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Centers and Generalized Centers of Near-Rings Without Identity Defined via Malone-Like Multiplications

G. Alan **Cannon**

Department of Mathematics Southeastern Louisiana University Hammond, LA 70402

Mark **Farag**

Department of Mathematics, Fairleigh Dickinson University, Teaneck, NJ, 07666

Lucyna **Kabza**

Department of Mathematics Southeastern Louisiana University Hammond, LA 70402

Kent M. **Neuerburg**

Department of Mathematics Southeastern Louisiana University Hammond, LA 70402

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Abstract: The multiplicative center of a right near-ring $(N, +, \cdot)$, $C(N) = \{x \in N \mid \text{for all } y \in N,\ x \cdot y = y \cdot x\}$, in general, is not a subnear-ring of N. On the other hand, the generalized center, $GC(N) = \{g \in N \mid \text{for all } d \in N_d,\ g \cdot d = d \cdot g\}$, where $N_d = \{d \in N \mid d \cdot (x + y) = d \cdot x + d \cdot y \text{ for all } x, y \in N\}$, is always a subnear-ring of N. We investigate four classes of zero-symmetric near-rings defined via special multiplications on groups. Three of these classes have not appeared in the literature, and nearly all near-rings investigated are near-rings without identity. The center and generalized center of each near-ring in these four classes are determined, with the center almost always being a subnear-ring of N. Numerous examples are given to illustrate the results.

E-mail addresses: acannon@southeastern.edu, mfarag@fdu.edu, lkabza@southeastern.edu, kneuerburg@southeastern.edu

1. Introduction

Given a right near-ring $(N,+,\cdot)$, its multiplicative center $C(N) = \{x \in N \mid \text{for all } y \in N,\ x\cdot y = y\cdot x\}$ is, when nonempty, a sub-semigroup of $(N,\cdot)$ that need not be a subnear-ring of N. The first systematic study of when multiplicative centers are subnear-rings is found in [2] where all near-rings with identity of order p^2, p a prime, having additively closed multiplicative center are determined, and the multiplicative centers of matrix near-rings (in the sense of Meldrum and van der Walt [9]) are described. In [6], the notion of multiplicative center of a near-ring is generalized as follows. Let $N_d = \{d \in N \mid d\cdot(x+y) = d\cdot x + d\cdot y \text{ for all } x,y \in N\}$ be the set of left distributive elements of N. Then the generalized center of N is defined as $GC(N) = \{g \in N \mid \text{for all } d \in N_d,\ g\cdot d = d\cdot g\}$; when nonempty, $GC(N)$ is always a subnear-ring of N that contains $C(N)$. Furthermore, $GC(N) = C(N)$ when N is a ring. The generalized center of polynomial near-rings (in the sense of Bagley [3]) is studied in [6], and the generalized centers of distributively generated near-rings with identity, centralizer near-rings determined by groups of automorphisms on nontrivial finite groups, matrix near-rings, and near-rings of polynomials with zero constant term over commutative rings with identity are studied and compared with their associated multiplicative centers in [4]. We observe that the majority of the existing work on centers and generalized centers focuses on near-rings with identity.

In this paper, we continue the study of centers and generalized centers in near-rings defined via special multiplications on groups. The first construction is given by Malone in [7]. Next, we define and study three similarly-structured multiplications and completely characterize the centers and generalized centers of all four classes of near-rings. In almost all cases, the near-rings we treat herein do not contain a two-sided identity and yield multiplicative centers that are subnear-rings. These new constructions might also be useful in studying other near-ring properties.

Along with our characterization results, we present a host of examples to illustrate the cases described in the paper. Many of these examples were discovered using the SONATA software [1]. We refer the reader to [5], [8], and [10] for basic definitions and references regarding near-rings. All groups used in examples have their usual addition. Hereafter, we shall denote $x\cdot y$ by xy.

2. Malone Trivial Near-Rings

We begin with a well-known construction of a near-ring N with an elementary multiplication in which $C(N)$ is always a subnear-ring of N. Let $(G, +)$ be a group, not necessarily abelian, with $|G| \geq 2$. Let $S \subseteq G^* := G \backslash \{0\}$ and define a multiplication on G by

$$a \cdot b = \begin{cases} a & \text{if } b \in S \\ 0 & \text{if } b \notin S \end{cases} .$$

Then $N = (G, +, \cdot)$ is a right, zero-symmetric near-ring [7], now called a *Malone trivial near-ring.* The next lemma demonstrates that Malone trivial near-rings rarely have a two-sided multiplicative identity.

Lemma 2.1. Let N be a Malone trivial near-ring. Then N has a two-sided multiplicative identity, 1, if and only if $N = \{0, 1\}$ and $S = \{1\}$, i.e., $N \cong \mathbb{Z}_2$.

Proof. Assume N has a two-sided multiplicative identity, 1. If $S = \emptyset$, then $ab = 0$ for all $a, b \in N$ and N does not have an identity. Thus, S is nonempty. Let $b \in S$. Then $b = 1 \cdot b = 1$ and $S = \{1\}$. Let $b \notin S$. Then $b = 1 \cdot b = 0$ and $N \setminus S = \{0\}$. By constructing the multiplication table for the Malone trivial near-ring using $N = \{0, 1\}$ and $S = \{1\}$, one can see the resulting near-ring is $\mathbb{Z}_2$. The converse is immediate. $\diamondsuit$

We now prove a sequence of lemmas which will be helpful in our characterization theorem and its ensuing examples.

Lemma 2.2. Let N be a Malone trivial near-ring. If $N_d \neq \{0\}$ and $x, y \in S$, then $x + y \notin S$.

Proof. Let $0 \neq a \in N_d$ and $x, y \in S$. Assume $x + y \in S$. Then $a(x+y) = ax + ay$ implies $a = a + a$. Hence $a = 0$, a contradiction. So $x + y \notin S$. $\diamondsuit$

Lemma 2.3. Let N be a Malone trivial near-ring with $S \neq \emptyset$. Then $N_d \subseteq \{a \in N \mid 2a = 0\}$.

Proof. If $N_d = \{0\}$, the result is clear. So assume $N_d \neq \{0\}$. Let $a \in N_d$. Since $S \neq \emptyset$, there exists $b \in S$. Since $N_d \neq \{0\}$, by the previous lemma, $b + b \notin S$. So $a(b + b) = ab + ab$ implies $0 = a + a$. Hence $N_d \subseteq \{a \in N \mid 2a = 0\}$. $\diamondsuit$

Lemma 2.4. Let N be a Malone trivial near-ring with $S \neq \emptyset$. Then $C(N) \neq \{0\}$ if and only if $C(N) = GC(N) = N = \{0, a\}$ for some $a \neq 0$.

Proof. Assume $0 \neq a \in C(N)$. If $a \notin S$, then for $b \in S$, $ab = ba$ implies $a = 0$, a contradiction. So $a \in S$. For $b \in S$, $ab = ba$ implies $a = b$. Thus $S = \{a\}$. For $b \notin S$, $ba = ab$ implies $b = 0$. So $N = S \cup \{0\}$. Hence, $N = \{0, a\} = C(N) = GC(N)$. The converse is clear. $\Diamond$

We now state the main characterization theorem for this section.

Theorem 2.5. Let N be a Malone trivial near-ring.

