)$. Thus, $q_Y^n(\mathcal{C}_n(f)((q^n_X)^{-1}(\chi))) = q_Y^n(\mathcal{C}_n(f)(X))$. Hence,

$$\text{PHS}_n(f)(\chi) = q_Y^n(\mathcal{C}_n(f)((q^n_X)^{-1}(\chi))) = q_Y^n(\mathcal{C}_n(f)(X)) =$$

$$q_Y^n(\mathcal{C}_n(f)((q^n_X)^{-1}(T^n_X))) = \text{PHS}_n(f)(T^n_X) = \xi.$$

Therefore, $q^n_X(\mathcal{C}_n(f)^{-1}(\mathcal{C}_n(f)(X))) \subset \text{PHS}_n(f)^{-1}(\xi)$, and $\text{PHS}_n(f)^{-1}(\xi) = q^n_X(\mathcal{C}_n(f)^{-1}(\mathcal{C}_n(f)(X)))$.

The fact that $\text{PHS}_n(f)^{-1}(\xi)$ is pathwise connected now follows from Lemma 3.2.

\begin{corollary}
Let $X$ and $Y$ be continua and let $n$ be a positive integer. If $f: X \to Y$ is a surjective map, then $\text{PHS}_n(f)^{-1}(T^n_Y)$ is a pathwise connected subcontinuum of $\text{PHS}_n(X)$.
\end{corollary}

\textbf{Proof.} Since $f$ is surjective, we have that $f(X) = Y$; i.e., $\mathcal{C}_n(f)(X) = Y$. Since (*) commutes, this implies that $\text{PHS}_n(f)(T^n_X) = T^n_Y$. Hence, $\text{PHS}_n(f)^{-1}(T^n_Y)$ is nonempty.

Since $\text{PHS}_n(f)$ is continuous, $\text{PHS}_n(f)^{-1}(T^n_Y)$ is a compact subset of $\text{PHS}_n(X)$. By Lemma 3.3, $\text{PHS}_n(f)^{-1}(T^n_Y)$ is pathwise connected. \hfill \Box

\begin{lemma}
Let $X$ and $Y$ be continua, let $n$ be a positive integer and let $f: X \to Y$ be a map. If $\text{PHS}_n(f)$ is surjective, then $f$ is surjective.
\end{lemma}

\textbf{Proof.} Since $\text{PHS}_n(f)$ is surjective, there exists $\chi \in \text{PHS}_n(X) \setminus \{F^n_X\}$ such that $\text{PHS}_n(f)(\chi) = T^n_Y$. Since (*) commutes and $\chi \neq F^n_X$, we have that $\mathcal{C}_n(f)((q^n_X)^{-1}(\chi)) = Y$. Since $\mathcal{C}_n(f)((q^n_X)^{-1}(\chi)) \subset X$, $Y = \mathcal{C}_n(f)((q^n_X)^{-1}(\chi)) \subset \mathcal{C}_n(f)(X)$. Hence, $\mathcal{C}_n(f)(X) = Y$; i.e., $f(X) = Y$.

Therefore, $f$ is surjective. \hfill \Box

\begin{theorem}
Let $X$ and $Y$ be continua and let $n$ be a positive integer. If $f: X \to Y$ is a map, then the following are equivalent:

1. $f$ is weakly confluent;
2. $\mathcal{C}_n(f)$ is surjective;
3. $\text{PHS}_n(f)$ is surjective.
\end{theorem}
**Proof.** By [3, Proposition 1], we have that $f$ is weakly confluent if and only if $C_n(f)$ is surjective. Since $(\ast)$ commutes and $q^n_Y$ is surjective, if $C_n(f)$ is surjective, then $PHS_n(f)$ is surjective.

