Mathematica Pannonica
25/1 (2014–2015), 147–155

A NOTE ON THE AXIOMATIZATION OF THE NASH EQUILIBRIUM CORRESPONDENCE

Ferenc **Forgó**

Department of Operations Research, Corvinus University of Budapest, Fővám tér 8, Budapest, H-1093 Hungary

Received: February 10, 2015

JEL Classification Number C72

Keywords: Nash equilibrium, axiomatization, independence of irrelevant strategies.

Abstract: A new axiomatization of the Nash equilibrium correspondence for n-person games based on independence of irrelevant strategies is given. Using a flexible general model, it is proved that the Nash equilibrium correspondence is the only solution to satisfy the axioms of non-emptiness, weak one-person rationality, independence of irrelevant strategies and converse independence of irrelevant strategies on the class of subgames of a fixed finite n-person game which admit at least one Nash equilibrium. It is also shown that these axioms are logically independent.

1. Introduction

Characterization of game theoretical concepts through axioms has been a standard approach in non-cooperative and especially in cooperative game theory. The earliest and most celebrated result is the axiomatization of the Nash bargaining solution, Nash [5]. For non-cooperative games, the axiomatization of the Nash equilibrium correspondence (*NE*), Nash [6] and its refinements was first studied by Peleg and Tijs [7]. The

E-mail address: ferenc.forgo@uni-corvinus.hu

key concept was that of a reduced game which is obtained by fixing the strategies of some players and letting the rest of the players play the original game. They called a solution consistent if a solution to the original game when restricted to a subset of players remains a solution to the reduced game. Their main result was that one-person rationality (*OPR*), consistency (*CONS*) and converse consistency (*COCONS*) uniquely determine *NE*. They also touched upon the subject of using independence of irrelevant strategies (*IIS*) in the axiomatization by showing that the dummy axiom (*DUM*) and *IIS* imply *CONS*. Ray [9] further studied the relationship among *CONS*, *IIS*, *DUM* and a weakening of the dummy axiom (*WDUM*). Peleg, Potters and Tijs [8] gave conditions under which they could do away with *COCONS*. In these works, however, consistency and not *IIS* was in the focus.

In this note we set up a scheme to axiomatize *NE* for finite n-person normal-form games where *IIS* and its converse, converse independence of irrelevant strategies (*CIIS*) play the central role. Of course, we also need non-emptiness (*NEMP*) and a weaker form of one-person rationality (*WOPR*). The general framework is flexible, allowing for all sorts of different truncated games (this is how we call the subgames introduced by Gilboa et al. [3] and studied subsequently by Ray [10] and Shinohara [13]).

The acceptability of *IIS*, let alone that of *CIIS*, is an issue that we do not want to address here. Even its close relative, independence of irrelevant alternatives (*IIA*) has been understood in many ways in various contexts and still is a subject of disagreement and debate. On one hand, its main appeal is that in the optimization context it reduces to the relaxation principle which is the basis of many algorithms and can hardly be questioned. On the other hand, in the human decisions context theoretical considerations as well experiments cast serious doubts on its plausibility. From an ocean of relevant literature on the subject we only refer to the classical works of Luce and Raiffa [4] and Sen [12]. We do not want to argue either in favor or against *IIS*, the purpose of this note is not more than show that an *IIS*-based axiomatization can be an alternative to the reduced game approach.

2. The main result

Let a finite n-person game $G = \{N, (S_i)_{i\in N}, (f_i)_{i\in N}\}$ be given in normal (strategic) form, where $N = \{1, \dots, n\}, n \geq 1$ is the finite set of players, $S_i, i \in N$ is the finite strategy-set of player i, and $f_i : S \longrightarrow \mathbb{R}$ is her payoff function defined on the profile-set $S = \times_{i\in N} S_i$. We use the standard notation $S_{-i} = \times_{j\in N, j\neq i} S_j$ and s_{-i} for an element of S_{-i}. The game G is kept fixed throughout and is referred to informally as the "large game". A set $T_i \subset S_i$, $i \in N$ is said to be a truncated strategy-set, $T = \times_{i\in N} T_i$ a truncated strategy profile-set and $G_T = \{N, (T_i)_{i\in N}, (f_i)_{i\in N}\}$ a truncated game. Obviously, $G_S = G$. A player whose strategy set in a truncated game is a singleton is called a dummy, and a game where every player is a dummy is said to be trivial. A game with $n-1$ dummies is called a one-person game. A one-person game with two strategies for the non-dummy player is said to be semi-trivial. A game is called simple if it is either trivial or semi-trivial. All other games are termed non-simple.