1. If $S = \emptyset$, then $C(N) = GC(N) = N$.
2. If $S \neq \emptyset$ and $|N| = 2$, then $\{0\} \neq C(N) = GC(N) = N$.
3. If $S \neq \emptyset$, $|N| > 2$, and $N_d = \{0\}$, then $\{0\} = C(N) \subsetneq GC(N) = N$.
4. If $S \neq \emptyset$, $|N| > 2$, $N_d \cap S = \emptyset$, and $N_d \neq \{0\}$, then $\{0\} = C(N) \subsetneq N \setminus S = GC(N) \subsetneq N$.
5. If $S \neq \emptyset$, $|N| > 2$, $N_d \cap S \neq \emptyset$, and $|N_d| = 2$, then $\{0\} = C(N) \subsetneq N_d = GC(N) \subsetneq N$.
6. If $S \neq \emptyset$, $|N| > 2$, $N_d \cap S \neq \emptyset$, and $|N_d| > 2$, then $\{0\} = C(N) = GC(N) \subsetneq N$.

In all cases, $C(N)$ is a subnear-ring of N.

Proof. The first two cases are straightforward to verify. Since $|N| > 2$ in cases (3) through (6), by the previous lemma, $C(N) = \{0\}$. To complete case (3), note that if $N_d = \{0\}$, then $GC(N) = N$.

For case (4), let $n \in N_d$. Then $n \notin S$. For all $t \in N \setminus S$, $nt = 0 = tn$. Thus, $N \setminus S \subseteq GC(N)$. Now let $s \in S$ and $0 \neq n \in N_d$. Then $ns = n \neq 0 = sn$. Hence, $s \notin GC(N)$ and $GC(N) = N \setminus S \neq N$. Since $N_d \neq \{0\}$ and $N_d \cap S = \emptyset$, we conclude $\{0\} \neq N \setminus S$. Case (4) now follows.

For cases (5) and (6), fix $y \in N_d \cap S$. Let $a \in GC(N)$. If $a \in S$, then $ay = ya$ implies $a = y$. If $a \notin S$, then $ay = ya$ implies $a = 0$. Thus $a = y$ or $a = 0$, and $GC(N) \subseteq \{0, y\}$.

If $|N_d| = 2$, then $N_d = \{0, y\}$. Thus $y \in GC(N)$, and $GC(N) = \{0, y\}$. This finishes case (5). For case (6), assume $|N_d| \geq 3$. Let $z \in N_d$

such that $z \notin \{0, y\}$. Then $zy = z$ and $yz \in \{0, y\}$. Thus $zy \neq yz$ and $y \notin GC(N)$. It follows that $GC(N) = \{0\}$, completing case (6).

Since $C(N) = \{0\}$ or $C(N) = N$ in all cases, $C(N)$ is a subnear-ring of N. $\Diamond$

We end the section by providing examples of each of the six cases in the characterization theorem.

Example 2.6. Examples of Malone trivial near-rings

Case (1). Let G be any group and $S = \emptyset$. Then $ab = 0$ for all $a, b \in G$ and $C(N) = GC(N) = N$.

Case (2). Let $G = \mathbb{Z}_2$ and $S = \{1\}$. Then $\{0\} \neq \{0, 1\} = C(N) = GC(N) = N$.

Case (3). Let $G = \mathbb{Z}_3$ and $S = \{1\}$. Since $N_d \subseteq \{a \in N \mid 2a = 0\}$ by Lemma 2.3 and G has no elements of additive order two, $N_d = \{0\}$. Hence $\{0\} = C(N) \subsetneq GC(N) = N$.

Case (4). Let $G = \mathbb{Z}_8$ and $S = \{1, 3, 5, 7\}$. By Lemma 2.3, $N_d \subseteq \{0, 4\}$. We now show containment in the other direction. Note that for $x, y \in N$, $x + y \notin S$ if and only if $x, y \in S$ or $x, y \notin S$. Thus, for $x, y \in S$, $4(x + y) = 0 = 4 + 4 = 4x + 4y$. For $x, y \notin S$, $4(x + y) = 0 = 0 + 0 = 4x + 4y$. For $x \in S$ and $y \notin S$, $4(x + y) = 4 = 4 + 0 = 4x + 4y$. By symmetry, $4(x + y) = 4x + 4y$ also follows when $x \notin S$ and $y \in S$. Thus, $4 \in N_d$ and $N_d = \{0, 4\}$. It follows that $\{0\} = C(N) \subsetneq \{0, 2, 4, 6\} = N \setminus S = GC(N) \subsetneq N$.

Case (5). Let $G = \mathbb{Z}_6$ and $S = \{1, 3, 5\}$. Using a similar technique as in case (4), one obtains $N_d = \{0, 3\}$. Thus, $\{0\} = C(N) \subsetneq \{0, 3\} = N_d = GC(N) \subsetneq N$.

Case (6). Let $G = S_3$ and $S = \{(1\,2), (1\,3), (2\,3)\}$. Again, using the technique as in case (4), one obtains $N_d = \{0, (1\,2), (1\,3), (2\,3)\}$. It follows from the characterization theorem that $\{0\} = C(N) = GC(N) \subsetneq N$.

3. Complemented Malone Near-Rings

Let $(G, +)$ be an abelian group and suppose $\emptyset \neq S \subseteq G^*$ such that for all $x \in S$, $-x \notin S$. Define a multiplication on G by

$$a \cdot b = \begin{cases} a \text{ if } b \in S \\ -a \text{ if } -b \in S \\ 0 \text{ if } b \notin S \text{ and } -b \notin S \end{cases} .$$

Here S is taken to be nonempty to avoid having the zero multiplication on G. In this section, we first show that $(G, +, \cdot)$ is always a near-ring and that there is only one such near-ring with identity. Next, we characterize the center and generalized center of the resulting near-ring. In particular, we find that the center of a near-ring with this multiplication is always a subnear-ring. We end this section with examples to illustrate the theory.

Theorem 3.1. Given an abelian group $(G, +)$ and a nonempty subset $S \subseteq G^*$ satisfying $x \in S$ implies $-x \notin S$ and using the multiplication defined above, $N = (G, +, \cdot)$ is a zero-symmetric right near-ring with $|N| \geq 3$.

Proof. It is straightforward to show $0a = a0 = 0$ for all $a \in N$, making N zero-symmetric. Next we establish that for all $a, b, c \in N$, $a(bc) = (ab)c$. If any of a, b, c equals 0, then $a(bc) = 0 = (ab)c$. So suppose $a, b, c \neq 0$.

1. If $c \notin S$ and $-c \notin S$, then $(ab)c = 0 = a(0) = a(bc)$.
2. If $c \in S$, then $(ab)c = ab = a(bc)$.
3. If $-c \in S$, then $(ab)c = -(ab)$, and $a(bc) = a(-b)$.
   (a) If $b \in S$, then $-(ab) = -a = a(-b)$.
   (b) If $-b \in S$, then $-(ab) = -(-a) = a = a(-b)$.
   (c) If $b, -b \notin S$, then $-(ab) = 0 = a(-b)$.

Then, in all cases, $(ab)c = a(bc)$, and hence the multiplication is associative. Now we show that for all $a, b, c \in N$, $(a + b)c = ac + bc$.

1. If $c \in S$, then $(a+b)c = a+b = ac+bc$.

2. If $-c \in S$, then $(a+b)c = -(a+b) = -a+(-b) = ac+bc$ since $(N,+)$ is abelian.

3. If $c \notin S$ and $-c \notin S$, then $(a+b)c = 0 = ac+bc$.

Thus, right distributivity holds in all cases and N is a right near-ring.