Suppose $PHS_n(f)$ is surjective, we see that $C_n(f)$ is surjective. By Lemma 3.5, $f$ is surjective. Thus, if $B$ is in $\mathcal{F}_1(Y)$, then there exists $A \in \mathcal{F}_1(X)$ such that $C_n(f)(A) = B$. Assume $B \in C_n(Y) \setminus \mathcal{F}_1(Y)$. Then $q^n_Y(B) \in PHS_n(Y) \setminus \{F^n_X\}$. Since $PHS_n(f)$ is surjective, there exists $\chi \in PHS_n(X) \setminus \{F^n_X\}$ such that $PHS_n(f)(\chi) = q^n_Y(B)$. Hence, $(q^n_X)^{-1}(\chi) \in C_n(X) \setminus \mathcal{F}_1(X)$. Since $(\ast)$ commutes, we have that $C_n(f)((q^n_X)^{-1}(\chi)) = B$. Therefore, $C_n(f)$ is surjective.

\[\diamondsuit\]

4. Homeomorphisms

**Theorem 4.1.** Let $X$ and $Y$ be continua and let $n$ be a positive integer. If $f \colon X \to Y$ is a map, then the following are equivalent:

1. $f$ is a homeomorphism;
2. $C_n(f)$ is a homeomorphism;
3. $C_n(f)$ is atomic;
4. $PHS_n(f)$ is a homeomorphism;
5. $PHS_n(f)$ is atomic.

**Proof.** By [3, Theorem 46], we have that (1), (2) and (3) are equivalent. Suppose $C_n(f)$ is a homeomorphism, we prove that $PHS_n(f)$ is a homeomorphism. Note that, by Theorem 3.6, $PHS_n(f)$ is surjective. To show that $PHS_n(f)$ is one-to-one, let $\chi \in PHS_n(Y)$. Suppose that $\chi \neq F^n_Y$. Since $(\ast)$ commutes, by Remark 2.2, we have that there exists a unique element $A \in C_n(X)$ such that $PHS_n(f) \circ q^n_X(A) = \chi$. Assume that $\chi = F^n_Y$ and that there exists $\chi' \in PHS_n(X) \setminus \{F^n_X\}$ such that $PHS_n(f)(\chi') = \chi$. Note that $(q^n_X)^{-1}(\chi') \in C_n(X) \setminus \mathcal{F}_1(X)$. Since $C_n(f)$ is a homeomorphism, we have that $C_n(f)((q^n_X)^{-1}(\chi')) \in C_n(Y) \setminus \mathcal{F}_1(Y)$. Thus, by Remark 2.2, $q^n_Y(C_n(f)((q^n_X)^{-1}(\chi'))) \not\in PHS_n(Y) \setminus \{F^n_Y\}$. Since $(\ast)$ commutes, we obtain that $PHS_n(f)(\chi') \not\in PHS_n(Y) \setminus \{F^n_Y\}$, a contradiction to our election of $\chi'$. Therefore, $PHS_n(f)$ is one-to-one and a homeomorphism.

Suppose $PHS_n(f)$ is a homeomorphism, we show that $C_n(f)$ is a homeomorphism. By [8, Theorem 3.4], this is true for $n = 1$. Assume $n \geq 2$. Observe that, by Theorem 3.6, $C_n(f)$ is surjective. To
prove $C_n(f)$ is one-to-one, note that, by Remark 2.2, the commutativity of $(\ast)$ and the fact that $PHS_n(f)$ is a homeomorphism, we have that $C_n(f)|_{\mathcal{C}_n(X)\setminus \mathcal{F}_1(X)}: C_n(X) \setminus \mathcal{F}_1(X) \to C_n(Y) \setminus \mathcal{F}_1(Y)$ is a homeomorphism. We only need to prove that $C_n(f)|_{\mathcal{F}_1(X)}$ is one-to-one. Suppose there exist $\{x_1\}, \{x_2\} \in \mathcal{F}_1(X)$, with $\{x_1\} \neq \{x_2\}$, such that $C_n(f)(\{x_1\}) = C_n(f)(\{x_2\})$. This implies that $C_n(f)(\{x_1, x_2\}) \in \mathcal{F}_1(Y)$, a contradiction to the fact that $C_n(f)|_{\mathcal{C}_n(X)\setminus \mathcal{F}_1(X)}$ is a homeomorphism. Therefore, $C_n(f)$ is a homeomorphism.