A crucial role in the axiomatization is played by a family of truncated strategy profile-sets Ω with the following properties:

Property 1. $S \in \Omega$.

Property 2. $\{s\} \in \Omega$ for all $s \in S$.

Property 3. For every non-simple game G, for the non-dummy player $i \in N$, to any $y \in S$ and $z_i \in S_i, z_i \neq y_i$ there exists a one-person game $G_T = \{N, (\{y_j\}_{j\in N, j\neq i}, T_i), (f_i)_{i\in N}\}$, $T = \Pi_{j\in N, j\neq i}\{y_j\} \times T_i \in \Omega$, $T_i \neq S_i$ such that $\{(y_i, y_{-i}), (z_i, y_{-i})\} \subset T_i$.

An example of Ω satisfying Properties 1–3 is the set of all truncated profiles $T = \times_{i\in N} T_i$ where T_i consists of all one- and two-element subsets of S_i and S_i itself.

Let Γ be the set of truncated games G_T where $T \in \Omega$. We call a set-valued function $\varphi : \Gamma \to S$ a solution (correspondence) if to any game $G_T \in \Gamma$ it assigns a set $\varphi(G_T) \subset T$. We require of φ to satisfy the following four axioms for any game $G_T \in \Gamma$:

Axiom 1 (Non-emptiness, *NEMP*). $\varphi(G_T) \neq \varnothing$.

Axiom 2 (Weak one-person rationality, *WOPR*).

a) For every semi-trivial game G_T where the (single) non-dummy player i has strategies $r_i, q_i \in T_i$

$$\varphi(G_T) = \{(x, t_{-i}) \in T : f_i(x, t_{-i}) = \max\{f_i(r_i, t_{-i}), f_i(q_i, t_{-i})\}.$$

b) For every non-simple one-person game

$$G_T = \{N, (\{y_j\}_{j\in N, j\neq i}, T_i), (f_i)_{i\in N}\}, \quad T_i \neq S_i,$$

$$\varphi(G_T) = \{(x, t_{-i}) \in T : f_i(x, t_{-i}) = \max_{r_i \in T_i} f_i(r_i, t_{-i})\}.$$

Axiom 3 (Independence of irrelevant strategies, *IIS*). If $x \in \varphi(G_T)$ and $x \in R \subset T$, $R \in \Omega$, then $x \in \varphi(G_R)$.

In order to formulate Axiom 4 we need the following definition. Given the game $G_T \in \Gamma$ and solution φ, define

$$\varphi^*(G_T) = \{x \in T : R \in \Omega,\ R \neq T,\ x \in R \implies x \in \varphi(G_R)\}$$

if G is non-simple,

$$\varphi^*(G_T) = \varphi(G_T)$$

if G is simple.

Axiom 4 (Converse independence of irrelevant strategies, *CIIS*).

$$\varphi^*(G_T) \subset \varphi(G_T).$$

WOPR is weaker than the usual *OPR* because it only requires individual rationality in "smaller" one-person games and not in the whole large game. In the special case when T_i consists of all one- and two-element subsets of S_i and S_i itself, *WOPR* amounts to the rationality of pairwise comparisons.

We mention that *IIS* as defined in Axiom 3 coincides with the classical definition only if Ω contains all subsets of S. Axiom 3 provides flexibility in the choice of Ω to accommodate for needs of special classes of games.

CIIS requires that a solution obtained by putting together solutions of "smaller" games in a coherent way should be a solution of a "larger" game. The analog in decision theory (the case $n = 1$) when the "best" alternatives are to be selected from a finite list is that if an alternative is "best" in all properly selected sublists it is an element of, then it should be "best" in the entire list. In other contexts this is called "basic expansion consistency", see Sen [12].

Denote by *NE* the solution that assigns to any game $G \in \Gamma$ the (possibly empty) set of *NE*'s. From now on we define Γ to be the set of finite games that admit at least one *NE*. The following is our main result.