Since S is nonempty, there exists $0 \neq x \in S$. It follows that $0 \neq -x \notin S$. So $\{x, -x, 0\} \subseteq N$, and $|N| \geq 3$. The proof is now complete. $\lozenge$

We refer to $(N, +, \cdot)$ as a *complemented Malone near-ring* since its multiplication is similar to that of ordinary Malone trivial near-rings, but with the additional condition that negatives of elements of S must be in the complement of S. This results in corresponding extra cases for multiplication.

Lemma 3.2. Let N be a complemented Malone near-ring. The following are equivalent:

1. $C(N) \neq \{0\}$;

2. N has a multiplicative left identity;

3. $|N| = 3$;

4. $N \cong \mathbb{Z}_3$.

Proof. It is obvious that condition (4) implies conditions (1), (2), and (3). Assume condition (1). So there exists $0 \neq a \in C(N)$. Either $a \notin S$ or $a \in S$.

Assume $a \notin S$. If $-a \notin S$, then for $b \in S$, $a = ab = ba = 0$, a contradiction. Thus, $-a \in S$ and $a \neq -a$. For $c \in S$, $a = ac = ca = -c$, or equivalently, $c = -a$. Since $c \in S$ is arbitrary, it follows that $S = \{-a\}$.

Consider $d \notin S$. If $-d \in S$, then $-a = ad = da = -d$. Thus $d = a$. If $-d \notin S$, then $0 = ad = da = -d$. Thus $d = 0$. It follows that $N \setminus S \subseteq \{0, a\}$ and $N = S \cup (N \setminus S) = \{0, a, -a\}$.

Now assume $a \in S$. By definition of S, $-a \notin S$ and $a \neq -a$. For $b \in S$, $a = ab = ba = b$. Since $b \in S$ is arbitrary, $S = \{a\}$. Now consider $d \notin S$. Using the cases $-d \in S$ and $-d \notin S$ with $ad = da$

as above, we conclude $d = -a$ or $d = 0$. Hence, $N \setminus S \subseteq \{0, -a\}$ and $N = S \cup (N \setminus S) = \{0, a, -a\}$. In both cases, $N = \{0, a, -a\}$ and $|N| = 3$, giving condition (3). So (1) implies (3).

For condition (2), let $1 \in N$ be a multiplicative left identity. Then for $x \in S$, $1x = x$ implies $1 = x$. So $1 \in S$, and since $x \in S$ is arbitrary, $S = \{1\}$ and $-S = \{-1\}$. Now let $y \in N \setminus \{-1, 1\}$. Then $1y = y$ implies $0 = y$. Thus, $N = \{0, -1, 1\}$ and $|N| = 3$. Hence (2) implies (3).

Lastly, assume condition (3) holds. For $0 \neq a \in N$, either $a \notin S$ and $-a \in S$, or $a \in S$ and $-a \notin S$. So $N = \{0, a, -a\}$. Using the definition of complemented Malone near-rings, one can construct the multiplication table for N in each case and see that $N \cong \mathbb{Z}_3$. This gives (3) implies (4), completing the proof. $\Diamond$

The lemma shows that the only complemented Malone near-ring with identity is the ring $\mathbb{Z}_3$. Furthermore, the only complemented Malone near-ring with nontrivial center is also the ring $\mathbb{Z}_3$.

We now state our main characterization theorem on complemented Malone near-rings.

Theorem 3.3. Let N be a complemented Malone near-ring.

1. If $|N| = 3$, then $N \cong \mathbb{Z}_3$.
2. If $|N| > 3$ and $N_d = \{0\}$, then $\{0\} = C(N) \subsetneq GC(N) = N$.
3. If $|N| > 3$, $N_d \neq \{0\}$, and $N_d \cap S = \emptyset$, then $\{0\} = C(N) \subsetneq N \setminus (S \cup (-S)) = GC(N) \subsetneq N$.
4. If $|N| > 3$, $|N_d| = 3$, and $N_d \cap S \neq \emptyset$, then $\{0\} = C(N) \subsetneq N_d = GC(N) = \{0, y, -y\} \subsetneq N$ for some $y \neq -y$.
5. If $|N| > 3$, $|N_d| > 3$, and $N_d \cap S \neq \emptyset$, then $\{0\} = C(N) = GC(N) \subsetneq N$.

In all cases, $C(N)$ is a subnear-ring of N.

Proof. First note that $|N_d| = 2$ and $N_d \cap S \neq \emptyset$ is an impossibility since $y \in N_d \cap S$ implies $N_d = \{0, y\}$. Since N is abelian, $-y \in N_d$, giving $y = -y \notin S$, a contradiction. So the five cases presented in the theorem are exhaustive.

Case (1) is immediate from the previous lemma. Since $|N| > 3$ in (2) through (5), by the previous lemma, $C(N) = \{0\}$. To complete (2), note that if $N_d = \{0\}$, then $GC(N) = N$.

For (3), let $0 \neq y \in N_d$. Then $0 \neq -y \in N_d$. Thus $y, -y \notin S$. Let $t \in N \setminus (S \cup (-S))$. Then $t, -t \notin S$. So $yt = 0 = ty$ and $t \in GC(N)$. Hence, $N \setminus (S \cup (-S)) \subseteq GC(N)$. Now let $t \in GC(N)$. If $t \in S$, then $ty = 0 \neq y = yt$, and $t \notin GC(N)$, a contradiction. So $t \notin S$. If $t \in -S$, then $-t \in S$ and $ty = 0 \neq -y = yt$. Hence $t \notin GC(N)$, a contradiction. So $t \notin -S$. It follows that $t \in N \setminus (S \cup (-S))$ and $GC(N) \subseteq N \setminus (S \cup (-S))$.

To prove (4), let $y \in N_d \cap S$. Note that $-y \notin S$ since $y \in S$, making $y \neq -y$. Given N is abelian, we know that $-y \in N_d$, thus $N_d = \{0, y, -y\}$ since $|N_d| = 3$. Let $a \in GC(N)$. The three cases (i) $a \in S$, (ii) $-a \in S$, and (iii) $a \notin S$, $-a \notin S$ used in conjunction with $ay = ya$ yield $a \in \{0, y, -y\}$. Thus, $GC(N) \subseteq \{0, y, -y\}$. As $y(-y) = -y = (-y)y$, all elements of $\{0, y, -y\}$ commute with one another. Hence, $GC(N) = \{0, y, -y\} = N_d$.

For the last case, let $y \in N_d \cap S$. Using similar arguments to those in case (4), we get $GC(N) \subseteq \{0, y, -y\} \subsetneq N_d$. Since $|N_d| > 3$, there exists $z \in N_d \setminus \{0, y, -y\}$. If $z \in S$, then $GC(N) \subseteq \{0, z, -z\}$, so that $GC(N) \subseteq \{0, y, -y\} \cap \{0, z, -z\} = \{0\}$. For $z \notin S$, we consider two subcases. If $-z \in S$, then $zy = z \neq -y = yz$ and $z(-y) = -z \neq y = (-y)z$. If $-z \notin S$, then $zy = z \neq 0 = yz$ and $z(-y) = -z \neq 0 = (-y)z$. In both subcases, $zy \neq yz$ and $z(-y) \neq (-y)z$. Since $z \in N_d$, it follows that $y, -y \notin GC(N)$. This leaves $GC(N) = \{0\}$.