Assume that $PHS_n(f)$ is atomic, we see that $f$ is a homeomorphism. Since $PHS_n(f)$ is an atomic map, for each $\chi \in PHS_n(Y)$, we have that $PHS_n(f)^{-1}(\chi)$ is a terminal subcontinuum of $PHS_n(X)$ [11, (2.1)]. Since $C_n(X)$ is arcwise connected [9, 1.8.12], $PHS_n(X)$ is arcwise connected. It is easy to see that the only terminal subcontinua of an arcwise connected continuum are the continuum itself and singletons. Hence, if $\chi \in PHS_n(Y)$, then either $PHS_n(f)^{-1}(\chi) = PHS_n(X)$ or $PHS_n(f)^{-1}(\chi)$ is a singleton of $PHS_n(X)$. Note that $PHS_n(f)^{-1}(\chi) = PHS_n(X)$ is impossible, otherwise $PHS_n(f)$ would be a constant map, in particular, $PHS_n(f)$ would not be surjective. Thus, for each $\chi \in PHS_n(Y)$, $PHS_n(f)^{-1}(\chi)$ is a singleton. This implies that $PHS_n(f)$ is a homeomorphism, and, by the above argument, $f$ is a homeomorphism. ♦

5. Monotone maps

Theorem 5.1. Let $X$ and $Y$ be continua and let $n$ be a positive integer. If $f: X \to Y$ is a map, then the following are equivalent:

(1) $f$ is monotone;
(2) $C_n(f)$ is monotone;
(3) $PHS_n(f)$ is monotone.

Proof. By [3, Theorem 4], we have that $f$ is monotone if and only if $C_n(f)$ is monotone.

Suppose $C_n(f)$ is monotone. Since $(\ast)$ commutes and $q_n^X$ is monotone, $q_n^Y \circ C_n(f)$ is monotone [10, (5.1)]. Hence, by [10, (5.15)], we obtain that $PHS_n(f)$ is monotone.

Assume $PHS_n(f)$ is monotone, we show that $f$ is monotone. Let $B \in \mathcal{C}_1(Y) \setminus \mathcal{F}_1(Y)$. By definition, $PHS_n(f)$ is surjective. Thus, by Theorem 3.6, $f$ is weakly confluent. Hence, there exists $A \in \mathcal{C}_1(X)$ such that
we have that
\[ f(A) = B. \]
Since (⋆) commutes and \( q^n_X|_{\mathcal{C}_n(X) \setminus \mathcal{F}_1(X)} \) is a homeomorphism, we have that
\[ \mathcal{C}_n(f)^{-1}(B) = (q^n_X)^{-1}(PHS_n(f)^{-1}(q^n_Y(B))). \]

Since \( PHS_n(f) \) and \( q^n_X \) are monotone, by [9, 2.1.12], \( \mathcal{C}_n(f)^{-1}(B) \) is connected. Since \( A \in \mathcal{C}_1(X) \cap \mathcal{C}_n(f)^{-1}(B) \), by [12, (1.49)], \( \bigcup \mathcal{C}_n(f)^{-1}(B) \in \mathcal{C}_1(X) \). Note that \( \bigcup \mathcal{C}_n(f)^{-1}(B) \subset f^{-1}(B) \). To prove the other inclusion, let \( x \in f^{-1}(B) \). Then \( A \cup \{x\} \in \mathcal{C}_n(X) \) and \( \mathcal{C}_n(f)(A \cup \{x\}) = B \). Thus, \( A \cup \{x\} \in \mathcal{C}_n(f)^{-1}(B) \). In particular, \( x \in \bigcup \mathcal{C}_n(f)^{-1}(B) \). Hence, \( \bigcup \mathcal{C}_n(f)^{-1}(B) = f^{-1}(B) \). In particular, \( f^{-1}(B) \) is connected.

Now, let \( y \in Y \) and let \( \{K_m\}_{m=1}^\infty \) be a decreasing sequence of subcontinua of \( Y \) such that \( \bigcap_{m=1}^\infty K_m = \{y\} \).

