Mathematica Pannonica
25/1 (2014–2015), 113–133

SMALL DEGREE CNS POLYNOMIALS WITH DOMINANT CONDITION

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Received: December 10, 2014

MSC 2010: 12 D 99, 11 A 63

Keywords: Canonical number system, radix representation.

Abstract: CNS polynomials whose constant term is large in comparison to their other coefficients are studied. Characterization results for this class of polynomials are given provided that their degrees do not exceed five.

1. Introduction

Many generalizations of our ordinary decimal number system have been known and investigated for a long time. Here we deal with canonical number systems (commonly abbreviated by CNS) special cases of which were already presented by Grünwald [16], Knuth [24], Penney [28] and Gilbert [15]. The systematic study of canonical number systems has been initiated by the Hungarian school some decades ago (see [22, 20, 21, 26]). The works [7, 8, 19] are recommended as profound surveys on this subject in a broader context. Detailed background information on the historical development and relations of CNS polynomials to other areas such as shift radix systems, finite automata or fractal tilings can be found in the surveys by Barat et al. [7] and by Kirschenhofer and Thuswaldner [23].

The general notion of canonical number systems and the concept of CNS polynomials (see Sec. 2 for the definition[1]) were introduced by

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[1] CNS polynomials are named complete base polynomials in [13].

Pethő [30] and extended in the sequel (see for example [1, 6, 33]). The CNS property of a given polynomial can algorithmically be decided [34, 9, 13], and some characterization results on these polynomials are known (for instance, see [20, 15] for quadratic polynomials, [26, 4, 10, 5] for some other classes of polynomials and [27, 18] for more general results). Several open questions in a more general framework are listed in [2].

In particular, there is strong evidence that the CNS property of polynomials of degree larger than 2 having relatively small constant term can hardly be predicted. In fact, several examples illustrating this phenomenon can be found in [25, 3, 12, 11]. In view of these observations we here concentrate on monic integer polynomials whose constant term is large compared to the remaining coefficients and provide some characterizing properties for this class of polynomials. In particular, we describe certain CNS quadrinomials and extend the respective results of Akiyama and Rao [5] on cubic, quartic and quintic polynomials.

2. Definition and fundamental properties of CNS polynomials

Let

$$P = X^d + p_{d-1}X^{d-1} + \cdots + p_1X + p_0 \in \mathbb{Z}[X] \tag{1}$$

be a monic integer polynomial of positive degree with non-vanishing constant term. Recall that P is a CNS polynomial if for every $A \in \mathbb{Z}[X]$ there exists a polynomial $B \in \{0, \ldots, |P(0)| - 1\}[X]$ such that $A \equiv B$ (mod P). Throughout we denote by $\mathbb{Z}$ ($\mathbb{N}$, $\mathcal{C}$, respectively) the set of rational integers (the set of nonnegative rational integers, CNS polynomials, respectively).

We briefly illustrate the definition of a CNS polynomial by exhibiting some examples.

Example 2.1. (i) Let $b \geq 2$ and $m \in \mathbb{Z}$. By [16] we can write

$$m = \sum_{j=0}^{\ell} e_i(-b)^i \qquad (\ell \in \mathbb{N},\ e_0, \ldots, e_\ell \in \{0, \ldots, b-1\}).$$

The polynomial $\sum_{i=0}^{\ell} e_iX^i - m$ vanishes at $-b$, hence it is divisible by $P := X + b$. In other words, m is canonically represented by $\sum_{j=0}^{\ell} e_iX^i$

modulo P. Using the Euclidean algorithm every integer polynomial is congruent to an integer modulo P, hence P is a CNS polynomial (in fact, every linear CNS polynomial is of this form [1, Rem. 4.5]).

(ii) Consider the quadratic polynomial $P := X^2 + 2X + 2$. In view of

$$(X - 1) \cdot P = X^3 + X^2 - 2$$

we have $2 \equiv X^3 + X^2 \pmod{P}$, i.e., 2 is canonically represented by $X^3 + X^2$ modulo P. With some more effort one can show that P is indeed a CNS polynomial [20, 21, 15, 10, 35, 5].

(iii) Take $P := X^2 - 2X + 2$ and assume that there exist $e_0, ..., e_\ell \in \in \{0, 1\}$ such that

$$2 \equiv \sum_{j=0}^{\ell} e_i X^i \pmod{P}.$$

Then there are $t_0, \ldots, t_d \in \mathbb{Z}$ such that

$$\sum_{i=0}^{\ell} e_i X^i - 2 = P \cdot \sum_{i=0}^{d} t_i X^i \ .$$

Comparing coefficients we successively find

$$e_0 = e_1 = 0,\ t_0 = t_1 = -1,\ e_2 = 1,\ t_2 = 0,\ e_3 = t_3 = 1,\ e_4 = 0,\ t_4 = 1$$

and $e_n = t_n = 1$ for $n \geq 5$ which is absurd. Therefore, 2 does not admit a canonical representation modulo P, hence P is not a CNS polynomial.