In all cases, $C(N) = \{0\}$ or $C(N) = N$, making $C(N)$ a subnear-ring of N. $\diamondsuit$

We now illustrate the characterization theorem through several examples.

Example 3.4. Examples of complemented Malone near-rings

Case (1). Let $G = \mathbb{Z}_3$ and $S = \{1\}$. Then N is the ring $\mathbb{Z}_3$ with the usual multiplication. So $\{0\} \neq C(N) = GC(N) = N$.

Case (2). Let $G = \mathbb{Z}_6$ and $S = \{1\}$. Let $x \in N_d$. Then $0 = x \cdot 3 = x(1 + 2) = x \cdot 1 + x \cdot 2 = x + 0 = x$. Thus, $N_d = \{0\}$ and $\{0\} = C(N) \subsetneq GC(N) = N$ follows.

Case (3). Let $G = \mathbb{Z}_4$ and $S = \{1\}$. So $3 \in -S$. It follows that $2(1) = 2 = -2 = 2(3)$. First we show that $2 \in N_d$. Let $a, b \in N$. If a and

b are odd, then $a + b \in \{0, 2\}$ and $2(a + b) = 0 = 2 + 2 = 2a + 2b$. If a and b are even, then $a+b \in \{0, 2\}$ and $2(a+b) = 0 = 0+0 = 2a+2b$. If a and b have opposite parity, since G is abelian, we can assume without a loss of generality that a is odd and b is even. Then $a + b \in \{1, 3\}$ and $2(a + b) = 2 = 2 + 0 = 2a + 2b$. Thus, $2 \in N_d$ and $N_d \neq \{0\}$. Note that $1 \cdot (1+2) = 1 \cdot 3 = 3 \neq 1 = 1+0 = 1 \cdot 1+1 \cdot 2$ and $1 \notin N_d$. So $N_d \cap S = \emptyset$. The conclusion of case (3) yields $\{0\} = C(N) \subsetneq N \setminus (S \cup (-S)) = GC(N) \subsetneq N$.

Case (4). Let $G = \mathbb{Z}_6$ and $S = \{2, 5\}$. Let $a, b \in N$. We leave it to the reader to verify $2(a + b) = 2a + 2b$ with the following combinations of choices for a and b: either a or b is zero; $a, b \in S$; $a, b \in -S$; $a \in S$, $b \in -S$; $a = 3$, $b \in S$; $a = 3$, $b \in -S$; $a = b = 3$. Thus $2 \in N_d$. Since G is abelian, N_d is a subgroup of G and $-2 = 4 \in N_d$ as well. Note that $1 \cdot (2 + 2) = 1 \cdot 4 = 5 \neq 2 = 1 + 1 = 1 \cdot 2 + 1 \cdot 2$ and $1 \notin N_d$. So $|N_d| = 3$. It follows that $N_d = \{0, 2, 4\}$ and $N_d \cap S = \{2\} \neq \emptyset$. Hence, $\{0\} = C(N) \subsetneq N_d = GC(N) = \{0, y, -y\} \subsetneq N$ for some $y \neq -y$.

Case (5). Let $G = \mathbb{Z}_3 \times \mathbb{Z}_3$ and $S = \{(2, 0), (2, 1), (2, 2)\}$. We leave it to the reader to verify $(1, 0)(a+b) = (1, 0)a+(1, 0)b$ and $(0, 1)(a+b) = (0, 1)a + (0, 1)b$ with the following combinations of choices for a and b: $a, b \in S$; $a, b \in -S$; $a \in S$, $b \in -S$; $a \in S$, $b \in S + (-S)$; $a \in -S$, $b \in S + (-S)$; $a, b \in S + (-S)$. Thus $(1, 0), (0, 1) \in N_d$. Since G is abelian, N_d is a subgroup of G. So $N_d = N$ and $N_d \cap S \neq \emptyset$. It follows that N is a ring. Therefore, $\{0\} = C(N) = GC(N) \subsetneq N$.

4. TS Near-Rings

In this section, we construct a near-ring N from a given finite group of even order. As with Malone trivial near-rings, a product $a \cdot b$ in N is defined in terms of the membership of b in a certain set S. Unlike multiplication in Malone trivial near-rings, however, the product $a \cdot b$ requires consideration of the membership of b in different subsets of S, and also depends on the set membership of a in a superset T of S. We show that for the near-ring N constructed in this section, $C(N)$ is always a subnear-ring of N.

Theorem 4.1. Let $(G, +)$ be a finite group of even order, not necessarily abelian. Suppose there exists $\emptyset \neq T \subseteq G^*$ such that $G \backslash T$ is a (normal) subgroup of G of index 2. Further suppose there is $\emptyset \neq S \subseteq T$ with $S = S_1 \dot{\cup} S_2 \dot{\cup} \cdots \dot{\cup} S_n$, a partition of S, and that there are distinct elements $q_1, \ldots, q_n$ of order 2 in $G \backslash (T \cup \{0\})$.

Define a multiplication on G by

$$a \cdot b = \begin{cases} q_1 & \text{if } a \in T, b \in S_1 \\ q_2 & \text{if } a \in T, b \in S_2 \\ \vdots & \\ q_n & \text{if } a \in T, b \in S_n \\ 0 & \text{otherwise} \end{cases} .$$

Then $N = (G, +, \cdot)$ is a right, zero-symmetric near-ring without multiplicative identity.

Proof. Since $(G, +)$ is a group, we only need to show associativity of multiplication and right distributivity of multiplication over addition. To show associativity, let $a, b, c \in N$. If $a \notin T$, $b \notin T$ or $c \notin T$, then $(ab)c = 0 = a(bc)$. So assume $a, b, c \in T$. We consider four cases. (Note that if one assumes $x \in S$, then $x \in S_i$ for some $i = 1, 2, \ldots, n$. For ease of notation, throughout this section we will immediately assume $x \in S_i$.)

1. If $b \in S_j$ and $c \in S_i$, then $(ab)c = q_j c = 0 = aq_i = a(bc)$.
2. If $b, c \in T \setminus S$, then $(ab)c = 0 \cdot c = 0 = a \cdot 0 = a(bc)$.
3. If $b \in T \setminus S$ and $c \in S_i$, then $(ab)c = 0 \cdot c = 0 = aq_i = a(bc)$.
4. If $b \in S_j$ and $c \in T \setminus S$, then $(ab)c = q_j c = 0 = a \cdot 0 = a(bc)$.

So all cases are exhausted and multiplication is associative.

Now we verify the right distributive law. We note that $G \backslash T$ is a normal subgroup of index 2 in G, making T the other coset of G determined by $G \backslash T$. It follows that:

1. If $a, b \in T$, then $a + b \notin T$.
2. If $a \in T$ and $b \notin T$, then $a + b \in T$ and $b + a \in T$.
3. If $a, b \notin T$, then $a + b \notin T$.

Let $a, b, c \in N$. If $c \notin S$, then $(a + b)c = 0 = 0 + 0 = ac + bc$. So assume $c \in S_i$. We consider four cases.

1. If $a, b \in T$, then $a + b \notin T$ and $(a + b)c = 0 = q_i + q_i = ac + bc$.

2. If $a, b \notin T$, then $a + b \notin T$ and $(a + b)c = 0 = 0 + 0 = ac + bc$.
3. If $a \in T$ and $b \notin T$, then $a+b \in T$ and $(a+b)c = q_i = q_i+0 = ac+bc$.
4. If $a \notin T$ and $b \in T$, then $a+b \in T$ and $(a+b)c = q_i = 0+q_i = ac+bc$.