This implies that \( \bigcap_{m=1}^\infty f^{-1}(K_m) = f^{-1}(y) \). Hence, by the previous paragraph, we obtain that \( \{f^{-1}(K_m)\}_{m=1}^\infty \) is a decreasing sequence of subcontinua. Therefore, \( f^{-1}(y) \) is connected [9, 1.7.2], and \( f \) is monotone.

\[ \Box \]

6. Refinable Maps

Let \( X \) be a continuum and let \( n \) be a positive integer. We follow Professor Sam B. Nadler, Jr. [13] (compare with [8, p. 149]) to define a metric on \( PHS_n(X) \). Let
\[ \mathcal{S}_n(X) = \{\mathcal{F}_1(X) \cup \{A\} \mid A \in \mathcal{C}_n(X)\}. \]

Note that \( \mathcal{S}_n(X) \subset \mathcal{C}_2(\mathcal{C}_n(X)) \). Let \( G_n : PHS_n(X) \to \mathcal{S}_n(X) \) be given by
\[ G_n(\chi) = \mathcal{F}_1(X) \cup (q^n_X)^{-1}(\chi). \]

Then \( G_n \) is a homeomorphism. Next, define
\[ \rho^n_X : PHS_n(X) \times PHS_n(X) \to [0, \infty) \]
by
\[ \rho^n_X(\chi_1, \chi_2) = \mathcal{H}^2_X(G_n(\chi_1), G_n(\chi_2)), \]
where \( \mathcal{H}^2_X \) is the Hausdorff metric on \( \mathcal{C}_2(\mathcal{C}_n(X)) \) induced by the Hausdorff metric \( \mathcal{H}_X \) on \( \mathcal{C}_n(X) \). Then \( \rho^n_X \) is a metric. We rewrite [13, (2.3)], with our terminology, to obtain:
**Theorem 6.1.** Let $X$ and $Y$ be continua, let $n$ be a positive integer, let $f : X \to Y$ be a map, and let $\varepsilon > 0$. If $C_n(f)$ is an $\varepsilon$-map, then $PHS_n(f)$ is an $\varepsilon$-map.

**Lemma 6.2.** Let $X$ and $Y$ be continua, let $n$ be a positive integer, let $g, f : X \to Y$ be two maps, and let $\varepsilon > 0$. If $d(f(x), g(x)) < \varepsilon$ for all $x \in X$, then $\rho_\varepsilon^X(PHS_n(f)(\chi), PHS_n(g)(\chi)) < \varepsilon$ for each $\chi \in PHS_n(X)$.

**Proof.** Let $\chi \in PHS_n(X)$. If $\chi = F_X^n$, then

$$
\rho_\varepsilon^X(PHS_n(f)(\chi), PHS_n(g)(\chi)) = \rho_\varepsilon^X(PHS_n(f)(F_X^n), PHS_n(g)(F_X^n)) = \\
\rho_\varepsilon^X(F_X^n, F_X^n) = 0 < \varepsilon.
$$

Suppose that $\chi \neq F_X^n$. Then $(q_X^n)^{-1}(\chi) \in C_n(X) \setminus F_1(X)$, and

$$
\rho_\varepsilon^X(PHS_n(f)(\chi), PHS_n(g)(\chi)) =
\mathcal{H}_\varepsilon^X(\mathcal{F}_1(Y) \cup C_n(f)((q_X^n)^{-1}(\chi)), \mathcal{F}_1(Y) \cup C_n(g)((q_X^n)^{-1}(\chi))) =
\mathcal{H}_\varepsilon(Y)(C_n(f)((q_X^n)^{-1}(\chi)), C_n(g)((q_X^n)^{-1}(\chi))) < \varepsilon.
$$

The last inequality follows from [3, Lemma 37].

A continuum $Y$ is in $\text{Class}(W)$ provided that for each continuum $X$, each surjective map $f : X \to Y$ is weakly confluent.

**Theorem 6.3.** Let $X$ and $Y$ be continua, where $Y$ is in $\text{Class}(W)$, and let $n$ be a positive integer. If $f : X \to Y$ is a refinable map, then $PHS_n(f)$ is a refinable map.