Theorem 1. *On Γ the Nash equilibrium solution NE is uniquely determined by NEMP, WOPR, IIS and CIIS.*

The proof of the theorem goes through a series of lemmas.

Lemma 1. *If a solution φ on Γ satisfies NEMP, WOPR, IIS, then for any $G_T \in \Gamma$ we have*

$$\varphi(G_T) \subset \varphi^*(G_T) \subset NE(G_T) \subset NE^*(G_T).$$

Proof. Observe that *IIS* can be reformulated as $\varphi(G_T) \subset \varphi^*(G_T)$. We claim that $\varphi^*(G_T) \subset NE(G_T)$. If G_T is simple, then the claim is obviously true, since by definition, $\varphi^*(G_T) = NE(G_T)$. In the case G_T is non-simple, assume on the contrary that there exists a $y \in \varphi^*(G_T)$ that is not an *NE*. Then there is a player $i \in N$ and a strategy $z_i \in T_i$ to satisfy $f_i(z_i, y_{-i}) > f_i(y_i, y_{-i})$. By Property 3, there exists a one-person game $G_R = \{N, (\{y_j\}_{j\in N, j\neq i}, R_i), (f_i)_{i\in N}\}$, $R_i \neq T_i$ such that $\{(y_i, y_{-i}), (z_i, y_{-i})\} \subset R_i$. By *WOPR* we have $y \notin \varphi(G_R)$ and thus by the definition of $\varphi^*(G_T)$ we get $y \notin \varphi^*(G_T)$, a contradiction. Since $y \in NE(G_T)$ is also an *NE* of any game G_R if $R \subset T$, by the relaxation principle of optimization theory we have $NE(G_T) \subset NE^*(G_T)$, and the proof is complete. $\Diamond$

Lemma 2. *If the solution φ satisfies NEMP, WOPR and CIIS on Γ, then $NE(G_T) \subset \varphi(G_T)$ for any $G_T \in \Gamma$.*

Proof. Denote $M = \mid S \mid$. We call a natural number t admissible if there is a $T \in \Omega$ such that $\mid T \mid = t$. Put the admissible numbers in increasing order $1 = t_1 < t_2 < t_3 <, \dots, < t_k <, \dots, < t_q = M$.

The proof goes by induction on the indices k of the admissible numbers. If $k = 1$, i.e. for trivial games, the claim obviously holds, since all strategy sets are singletons and thus $NE(G_T) = \varphi(G_T)$. For semi-trivial games, i.e. if $t_2 = 2, k = 2$ we have $NE(G_T) = \varphi(G_T)$ by *WOPR*. Assume that $NE(G_T) \subset \varphi(G_T)$ for any $1 \leq k < r$ and let $G_R \in \Gamma$ be a game for which $\mid R \mid = t_r \geq 3$. Thus

$$\varphi^*(G_R) = \{x \in R : P \in \Omega, P \subset R, P \neq R,\ x \in P \Longrightarrow x \in \varphi(G_P)\}.$$

By induction and the definition of $\varphi^*(G_R)$, if for all $1 \leq k < r$, that is, for all $P \in \Omega, P \subset R, P \neq R, \mid P \mid \geq 1$ we have $NE(G_P) \subset \varphi(G_P)$, then $NE^*(G_R) \subset \varphi^*(G_R)$. By *CIIS*, $\varphi^*(G_R) \subset \varphi(G_R)$. From the proof of Lemma 1 we know that $NE(G_R) \subset NE^*(G_R)$. So we come to the conclusion that $NE(G_R) \subset \varphi(G_R)$ for any r, in particular for $r = q$, and the claim of the lemma follows. $\Diamond$

Lemma 3. *The solution NE satisfies NEMP, WOPR, IIS and CIIS on Γ.*

Proof. Satisfying *NEMP, WOPR, IIS* is trivial. By using the same argument as in the proof of Lemma 1 one can prove that *CIIS* is also satisfied. $\Diamond$

Proof of Theorem 1. From Lemmas 1 and 2 we get $\varphi(G) \subset NE(G) \subset \subset \varphi(G)$ implying $\varphi(G) = NE(G)$ which together with Lemma 3 establishes the claim of the theorem. $\Diamond$

Corollary 1. *Under the conditions of Lemma 1, φ is a refinement of NE.*

Corollary 2. *Under the conditions of Lemma 2, φ is a generalization of NE.*

Now we will show by three examples that *WOPR*, *IIS* and *CIIS* are logically independent. (*NEMP* is assumed throughout to make the other three axioms meaningful).