So all cases are exhausted and multiplication distributes over addition on the right. We conclude that N is a right near-ring.

Suppose N has a multiplicative identity 1. If $1 \in T$, then for $b \in S_1$, $b = 1 \cdot b = q_1 \notin S_1$, a contradiction. So $1 \notin T$. Thus, for $b \in S_1$, $b = 1 \cdot b = 0 \notin S_1$, a contradiction. It follows that N does not have a multiplicative identity. This completes the proof. $\Diamond$

We call the near-ring N above a *TS near-ring.* Our characterization theorem for this section is given by the following.

Theorem 4.2. Let N be a TS near-ring with $S = S_1 \dot{\cup} S_2 \dot{\cup} \cdots \dot{\cup} S_n$ as described above.

1. If $n = 1$ and $S = T$, then $C(N) = N_d = GC(N) = N$, making N a commutative near-ring.
2. If $n = 1$ and $S \subsetneq T$, then $N \setminus T = C(N) = N_d \subsetneq GC(N) = N$.
3. If $n \geq 2$, then $N \setminus T = C(N) = N_d \subsetneq GC(N) = N$.

In all cases, $C(N)$ is a subnear-ring of N.

Proof. Note that in all cases if $x \in N \setminus T$ and $a \in N$, then $xa = 0 = ax$. Thus $N \setminus T \subseteq C(N)$.

(1) Assume $n = 1$ and $S = T$. Let $x \in T$. For $a \in T = S_1$, $xa = 0 = ax$. For $a \notin T$, $xa = 0 = xa$. So $x \in C(N)$ and $T \subseteq C(N)$. From the remark above, $N \setminus T \subseteq C(N)$ as well, giving $C(N) = N$. Since $C(N) \subseteq N_d$ in any near-ring N, it follows that $N_d = N$. Thus $C(N) = N_d = GC(N) = N$.

(2) Assume $n = 1$ and $S \subsetneq T$. Let $x \in C(N)$ and assume $x \in T$. Then for $a \in S$, $xa = ax$ implies $q_1 = ax$. We conclude that $x \in S$. Now let $y \in T \setminus S$. Then $xy = 0 \neq q_1 = yx$, which contradicts $x \in C(N)$. It follows that $x \notin T$ and $C(N) \subseteq N \setminus T$. Using the comment at the beginning of the proof, we get $N \setminus T = C(N)$.

As above, since $C(N) \subseteq N_d$ for any near-ring, it follows that $N \setminus T \subseteq N_d$. To show containment in the other direction, let $x \in N_d$ and assume

$x \in T$. Let $a \in S$ and $b \in T \setminus S$. Then $a + b \notin T$ as noted above when proving the right distributive law. Hence, $x(a + b) = 0 \neq q_1 = q_1 + 0 = xa + xb$, contradicting $x \in N_d$. Thus $x \notin T$ and $N \setminus T = N_d$. Since $C(N) = N_d$, $GC(N) = N$. Proper containment in the chain follows since $N \setminus T \subsetneq N$.

(3) Assume $n \geq 2$. Suppose first that $S = T$. Let $x \in C(N)$ and assume $x \in T = S$. Suppose $x \in S_j$ and choose any $a \in S_i \neq S_j$. Then $xa = q_i \neq q_j = ax$, contradicting $x \in C(N)$. Hence, $x \notin T$ and $C(N) \subseteq N \setminus T$. It follows that $N \setminus T = C(N) \subseteq N_d$. For containment the other way, let $x \in N_d$ and assume $x \in T$. Then for $a \in S_1$ and $b \in S_2$, $a + b \notin T = S$ and $x(a + b) = 0 \neq q_1 + q_2 = xa + xb$. Thus, $x \notin N_d$, a contradiction. So $x \notin T$ and $N_d = N \setminus T = C(N)$. Since $N_d = C(N)$, we conclude $GC(N) = N$.

Now suppose $S \subsetneq T$. Let $x \in C(N)$ and assume $x \in T$. If $x \in S$, then using the same proof in the $S = T$ case above, we contradict $x \in C(N)$. If $x \notin S$, then for $a \in S_1$, $xa = q_1 \neq 0 = ax$ and $x \notin C(N)$, a contradiction as well. Thus, $x \notin T$ and $N \setminus T = C(N) \subseteq N_d$. A similar argument as above shows containment in the other direction.

Lastly, we show that $C(N)$ is always a subnear-ring of N by considering the three cases given in the theorem. In case (1), since $C(N) = GC(N)$, $C(N)$ is a subnear-ring of N. For cases (2) and (3), we have $C(N) = N \setminus T$. Since $N \setminus T$ is a subgroup of N, $C(N)$ is closed under addition. As $C(N)$ is closed under multiplication,and N is finite, it follows that $C(N)$ is a subnear-ring of N. $\diamondsuit$

Examples of each case of Theorem 4.2 may be easily constructed following the definition.

5. TSI Near-Rings

Theorem 5.1. Let $(G, +)$ be a group of even order, not necessarily abelian. Suppose there exists $\emptyset \neq T \subseteq G^*$ such that $G \setminus T$ is a (normal) subgroup of G of index 2. Let $\emptyset \neq I \subseteq T$ and $\emptyset \neq S \subseteq G^* \setminus I$ such that $T = I \cup (S \cap T)$. Partition S into $S = S_1 \dot\cup S_2 \dot\cup \cdots \dot\cup S_n$ such that for each $1 \leq i \leq n$, $S_i \subseteq S \cap T$ or $S_i \subseteq S \setminus T$. Furthermore, choose distinct $q_i \in S_i$ such that $2q_i = 0$ for each $1 \leq i \leq n$.

Define a multiplication on G by

$$ab = \begin{cases} a & \text{if } b \in I \\ q_1 & \text{if } a \in T, b \in S_1 \\ \vdots & \\ q_n & \text{if } a \in T, b \in S_n \\ 0 & \text{otherwise} \end{cases}.$$

Then $N = (G, +, \cdot)$ is a right zero-symmetric near-ring. Furthermore, N has a two-sided identity, 1, if and only if $I = \{1\}$, $S = \{q_1, q_2, \ldots, q_n\}$, and $N \setminus (S \cup T) = \{0\}$.

Proof. Since $(N, +)$ is a group, we only need to show that the given multiplication is associative and that multiplication distributes from the right over the addition of N. First we need a lemma.

Lemma 5.2. The product $ab \in T$ if and only if $a \in T$ and $b \in T$.

Proof. Assume $a \in T$ and $b \in T$. Since $b \in T$, either $b \in I$ or $b \in S$. If $b \in I$, then $ab = a \in T$. If $b \in S$, then $b \in S_j \cap T$ for some j and $ab = q_j \in S_j \subseteq T$. Thus, $ab \in T$. For the converse, first assume $a \notin T$. Either $b \in I$ or $b \notin I$. If $b \in I$, then $ab = a \notin T$. If $b \notin I$, then $ab = 0 \notin T$. Now assume $b \notin T$. If $b \in S_j \setminus T$ and $a \in T$, then $ab = q_j \in S_j$ and $ab \notin T$. Otherwise, $ab = 0 \notin T$. Hence, if $a \notin T$ or $b \notin T$, then $ab \notin T$, and the proof of the lemma is complete. $\Diamond$

To show associativity of multiplication, let $a, b, c \in N$. We consider five cases.