**Proof.** Let $\varepsilon > 0$. Since $f$ is refinable, there exists an $\varepsilon$-map $g : X \to Y$ such that $d(f(x), g(x)) < \varepsilon$ for all $x \in X$. Since $g$ is surjective and $Y$ is in $\text{Class}(W)$, we have that $g$ is weakly confluent. Hence, $C_n(g)$ and $PHS_n(g)$ are both surjective (Theorem 3.6). By [3, Lemma 36] and Theorem 6.1, $PHS_n(g)$ is an $\varepsilon$-map.

By Lemma 6.2, $\rho_\varepsilon^X(PHS_n(f)(\chi), PHS_n(g)(\chi)) < \varepsilon$, for each $\chi \in PHS_n(X)$. Therefore, $PHS_n(f)$ is a refinable map.

**Theorem 6.4.** Let $X$ and $Y$ be continua and let $n$ be a positive integer. If $f : X \to Y$ is a monotonically refinable map, then $PHS_n(f)$ is a monotonically refinable map.
Proof. Let \( \varepsilon > 0 \). Since \( f \) is monotonely refinable, there exists a monotone \( \varepsilon \)-map \( g : X \to Y \) such that \( d(f(x), g(x)) < \varepsilon \) for every \( x \in X \). By Theorem 5.1 and the argument given in the proof of Theorem 6.3, we obtain that \( \text{PHS}_n(g) \) is a monotone \( \varepsilon \)-map such that

\[
\rho_Y^n(\text{PHS}_n(f)(\chi), \text{PHS}_n(g)(\chi)) < \varepsilon
\]

for each \( \chi \in \text{PHS}_n(X) \). Therefore, \( \text{PHS}_n(f) \) is a monotonely refinable map. ♦

In [3, Theorems 38 and 40] it is shown that if \( f \) is a (monotonically) refinable map, then \( C_n(f) \) is a (monotonically) refinable map.

Questions 6.5. Let \( X \) and \( Y \) be continua, let \( n \) be a positive integer and let \( f : X \to Y \).

(i) If \( C_n(f) \) is (monotonically) refinable, then is \( f \) (monotonically) refinable?

(ii) If \( C_n(f) \) is (monotonically) refinable, then is \( \text{PHS}_n(f) \) (monotonically) refinable?

(iii) If \( \text{PHS}_n(f) \) is (monotonically) refinable, then \( f \) is (monotonically) refinable?

(iv) If \( \text{PHS}_n(f) \) is (monotonically) refinable, then \( C_n(f) \) is (monotonically) refinable?

7. Light Maps

We begin with the following easy lemma.

Lemma 7.1. Let \( X \) and \( Y \) be continua, let \( f : X \to Y \) be a map, and let \( A \) be a nonempty subset of \( X \). If \( f \) is light, then \( f|_A : A \to f(A) \) is light.

Theorem 7.2. Let \( X \) and \( Y \) be continua, let \( n \) be a positive integer and let \( f : X \to Y \) be a surjective map. Consider the following statements:

1. \( f \) is light;
2. \( C_n(f) \) is light;
3. \( \text{PHS}_n(f) \) is light.

Then (3) implies (2) and (2) implies (1). Hence, (3) implies (1).
Proof. Assume $PHS_n(f)$ is light, we show that $C_n(f)$ is light. Let $B \in C_n(Y)$. Suppose first that $B \not\in \mathcal{F}_1(Y)$. Since (*) commutes,

$$PHS_n(f)^{-1}(q^n_Y(B)) = q^n_X(C_n(f)^{-1}(B)).$$

Since $PHS_n(f)$ is light and $q^n_X|_{C_n(X) \setminus \mathcal{F}_1(X)}$ is a homeomorphism, we have that $C_n(f)^{-1}(B)$ is totally disconnected.