Example 1. Define the solution ψ by $\psi(G_T) = T$. Then obviously ψ does not satisfy *WOPR*. Since $\psi'(G_T) = \psi(G_T) = T$, therefore φ satisfies *IIS* and *CIIS*.

Example 2. Let $B(x)$ be the set of best replies to $x \in T$. Define the solution ψ by $\psi(G_T) = \cup_{x\in T} B(x)$. Take the one-person game G_T defined in Axiom 2. Then $\psi(G_T) = NE(G_T)$ as required by *WOPR* and thus ψ satisfies *WOPR*. The solution ψ obviously satisfies *CIIS* if G_T is simple. If G_T is non-simple, then assume by negation that it does not satisfy *CIIS*. Then there is a $y \in \psi^*(G_T)$ such that $y \notin \cup_{x\in T} B(x)$. Since $y \in \psi^*(G_T)$, by the definition of $\psi^*(G_T)$ there is a set $R \in \Omega$, $R \subset T$, $R \neq T$ such that $y \in \cup_{x\in R} B(x) \subset \cup_{x\in R} B(x)$, a contradiction. Since $\psi \neq NE$ ($NE(G)$ contains only fixed points of the best reply mapping B, while $\psi(G)$ may be a superset of $NE(G)$), ψ does not satisfy *IIS*.

Example 3. Let K be any strict refinement of *NE* on Γ, i.e., $K(G_T) \subset \subset NE(G_T)$ for all $G_T \in \Gamma$ and this inclusion is strict for at least one game $H \in \Gamma$. Define the solution ψ as

$$\psi(H) = K(H) \text{ and } \psi(G_T) = NE(G_T) \text{ if } G_T \neq H.$$

Clearly, ψ is a refinement of *NE*, therefore it satisfies *WOPR* and *IIS*. Since $\psi \neq NE$, it does not satisfy *CIIS*.

Since this axiomatization (and the proof of Th. 1) was inspired by the landmark work of Peleg and Tijs [7], one might wonder how their consistency-based axiomatization relates to ours. A valid argument could be that staying within the class of n-person games with n fixed instead allowing the number of players to be any natural number not exceeding n is not a matter of substance since a game with some dummies is a reduced game though formally it is defined as an n-person game. This cannot be done, however, if in the class of games the number of players is not bounded. Disregarding this "nuance", the two axiomatizations are different in substance.

Most importantly, the domain of the axiomatization is different in the two approaches. Here it is a subset of the truncated games of a fixed large game and this subset could be much smaller than the set of all games even if the number of players is not fixed. Disregarding this fact, one might try in order to bring the two axiomatizations together to define the family of games Ω as all games where every strategy set is either the entire S_i or a single strategy $\{s_i\}$ for all $i \in N$. In this setup, Property 3 cannot be satisfied and should therefore be abandoned. Then Axioms 3 and 4 correspond to *CONS* and *COCONS*, respectively, in Peleg and Tijs [7], but a difference remains between *WOPR* and *OPR* since *WOPR* only requires one-person rationality of "smaller" games, as argued earlier. The general framework allows for Ω to be a larger set than Peleg and Tijs'es. Then *IIS* is stronger, *CIIS* is weaker than *CONS* and *COCONS*, respectively, and we really have a different axiomatization. Ray [9] demonstrates that *CONS* in the reduced game framework cannot simply be replaced by *DUM* (or *WDUM* for that matter) and *IIS*, since, as he shows, *CONS* does not imply *IIS*. Our result suggests that if *CONS* is to be replaced by *IIS*, then *COCONS* also is to be changed to *CIIS*, or something similar. These axioms of the reduced games based and truncated (subgame) based approaches do not mix in the axiomatization of *NE*.

As Peleg, Potters and Tijs [8] point out, an axiomatization as stated in Th. 1 is not quite satisfactory because the solution concept φ (in this case *NE*) to be characterized explicitly plays a role in the definition of the domain of games φ is defined on. Though this approach is used e.g. in Aumann [2], a characterization where the definition of the domain of games is independent of φ is preferable. This is the case for ordinal potential games.