1. If $c \in I$, then $(ab)c = ab = a(bc)$.

2. If $c \notin I$ and $c \notin S$, then $(ab)c = 0 = a(bc)$.

3. If $c \notin I$, $c \in S_i$, and $a, b \in T$, then by the previous lemma, $ab \in T$. Thus $(ab)c = q_i = aq_i = a(bc)$.

4. If $c \notin I$, $c \in S_i$, and $a \notin T$, then by the previous lemma, $ab \notin T$. Since $c \in S_i$, it follows that $bc \notin I$. Therefore $(ab)c = 0 = a(bc)$.

5. If $c \notin I$, $c \in S_i$, and $b \notin T$, then by the previous lemma, $ab \notin T$. So $(ab)c = 0 = a \cdot 0 = a(bc)$.

Associativity of multiplication now follows.

Since $G \backslash T$ is a normal subgroup of index 2 in G, we have the same conditions as in TS near-rings:
1. If $a, b \in T$, then $a + b \notin T$.
2. If $a \in T$ and $b \notin T$, then $a + b \in T$ and $b + a \in T$.
3. If $a, b \notin T$, then $a + b \notin T$.

To show distributivity, again let $a, b, c \in N$. If $c \in I$, then $(a+b)c = a + b = ac + bc$. If $c \notin I$, but $c \in S_i$, then:

1. If $a, b \in T$, then $a + b \notin T$ and $(a + b)c = 0 = q_i + q_i = ac + bc$.
2. If $a, b \notin T$, then $a + b \notin T$ and $(a + b)c = 0 = 0 + 0 = ac + bc$.
3. If $a \in T$, $b \notin T$, then $a+b \in T$ and $(a+b)c = q_i = q_i + 0 = ac+bc$.
4. If $a \notin T$, $b \in T$, then $a+b \in T$ and $(a+b)c = q_i = 0+q_i = ac+bc$.

Finally, if $c \notin I$ and $c \notin S$, $(a + b)c = 0 = 0 + 0 = ac + bc$. The right distributive law now follows and N is a right near-ring.

Assume 1 is a two-sided multiplicative identity for N. Let $b \in I$. Then $b = 1 \cdot b = 1$. So $I = \{1\}$. Now let $b \in S_i$. Then $b = 1 \cdot b = q_i$ since $1 \in T$. Thus $S_i = \{q_i\}$ for every i. Finally, let $b \in N \setminus (S \cup T)$. Then $b = 1 \cdot b = 0$. So $N \setminus (S \cup T) = \{0\}$. Now assume $I = \{1\}$, $S = \{q_1, q_2, \ldots, q_n\}$, and $N \setminus (S \cup T) = \{0\}$. Since $1 \cdot q_i = q_i = q_i \cdot 1$ for all i, it follows that 1 is a two-sided identity for N. $\diamondsuit$

We call the near-ring N above a TSI *near-ring*. Note that I is the set of right identities in N. Throughout this section, let $Q = \{q_1, q_2, \ldots, q_n\}$. We consider three cases for S and T: $S \cap T = \emptyset$, $S \subsetneq T$, and $S \cap T \neq \emptyset$ with $S \not\subseteq T$.

Theorem 5.3. Let N be a TSI near-ring such that $S \cap T = \emptyset$. Then:

1. $C(N) = Q \cup \{0\}$, which is a subnear-ring of N if and only if $Q \cup \{0\}$ is a subgroup of $G \setminus T$;
2. $N_d = \{d \in N \backslash T \mid \text{order of } d \leq 2\}$;
3. If $N_d = Q \cup \{0\}$, then $GC(N) = N$. If $N_d \neq Q \cup \{0\}$, then $GC(N) = N \backslash T$.

Proof. Note that if $S \cap T = \emptyset$, then $T = I$.

(1) Let $c \in C(N)$. Assume $c \notin S$. Let $t \in T$. Then $ct = tc$ implies $c = 0$. Hence $C(N) \subseteq S \cup \{0\}$. Now assume $c \in S$. Thus, $c \in S_i$ for some i. Then for $t \in T$, $ct = tc$ implies $c = q_i$. Hence $C(N) \subseteq Q \cup \{0\}$. For containment in the other direction, let $q \in Q$ and $n \in N$. If $n \in T$, then $qn = q = nq$. If $n \notin T$, then $qn = 0 = nq$. Thus $q \in C(N)$ and $C(N) = Q \cup \{0\}$. The last part of (1) is a restatement of the additive closure of $C(N)$.

(2) First we show that $N_d \subseteq N \backslash T$. Assume $d \in N_d \cap T$. Let $a \in T$ and $b \in S_i$. Then $a + b \in T$. So $d(a+b) = d$ and $da + db = d + q_i$ imply $q_i = 0$, a contradiction. Hence, $d \notin T$. Thus, by contradiction, $N_d \subseteq N \backslash T$. Now let $d \in N_d$ and $a, b \in T$. So $d, a+b \notin T$. Thus, $d(a+b) = 0$ and $da + db = d + d$ imply $d + d = 0$, and every element of N_d has order at most two. We conclude that $N_d \subseteq \{d \in N \backslash T \mid \text{order of } d \leq 2\}$.

Now we show containment in the other direction. We know $0 \in N_d$, so let $0 \neq d \in \{d \in G \backslash T \mid \text{order of } d \leq 2\}$. If $a, b \in T$, then $a + b \notin T$ and $d(a+b) = 0 = d + d = da + db$. If $a, b \notin T$, then $a + b \notin T$ and $d(a+b) = 0 = 0 + 0 = da + db$. If $a \in T$ and $b \notin T$, then $a + b \in T$ and $d(a+b) = d = d + 0 = da + db$. The case $a \notin T$ and $b \in T$ follows by symmetry. Thus $d \in N_d$ and $\{d \in N \backslash T \mid \text{order of } d \leq 2\} \subseteq N_d$. The result now follows.

(3) If $N_d = Q \cup \{0\} = C(N)$, then $GC(N) = N$ is clear. Assume $N_d \neq Q \cup \{0\}$. Let $d \in N_d$. Then $d \notin T$. Let $x \notin T$. Then $dx = 0 = xd$, and $x \in GC(N)$. Thus, $N \backslash T \subseteq GC(N)$. Now let $x \in GC(N)$. Since $N_d \neq Q \cup \{0\}$, there exists $0 \neq d \in N_d \backslash Q$. Assume $x \in T$. Then $xd = dx = d$. If $d \in S$, then $xd = q_i$ for some $q_i \in Q$. So $d = q_i \in Q$, a contradiction. If $d \notin S$, then $xd = 0$. So $d = 0$, a contradiction. We conclude $x \notin T$. Thus $GC(N) \subseteq N \backslash T$, hence, equality. $\diamondsuit$

Example 5.4. Examples of TSI near-rings with $S \cap T = \emptyset$

Example 1. Let $G = \mathbb{Z}_4$, $T = I = \{1, 3\}$, and $S = S_1 = Q = \{2\}$. Then $S \cap T = \emptyset$ and by the previous theorem, the resulting TSI near-ring has $C(N) = Q \cup \{0\} = \{0, 2\} = N_d$ and $GC(N) = N$. Here, $C(N)$ is a subnear-ring of N.