(iv) The reader might find it instructive to notice that

$$P := X^3 + 50X^2 + 73X + 55 \in \mathcal{C},$$

but $P + 1 \notin \mathcal{C}$ (e.g., see [11, Prop. 10]).

To P we associate the mapping $\tau_P\colon\ \mathbb{Z}^d \mapsto \mathbb{Z}^d$ by[2]

$$\tau_P(a_1, \ldots, a_d) := \left(a_2, \ldots, a_d, -\left\lfloor \left(\sum_{j=0}^{d-1} p_{d-j}\, a_{j+1} \right) / |p_0| \right\rfloor \right) \qquad (p_d := 1)$$

and the sets

$$\mathcal{N}_P := \{z \in \mathbb{Z}^d\ :\ \tau_P^k(z) = 0 \text{ for some } k \in \mathbb{N}\}$$

and $\mathcal{E}_P := \emptyset$ if all coefficients of P are nonnegative, and $\mathcal{E}_P := \{(e_1, ..., e_d) \in \in E_0\ :\ e_d = 1\}$, otherwise, where we put $E_0 := \{0, 1\}^d$. If there is no fear of confusion we will occasionally omit the subscript P. For more details and background the reader is referred to [23]; here we only recall the following important facts.

Theorem 2.2. *Let $P = \sum_{i=0}^{d} p_i X^i$ be a CNS polynomial of degree d.*

[2] We denote by $\lfloor \ldots \rfloor$ the usual floor function.

(i) *P is expansive, i.e., every root of P lies outside the closed unit disc.*

(ii) *If r is a real root of P then $r < -1$.*

(iii) *For every $k \in \{1, \dots, d\}$ the polynomial P satisfies the k-subsum condition, i.e., there exists $\ell \in \{0, \dots, k-1\}$ such that*

$$S_k(\ell) := \sum_{0 \le ki+\ell \le d} p_{ki+\ell} \notin [0, p_0 - 1] \, .$$

(iv) $\sum_{i=1}^{d} p_i \ge 0$.

Proof. (i) [29, Th. 6.1].

(ii) This was shown by Gilbert [15, Prop. 6] under the (unused) assumption of the irreducibility of P.

(iii) [5, Sec. 3].

(iv) Either see [4, Lemma 2] or use (ii) and the 1-subsum condition. $\Diamond$

3. CNS polynomials with dominant constant term

Following Dubickas [14] we say that the polynomial $f \in \mathbb{Z}[X]$ has a dominant constant term if

$$2|f(0)| \ \ge \ L(f), \tag{2}$$

where $L(f)$ denotes the length of f, i.e., the sum of the moduli of the coefficients of f. Scheicher and Thuswaldner [32] coined the notion that the monic integer polynomial f with positive constant term fulfills the AP-condition if strict inequality holds in (2).

We review and slightly extend some well-known properties of CNS polynomials with dominant constant term. Further, we prepare our main tool (see Lemma 3.8 below) and subsequent characterization results, e.g., the characterization of CNS quadrinomials with at least one negative coefficient and dominant constant term. Let us start with a reformulation of an important result by Akiyama and Pethő [4, Lemma 5].

Lemma 3.1. *Let P be given by* (1) *and assume that P has a positive dominant constant term, i.e.,*

$$p_0 \ge \sum_{i=1}^{d} |p_i|.$$

Then we have

$$\|\tau_P(a)\|_\infty \le \|a\|_\infty \qquad (a \in \mathbb{Z}^d).$$

In particular, the set $E := \{-1,0,1\}^d$ *is* τ_P*-invariant, i.e., we have* $\tau_P(E) \subseteq E$. *Moreover, if* P *satisfies the AP-condition then the set* E_0 *is* τ_P*- invariant, and for every* $e \in E$ *there is some* $n \in \mathbb{N}$ *such that* $\tau_P^n(e) \in E_0$.

Akiyama and Pethő [4] and Akiyama and Rao [5] established the following necessary conditions for CNS polynomials with dominant constant term.

Theorem 3.2. *Let* P *given by* (1) *be a CNS polynomial of degree* $d \ge 2$ *with dominant constant term.*

(i) *For every* $\ell \in \{1,\dots,d-1\}$ *we have* $p_\ell + \sum_{i=\ell+1}^d |p_i| \ge 0$.

(ii) $1 + p_{d-1} + p_{d-2} \ge 0$.

(iii) $\sum_{2\le i\le d,\ 2|i}\ p_i \ge 0$.