Assume that $B \in \mathcal{F}_1(Y)$. Then there exists $y \in Y$ such that $B = \{y\}$. Suppose that $C_n(f)^{-1}(\{y\})$ is not totally disconnected. Let $A$ be a nondegenerate component of $C_n(f)^{-1}(\{y\})$. Then $\bigcup A \in C_n(X)$ [9, 6.1.2]. Since $A$ is nondegenerate, at least one component, say $A$, of $\bigcup A$ is nondegenerate. Let $a \in A$ and let $\alpha : [0, 1] \to C_n(X)$ be an order arc such that $\alpha(0) = \{a\}$ and $\alpha(1) = A$. Note that for each $t \in [0, 1]$, $C_n(f)(\alpha(t)) = \{y\}$. Then $q^n_X(\alpha([0, 1]))$ is an arc in $PHS_n(X)$ such that $q^n_X(\alpha(0)) = F^n_Y$. Since (*) commutes, we have that

$$PHS_n(f)(q^n_X(\alpha([0, 1]))) = q^n_Y(C_n(f)(\alpha([0, 1]))) = q^n_Y(\{y\}) = \{F^n_Y\}.$$

A contradiction to the fact that $PHS_n(f)$ is light. Therefore, $C_n(f)^{-1}(\{y\})$ is totally disconnected, and $C_n(f)$ is light.

The fact that $C_n(f)$ implies $f$ is light follows easily from Lemma 7.1.

The following example shows a light map $f$ such that neither $C_n(f)$ nor $PHS_n(f)$ are light for any positive integer $n$.

Example 7.3. Let $f : [-1, 1] \to [0, 1]$ be given by $f(t) = |t|$. Clearly, $f$ is a light map. Note that if $A = \{A \in C([-1, 1]) \mid [0, 1] \subset A\}$, then $A = C_1(f)^{-1}([0, 1])$. Hence, $C_1(f)$ is not light. Thus, by Lemma 7.1, $C_n(f)$ is not light for any positive integer $n$. Also, by Theorem 7.2, it follows that $PHS_n(f)$ is not light for any positive integer $n$.

Theorem 7.4. Let $X$ and $Y$ be continua, let $n$ be a positive integer, and let $f : X \to Y$ be a map. If $k$ is a positive integer such that $PHS_n(f)$ is a $k$-to-1 map, then $k = 1$ and all $f, C_n(f)$ and $PHS_n(f)$ are homeomorphisms.

Proof. Suppose $k \geq 2$. Let $\chi \in PHS_n(f)^{-1}(T^n_Y) \setminus \{T^n_X\}$. Then $(q^n_X)^{-1}(\chi) \in C_n(X) \setminus \{X\}$. Since (*) commutes, $C_n(f)((q^n_X)^{-1}(\chi)) = Y$. Let $\alpha : [0, 1] \to C_n(X)$ be an order arc such that $\alpha(0) = (q^n_X)^{-1}(\chi)$ and $\alpha(1) = X$. Note that, for each $t \in [0, 1]$, $C_n(f)(\alpha(t)) = Y$. Thus,
C_n(f)(\alpha(0, 1]) \subseteq \{Y\}. Then q^\alpha_X \circ \alpha : [0, 1] \to PHS_n(X) is an arc such that q^\alpha_X \circ \alpha(0) = x, q^\alpha_X \circ \alpha(1) = T^\alpha_X, and PHS_n(f)(q^\alpha_X \circ \alpha([0, 1])) = \{T^\alpha_Y\}. Hence, q^\alpha_X \circ \alpha([0, 1]) \subset PHS_n(f)^{-1}(T^\alpha_Y), a contradiction to the fact that PHS_n(f) is k-to-1. Therefore, k = 1. Since PHS_n(f) is 1-to-1, PHS_n(f) is a homeomorphism. Thus, by Theorem 4.1, f and C_n(f) are both homeomorphisms.

Questions 7.5. Let X and Y be continua, let n be a positive integer and let f : X \to Y. If C_n(f) is light, then is PHS_n(f) light?

8. Open maps

In contrast to equivalences in previous sections, we give an example that shows that f being open does not imply that either C_n(f) or PHS_n(f) is open.