Example 2. Let $G = \mathbb{Z}_4 \times \mathbb{Z}_2$, $T = I = \{(1,0), (3,0), (1,1), (3,1)\}$, $S = S_1 = \{(2,0), (2,1), (0,1)\}$, and $Q = \{(0,1)\}$. Then $S \cap T = \emptyset$ and by the previous theorem, the resulting TSI near-ring has $C(N) = Q \cup \{(0,0)\} = \{(0,1), (0,0)\}$ and $N_d = S \cup \{(0,0)\}$. So $N_d \neq Q \cup \{0\}$

and $GC(N) = N \setminus T = N_d = \{(2,0),(2,1),(0,1),(0,0)\}$. Here, $C(N)$ is a subnear-ring of N.

Theorem 5.5. Let N be a TSI near-ring such that $S \subsetneq T$.

1. If $N_d \neq \{0\}$, then $S = S_1$, $Q = \{q_1\}$, $N_d = \{q_1, 0\} = C(N)$, and $GC(N) = N$.

2. If $N_d = \{0\}$, then $C(N) = \{0\}$ and $GC(N) = N$.

In both cases, $C(N)$ is a subnear-ring of N with $C(N) \subsetneq GC(N)$.

Proof. To show the first assertion, assume there exists $0 \neq d \in N_d$. Suppose $d \notin T$. Then for arbitrary $a \in I$ and $b \in S$, $a + b \notin T$. Thus $d(a+b) = 0$ and $da + db = d + 0$ imply $d = 0$, a contradiction. So $d \in T$. Now choose arbitrary $a \in I$ and $b \in S$. Then $a + b \notin T$. Thus $d(a+b) = 0$ and $da + db = d + q_i$ imply $d + q_i = 0$ and $d = q_i$. Since $b \in S$ is arbitrary, $S = S_1$ and $Q = \{q_1\}$. Thus, $N_d \subseteq \{q_1, 0\}$.

Now we show $q_1 \in C(N)$. Let $a \in N$. If $a \in S$, then $q_1 a = q_1 = a q_1$. If $a \in I$, then $q_1 a = q_1 = a q_1$. If $a \notin T$, then $q_1 a = 0 = a q_1$. So $q_1 \in C(N)$. This gives $\{0, q_1\} \subseteq C(N)$. Since $C(N) \subseteq N_d \subseteq \{0, q_1\} \subseteq C(N)$, we obtain equality of all three sets. It follows that $GC(N) = N$.

If $N_d = \{0\}$, then $C(N) \subseteq N_d$ implies $C(N) = \{0\}$. The rest of the proof follows immediately. $\Diamond$

Example 5.6. Examples of TSI near-rings with $S \subsetneq T$

Example 3. Let $G = \mathbb{Z}_6$, $T = \{1,3,5\}$, $I = \{5\}$, $S = S_1 = \{1,3\}$, and $Q = \{3\}$. Then the TSI near-ring N satisfies $S \subsetneq T$. One can verify that $C(N) = \{0,3\}$ so that $\{0\} \neq C(N) \subseteq N_d$. By the previous theorem, $N_d = \{0,3\} = C(N)$, and $GC(N) = N$.
Example 4. Let $G = S_3$, the symmetric group on 3 elements, $T = \{(2\,3),(1\,2),(1\,3)\}$, $I = \{(1\,3)\}$ and $S = \{(2\,3),(1\,2)\}$ with $S_1 = \{(2\,3)\}$ and $S_2 = \{(1\,2)\}$. By the previous theorem, $S = S_1 \cup S_2$ implies $C(N) = N_d = \{(1)\}$ and $GC(N) = N$.

Lemma 5.7. Let N be a TSI near-ring such that $S \cap T \neq \emptyset$ with $S \nsubseteq T$. Then $N_d \subseteq T \cup \{0\}$.

Proof. Assume $0 \neq x \in N_d$ such that $x \notin T$. Consider $q_k \in S \cap T$ and $i \in I$. Since $q_k, i \in T$, we know that $q_k + i \notin T$; hence $q_k + i \notin I$. Since $x \in N_d$, we have $x(q_k + (q_k + i)) = xq_k + x(q_k + i)$. Simplifying both sides of this equation yields $x = 0$, a contradiction. It follows that $x \in T$ and $N_d \subseteq T \cup \{0\}$. $\Diamond$

Note that if $N_d = \{0\}$, then $GC(N) = N$ and $C(N) = \{0\}$. So we turn our attention to the case where $N_d \neq \{0\}$.

Lemma 5.8. Let N be a TSI near-ring such that $S \cap T \neq \emptyset$ with $S \nsubseteq T$. If $N_d \neq \{0\}$, then $N_d = \{0, t\}$, for some $t \in T$.

Proof. Since $S \cap T \neq \emptyset$, there exists $q_j \in S_j \subseteq S \cap T$. Fix $i \in I$. Since $i \in T$ and $q_j \in T$, we have $i + q_j \notin T$. So $i + q_j \notin S \cup T$ or $i + q_j \in S \backslash T$.

Let $t \in N_d \backslash \{0\}$. It follows that $t = ti = t((i + q_j) + q_j) = t(i + q_j) + tq_j$. By the previous lemma, $t \in T$. If $i + q_j \notin S \cup T$, then the preceding equation simplifies to $t = q_j$. Since $t \in N_d \backslash \{0\}$ is arbitrary, we conclude that $N_d = \{0, q_j\}$. If $i + q_j \in S \backslash T$, the equation simplifies to $t = q_k + q_j$ for some $q_k \in S \backslash T$ which is independent of the choice of t. Since $t \in N_d \backslash \{0\}$ is arbitrary, we conclude that $N_d = \{0, q_k + q_j\}$. The result now follows. $\diamondsuit$

Theorem 5.9. Let N be a TSI near-ring such that $S \cap T \neq \emptyset$ with $S \nsubseteq T$ and $N_d \neq \{0\}$.

1. If $N_d = \{0, i\}$ for some $i \in I$, then $GC(N) = Q \cup \{0, i\}$. Furthermore, if $I = \{i\}$, $S = Q$, and $N \backslash (S \cup T) = \{0\}$, then $C(N) = \{0, i\}$; otherwise $C(N) = \{0\}$.
2. If $N_d = \{0, s\}$ for some $s \in (S_j \cap T) \backslash Q$, then $GC(N) = S_j \cup (N \backslash (S \cup T))$ and $C(N) = \{0\}$.
3. If $N_d = \{0, q_j\}$ for some $q_j \in S_j \cap T \cap Q$, then $GC(N) = I \cup S_j \cup (N \backslash (S \cup T))$ and $C(N) = \{0\}$.

The center $C(N)$ is a subnear-ring of N if and only if N does not have a two-sided multiplicative identity or N has a two-sided multiplicative identity of additive order two.

Proof. (1) Let $x \in GC(N)$. If $x \in I$, then $xi = ix$ implies $x = i$. If $x \in S$, then $xi = ix$ implies $x = q$ for some $q \in S$. If $x \notin S \cup T$, then $xi = ix$ implies $x = 0$. Hence, $GC(N) \subseteq Q \cup \{0, i\}$. Now assume $x \in Q \cup \{0, i\}$. If $x \in \{0, i\}$, then x clearly commutes with 0 and i. If $x = q \in Q$, then $x0 = 0 = 0x$ and $xi = x = q = ix$. Thus, $x \in GC(N)$ and $GC(N) = Q \cup \{0, i\}$. Since $C(N) \subseteq N_d$, we only need to determine if $i \in C(N)$ to complete the proof of the second statement. But if $I = \{i\}$, $S = Q$, and $N \backslash (S \cup T) = \{0\}$, by Theorem 5.1, i is a two-sided multiplicative identity for N. Thus $i \in C(N)$ and $C(N) = \{0, i\}$. For the last part of the theorem, assume $I \neq \{i\}$, $S \neq Q$, or $N\backslash(S \cup T) \neq \{0\}$. If $I \neq \{i\}$, then let $i \neq j \in I$. Then $ij = i \neq j = ji$ and $i \notin C(N)$. If $S \neq Q$, then let $s \in S_k\backslash Q$. Then $is = q_k \neq s = si$. Thus $i \notin C(N)$. If $N\backslash(S \cup T) \neq \{0\}$, then for $0 \neq x \notin S \cup T$, $ix = 0 \neq x = xi$, and $i \notin C(N)$. In all three cases, $i \notin C(N)$; hence, $C(N) = \{0\}$.