Proof. (i) [4, Th. 3]. (ii), (iii) [5, Th. 5.2]. $\diamondsuit$

Remark 3.3. In view of Th. 2.2 (iv) the property mentioned in Th. 3.2 (i) holds for every CNS polynomial and $\ell = 1$.

The characterization of CNS polynomials with dominant constant term relies on a result by Akiyama and Rao [5, Th. 4.3] which we reformulate in the following way.

Theorem 3.4. *Let* P *be monic integer polynomial with dominant constant term and* $P(0) \ge 2$. *Then* P *is a CNS polynomial if and only if* $E \subset \mathcal{N}_P$.

We apply this result to slightly sharpen the characterization of CNS polynomials with exactly one negative coefficient and dominant constant term given by Akiyama and Rao [5, Th. 3.5].

Theorem 3.5. *Let* $P = \sum_{i=0}^d p_i X^i \in \mathbb{Z}[X]$ *be monic,* $0 < k < d$, $p_i \ge 0$ *for* $i \in \{1,\dots,d-1\} \setminus \{k\}$ *and* $p_k < 0$. *If* P *has a positive dominant constant term then* P *is a CNS polynomial if and only if*

$$\sum_{1\le ki\le d} p_{ki} \ge 0.$$

Proof. Let $P \in \mathcal{C}$, thus by Th. 2.2 the k-subsum condition holds. Since $S_k(\ell) \in [0, p_0-1]$ for $\ell \in \{1,\dots,k-1\}$ we must have $S_k(0) \notin [0, p_0-1]$. In view of $S_k(0) \ge 0$ we then find $S_k(0) \ge p_0$, which implies our assertion.

To prove the converse we assume $P \notin \mathcal{C}$. Clearly,

$$p_0 \geq 1 + |p_k| > 1,$$

thus by Th. 3.4 and Lemma 3.1 the set E contains a non-zero element which is periodic under τ_P. Similarly as in the proof of [5, Th. 3.5] this leads to a contradiction. $\diamondsuit$

Focusing our attention upon the non-vanishing coefficients of polynomials we let $r \geq 3$ and introduce the set

$$\text{(3)} \qquad \mathcal{S}_r := \left\{ \sum_{i=1}^{r} p_{n_i} X^{n_i} \in \mathbb{Z}[X] \,:\, n_r > \cdots > n_2 > n_1 = 0, \right.$$
$$\left. p_{n_r} = 1,\ p_0 \geq 2,\ p_{n_i} \neq 0\ (i = 2, \ldots, r-1) \right\}$$

which is contained in the set of monic integer r-sparse polynomials with positive constant term; recall that a polynomial is called r-sparse if it has at most r nonzero terms.

A useful result by Akiyama and Rao [5, Th. 5.2] can immediately be extended.

Proposition 3.6. *If $\sum_{i=1}^{r} p_{n_i} X^{n_i} \in \mathcal{S}_r \cap \mathcal{C}$ has a dominant constant term then we have*

$$p_{n_{r-1}} \geq -1.$$

Proof. This is a direct consequence of Th. 3.2 since we have

$$p_{n_{r-1}} \geq -\sum_{i=n_{r-1}+1}^{n_r} |p_i| = -|p_{n_r}| = -1. \ \diamondsuit$$

For convenience, the following convention is adopted: For the word w of a (finite) alphabet[3] we let

$$w^n := \underbrace{w \cdots w}_{n}$$

if n is a positive integer; w^0 denotes the empty word. By abuse of notation we denote by ε^n (δ^n, respectively) a word of length n with arbitrary (possibly different) elements of $\{-1, 0, 1\}$ ($\{0, 1\}$, respectively). For instance, we apply notions of the shape $\varepsilon^3 = \varepsilon\varepsilon'\varepsilon''$. If there is no danger of confusion we symbolize the action of τ_P by an arrow connecting words. Two simple examples might clarify the use of this notation.

[3] In this connection we use the terminology of [31, Chapter 0].

Example 3.7. (i) Let $n \geq 2$, $a_1, \ldots, a_n \in \mathbb{Z}$, $a_n = 1$ and $\sum_{i=1}^{n} a_i > 0$. Then clearly

$$P := \sum_{i=1}^{n} a_i(X^{2i-1} + 1) \notin \mathcal{C},$$

by Th. 2.2 (ii), because $P(-1) = 0$. Alternatively, we might consider the periodic[4] τ_P-orbit

$$(1\,(-11)^{n-1}) \mapsto ((-11)^{n-1}\,(-1)) \mapsto (1\,(-11)^{n-1}).$$

(ii) Let $d > r > 3$ and $a_1, \ldots, a_{r-2} \in \mathbb{N}_{>0}$. Then the r-sparse polynomial with dominant constant term