Example 8.1. Let n be a positive integer. For each j \in \{1, \ldots, n + 1\}, let L_j = \{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} | x_j \in [0, 1] and x_k = 0 \text{ if } k \in \{1, \ldots, n + 1\} \setminus \{j\}\}. Let X = \bigcup_{j=1}^{n+1} L_j, and let f : X \to [0, 1] be given by

\[
f((x_1, \ldots, x_{n+1})) = \begin{cases} 
0, & \text{if } (x_1, \ldots, x_{n+1}) = (0, \ldots, 0); \\
x_j, & \text{if } (x_1, \ldots, x_{n+1}) \in L_j \setminus \{(0, \ldots, 0)\}, \\
\text{for some } j \in \{1, \ldots, n + 1\}.
\end{cases}
\]

Then f is an open map. We see that C_n(f) is not an open map. To this end, for each j \in \{1, \ldots, n + 1\}, let K_j = \{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} | x_j \in [0, \frac{1}{2}] \text{ and } x_k = 0 \text{ if } k \in \{1, \ldots, n + 1\} \setminus \{j\}\}. Let K = \bigcup_{j=1}^{n+1} K_j. Also, for every j \in \{1, \ldots, n + 1\}, let U_j = \{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} | x_j \in (0, \frac{1}{2}) \text{ and } x_k = 0 \text{ if } k \in \{1, \ldots, n + 1\} \setminus \{j\}\}. Let U = \{A \in C_n(X) | A \subset \bigcup_{j=1}^{n+1} (U_j \cup \{(0, \ldots, 0)\}) \text{ and } A \cap U_j \neq \emptyset \text{ for all } j \in \{1, \ldots, n + 1\}\}. It is easy to see that U is an open subset of C_n(X) and K \subset U. We prove that C_n(f) is not open. Suppose C_n(f) is open and consider the sequence \{\left(\frac{1}{m}, \frac{1}{2}\right)\}_{m=3}^{\infty}, which converges to [0, \frac{1}{2}]. Since C_n(f)(K) = [0, \frac{1}{2}], we obtain that [0, \frac{1}{2}] \subset C_n(f)(U). Since we are assuming that C_n(f)(U) is open, there exists a positive integer M such that for every m \geq M, \left[\frac{1}{m}, \frac{1}{2}\right] \subset C_n(f)(U). Let m \geq M. Then there exists B_m \in U such that C_n(f)(B_m) = \left[\frac{1}{m}, \frac{1}{2}\right]. This implies that f(B_m) = \left[\frac{1}{m}, \frac{1}{2}\right]. Observe that f^{-1}(\left[\frac{1}{m}, \frac{1}{2}\right]) = \bigcup_{j=1}^{n+1} \left[\frac{1}{m}, \frac{1}{2}\right], \text{ where } \left[\frac{1}{m}, \frac{1}{2}\right] = \{(x_1, \ldots, x_{n+1}) \in L_j | x_j \in \left[\frac{1}{m}, \frac{1}{2}\right]\}. Since B_m \in C_n(X) and B_m \subset f^{-1}(\left[\frac{1}{m}, \frac{1}{2}\right]), without loss of generality, we assume
that $B_m \subset \bigcup_{j=1}^{n} [\frac{1}{m+1}, \frac{1}{m}]$. Hence, $B_m \cap U_{m+1} = \emptyset$, a contradiction to the fact that $B_m \in \mathcal{U}$. Therefore, $C_n(f)$ is not open. To show that $PHS_n(f)$ is not open, note that $\mathcal{U} \cap \mathcal{F}_1(X) = \emptyset$ and $C_n(f)(\mathcal{U}) \cap \mathcal{F}_1(Y) = \emptyset$. This implies that $q^n_X(\mathcal{U})$ is an open subset of $PHS_n(X)$ and $q^n_Y(C_n(f)(\mathcal{U}))$ is not an open subset of $PHS_n(Y)$ (Remark 2.2). Since $(\ast)$ commutes, we obtain that $PHS_n(f)(q^n_X(\mathcal{U})) = q^n_Y(C_n(f)(\mathcal{U}))$. Therefore, $PHS_n(f)$ is not open.

**Questions 8.2.** Let $X$ and $Y$ be continua, let $n$ be a positive integer and let $f: X \to Y$ be a map. Is it true that $C_n(f)$ is open if and only if $PHS_n(f)$ is open?

**References**