(2) Let $x \in GC(N)$. If $x \notin S \cup T$, then $xs = 0 = sx$. Therefore, assuming $x \notin S \cup T$ imposes no restriction on x. If $x \in I$, then $xs = sx$ implies $q_j = s$, a contradiction. So $x \notin I$. If $x \in S_k \cap T$, then $xs = sx$ implies $q_j = q_k$. Thus $x \in S_j$. If $x \in S_k\backslash T$, then $xs = sx$ implies $0 = q_k$, a contradiction. So $x \notin S\backslash T$. Hence, $GC(N) \subseteq S_j \cup (N\backslash(S \cup T))$. For the reverse inclusion, assume $x \in S_j \cup (N\backslash(S \cup T))$. Clearly, $x0 = 0 = 0x$. If $x \in S_j$, then $xs = q_j = sx$. If $x \notin S \cup T$, then $xs = 0 = sx$. Thus, $x \in GC(N)$ and $GC(N) = S_j \cup (N\backslash(S \cup T))$. Since $C(N) \subseteq N_d = \{0, s\}$ and for $i \in I$, $si = s \neq q_j = is$, it follows that $C(N) = \{0\}$.

(3) Let $x \in GC(N)$. If $x \in I$, then $xq_j = q_j = q_jx$. If $x \notin S \cup T$, then $xq_j = 0 = q_jx$. Therefore, assuming $x \in I$ or $x \notin S \cup T$ imposes no restriction on x. If $x \in S_k \cap T$, then $xq_j = q_jx$ implies $q_j = q_k$, and $x \in S_j$. If $x \in S_k\backslash T$, then $xq_j = q_jx$ implies $0 = q_k$, a contradiction. So $x \notin S\backslash T$. Hence, $GC(N) \subseteq I \cup S_j \cup (N\backslash(S \cup T))$. Now assume $x \in I \cup S_j \cup (N\backslash(S \cup T))$. Clearly, $x0 = 0 = 0x$. If $x \in I \cup S_j$, then $xq_j = q_j = q_jx$. If $x \notin S \cup T$, then $xq_j = 0 = q_jx$. In all cases x commutes with q_j and $x \in GC(N)$. Thus $GC(N) = I \cup S_j \cup (N\backslash(S \cup T))$. Since $C(N) \subseteq N_d = \{0, q_j\}$ and for $q_k \in S\backslash T$, $q_jq_k = q_k \neq 0 = q_kq_j$, it follows that $C(N) = \{0\}$.

If N does not have a multiplicative identity, then $C(N) = \{0\}$. If N has a multiplicative identity i, then $C(N) = \{0, i\}$. The latter is closed under addition when i has additive order two. $\Diamond$

Example 5.10. Examples of TSI near-rings with $S \cap T \neq \emptyset$ and $S \not\subseteq T$

Example 5. Let $G = \mathbb{Z}_2 \times \mathbb{Z}_2$, $T = \{(1,0),(1,1)\}$, $I = \{(1,1)\}$, and $S = Q = \{(0,1),(1,0)\}$ with $S_1 = \{(0,1)\}$ and $S_2 = \{(1,0)\}$. Since I consists of a single element, $S = Q$, and $N\backslash(S \cup T) = \{0\}$, by part (1) of the previous theorem one sees that $C(N) = \{(0,0),(1,1)\} = N_d$ and $GC(N) = N$. Note that $C(N)$ is a subnear-ring of N.

Example 6. Let $G = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, $T = \{(1,0,0),(1,0,1),(1,1,0),(1,1,1)\}$, $I = \{(1,1,0),(1,1,1)\}$, $S_1 = \{(0,1,0),(0,1,1)\}$, $S_2 = \{(1,0,0),(1,0,1)\}$, and $Q = \{(0,1,0),(1,0,0)\}$. We claim that $(1,1,0) \in N_d$. To show this, we use various combinations of the following subsets of the TSI near-ring N: I, S_1, S_2, $N \setminus (S \cup T)$. First note that if $A \in \{I, S_1, S_2, N \setminus (S \cup T)\}$ and $x, y \in A$, then $(1,1,0)x = (1,1,0)y$. We consider four cases:

1. Let $A \in \{I, S_1, S_2, N \setminus (S \cup T)\}$. Consider $a \in A$ and $b \in N \setminus (S \cup T)$. Then $a + b \in A$. From the remark above, $(1,1,0)(a+b) = (1,1,0)a = (1,1,0)a + (0,0,0) = (1,1,0)a + (1,1,0)b$. Since G is an abelian group, the case $a \in N \setminus (S \cup T)$ and $b \in A$ follows. Throughout the remainder of the proof, we will employ this symmetry as well.

2. Let $A \in \{I, S_1, S_2, N \setminus (S \cup T)\}$. Consider $a, b \in A$. Then $a + b \in N \setminus (S \cup T)$. Since $a, b \in A$, it follows that $(1,1,0)a = (1,1,0)b$, which has order 2 in N. So $(1,1,0)(a+b) = (0,0,0) = (1,1,0)a + (1,1,0)b$.

3. Let $a \in I$ and $b \in S_i$, where $i \in \{1,2\}$. Then $a + b \in S_j$ where $j \in \{1,2\} - \{i\}$. So $(1,1,0)(a+b) = q_j = (1,1,0) + q_i = (1,1,0)a + (1,1,0)b$.

4. Let $a \in S_1$ and $b \in S_2$. Then $a + b \in I$. So $(1,1,0)(a+b) = (1,1,0) = (0,1,0) + (1,0,0) = (1,1,0)a + (1,1,0)b$.

It follows that $(1,1,0) \in N_d$. Since $(1,1,0) \in I$ and $I \neq \{(1,1,0)\}$, by (1) in the previous theorem, $C(N) = \{0\}$ and

$$GC(N) = \{(0,0,0), (1,1,0), (0,1,0), (1,0,0)\}.$$

Example 7. Let $G = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, $T = \{(1,0,0), (1,0,1), (1,1,0), (1,1,1)\}$, $I = \{(1,1,1)\}$, $S_1 = \{(0,1,0), (0,1,1)\}$, $S_2 = \{(1,0,0), (1,0,1)\}$, $S_3 = \{(1,1,0)\}$, and $Q = \{(0,1,0), (1,0,0), (1,1,0)\}$. As in the previous example, using the subsets I, S_1, S_2, S_3, and $N \setminus (S \cup T)$ of the TSI near-ring N in various combinations, one can show that $(1,1,0) \in N_d$. Since $(1,1,0) \in S_3 \cap T \cap Q$, by (3) in the previous theorem, $C(N) = \{0\}$ and $GC(N) = \{(0,0,0), (0,0,1), (1,1,0), (1,1,1)\}$.

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