A finite game $G = \{S_1, \ldots, S_n; f_1, \ldots, f_n\}$ is said to be an ordinal potential game if there is a (potential) function $P : S \to \mathbb{R}$ such that for all $i \in N, s_i, t_i \in S_i, s_{-i} \in S_{-i}$ we have

$$f_i(s_i, s_{-i}) - f_i(t_i, s_{-i}) > 0 \Leftrightarrow P(s_i, s_{-i}) - P(t_i, s_{-i}) > 0.$$

It is well known, Rosenthal [11], that a finite ordinal potential game always has at least one *NE*. Since for each $T \subset S$ the above equivalence when restricted to T holds, the game G_T is also a finite ordinal potential game. Thus Th. 1 holds, i.e., the axiomatization set forth in Sec. 2 works for finite ordinal potential games if the family of truncated strategy profile-sets Ω is properly chosen.

It is worth mentioning that the axiomatization works for any kind of finite potential games (cardinal, exact etc.). It is also known that congestion games as defined in Rosenthal [11] are finite potential games. Thus our axiomatization also bears on this important class of games. In the subclass of simple congestion games each player's strategy set is the same finite set (with cardinality of at least 2) of facilities and payoffs depend only on how many of the players use a particular facility. For details see e.g. Ashlagi et al. [1]. For the family of truncated strategy profile-sets in addition to the sets specified in Properties 1 and 2, we take all 2-facility subsets of the strategy sets. 2-facility simple congestion games exhibit special features, see Ashlagi et al. [1], which make the *IIS*-based axiomatization more appealing.

3. Conclusion

In this note an axiomatization of *NE* was given for subgames of finite n-person games which is based on the concept of independence of irrelevant strategies (*IIS*) and converse independence of irrelevant strategies (*CIIS*). Extensions of the results to infinite games will be the subject of a subsequent paper.

By analogy, one may wonder whether converse independence of irrelevant strategies (*CIIS*) can be got rid of in some classes of games, similarly as it is done in Peleg, Potters and Tijs [8]. It could also be the subject of further research to modify the general framework for the axiomatization of various refinements and generalizations of *NE*.

Acknowledgement. The support of research grant OTKA 101224 is gratefully acknowledged.

References

[1] ASHLAGI, I., MONDERER, D. and TENNENHOLTZ, M.: On the value of correlation, *Journal of Artificial Intelligence* **33** (2008), 575–613.

[2] AUMANN, R. J.: An axiomatization of the nontransferable utility value, *Econometrica* **53** (1987), 599–612.

[3] GILBOA, I., KALAI, E. and ZEMEL, E.: On the order of eliminating dominated strategies, *Operations Research Letters* **9** (1990), 85–89.

[4] LUCE, R. D. and RAIFFA, H.: *Games and Decisions: Introduction and Critical Survey*, Wiley, New York, 1957.

[5] NASH, J. F. JR.: The bargaining problem, *Econometrica* **18** (1950), 155–162.

[6] NASH, J. F.: Equilibrium points in n-person games, *Proceedings of the National Academy of Sciences* **36** (1950), 48–49.

[7] PELEG, B. and TIJS, S.: The consistency principle for games in strategic form, *International Journal of Game Theory* **25** (1996), 13–34.

[8] PELEG, B., POTTERS J. and TIJS, S.: Minimality of consistent solutions for strategic games, in particular for potential games, *Economic Theory* **7** (1996), 81–93.

[9] RAY, I.: Consistency, dummy and independence of irrelevant strategies, *Social Choice and Welfare* **17** (2000), 217–222.

[10] RAY, I.: On games with identical payoffs, *Economic Theory* **17** (2001), 223–231.

[11] ROSENTHAL, R. W.: A class of games possessing pure-strategy Nash equilibria, *International Journal of Game Theory* **2** (1973), 65–67.

[12] SEN, A.: Internal consistency of choice, *Econometrica* **61** (1993), 495–521.

[13] SHINOHARA, R.: Coalition-proof Nash equilibria in a normal form game and its subgames, *International Game Theory Review* **12** (2010), 253–261.