$$P := X^d + \left(\sum_{i=1}^{r-2} \bigl(a_i(X^{d-i} + 1)\bigr)\right) + 1$$

is not a CNS polynomial. Indeed, we see

$$1^{r-1}0^{d-(r-1)} \mapsto 1^{r-2}0^{d-(r-1)}(-1) \mapsto \cdots \mapsto 0^{d-(r-1)}(-1)0^{r-2} \mapsto \cdots$$
$$\cdots \mapsto 0^{r-2}(-1)0^{d-(r-1)} \mapsto 0^{r-3}(-1)0^{d-(r-1)}1 \mapsto \cdots$$
$$\cdots \mapsto 0^{d-(r-1)}1_{r-1} \mapsto \cdots \mapsto 1^{r-1}0^{d-(r-1)},$$

i.e., there exists a non-zero periodic orbit under τ_P.

For $P \in \mathcal{S}_r$ we need the set

$$\mathcal{F}_P := \bigl\{(\varepsilon^{n_r - n_{r-1} - 1}\, w\, \varepsilon^{n_2 - 1}\,(-1)) \in \mathbb{Z}^{n_r} \ :\ w \in \Pi_P\bigr\}$$

with

$$\Pi_P := \bigl\{\operatorname{sgn}(p_{n_{r-1}})\,\varepsilon^{n_{r-1} - n_{r-2} - 1}\,\operatorname{sgn}(p_{n_{r-2}})\,\varepsilon^{n_{r-2} - n_{r-3} - 1}\,\cdots$$
$$\cdots\varepsilon^{n_3 - n_2 - 1}\,\operatorname{sgn}(p_{n_2})\bigr\}.$$

The following criterion will play a central role in our subsequent considerations.

Lemma 3.8. *Let $P = \sum_{i=1}^{r} p_{n_i} X^{n_i} \in \mathcal{S}_r$ such that $p_0 = \sum_{i=2}^{r} |p_{n_i}|$. Then P is a CNS polynomial if and only if*

$$\mathcal{E}_P \cup \mathcal{F}_P \subset \mathcal{N}_P\,. \tag{4}$$

Proof. If P is a CNS polynomial then (4) clearly holds. Therefore, in view of Th. 3.4 it suffices to show that (4) implies $E \subset \mathcal{N}_P$. Let $d := n_r$ and assume that there exists

$$e \in E \setminus \mathcal{N}_P\,. \tag{5}$$

[4] For the definition the reader is referred to [1].

Then we have

$$(\tau_P^k(e))_d \geq 0 \qquad (k > 0), \tag{6}$$

because otherwise $(\tau_P^k(e))_d = -1$ for some positive k. We easily check

$$\tau_P^{k-1}(e) = (\, 1\, \varepsilon^{n_r - n_{r-1} - 1}\, w\, \varepsilon^{n_2 - 1}\,)$$

with some $w \in \Pi_P$ which implies

$$\tau_P^k(e) = (\, \varepsilon^{n_r - n_{r-1} - 1}\, w\, \varepsilon^{n_2 - 1}\, (-1)) \in \mathcal{F}_P$$

and then the contradiction $e \in \mathcal{N}_P$.

From (6) and Lemma 3.1 we deduce that $\tau_P^k(e) \in E_0$ for all sufficiently large k, and then (5) tells us that there is some $m \in \mathbb{N}$ such that $f := \tau_P^m(e)$ satisfies $f \in E_0$ and $f_d = 1$. Thus $f \in \mathcal{E}_P \subset \mathcal{N}_P$ which implies $e \in \mathcal{N}_P$ contradicting our assumption. $\Diamond$

We point out an immediate consequence of this result.

Corollary 3.9. *Let $P = \sum_{i=0}^d p_i X^i \in \mathbb{Z}[X]$ be a monic polynomial of degree $d \geq 2$ with $p_0 = \sum_{i=2}^r |p_{n_i}|$ and $p_i \neq 0$ for all $i = 1, \ldots, d-1$. If $\mathcal{E}_P \subset \mathcal{N}_P$ and*

$$(\mathrm{sgn}(p_{d-1}),\, \mathrm{sgn}(p_{d-2}),\, \cdots,\, \mathrm{sgn}(p_1),\, -1) \in \mathcal{N}_P$$

then P is a CNS polynomial.

Proof. Note that $\mathcal{F}_P = \{(\mathrm{sgn}(p_{d-1})\, \mathrm{sgn}(p_{d-2})\, \cdots\, \mathrm{sgn}(p_1)\, (-1))\}$. $\Diamond$

In favorable cases the verification of the prerequisites of Lemma 3.8 can slightly be simplified.

Lemma 3.10. *Let $P = \sum_{i=0}^d p_i X^i \in \mathbb{Z}[X]$ be monic with $d \geq 2$ and $p_0 = \sum_{i=1}^d |p_i|$. Then we have*

$$(\tau_P(\mathrm{sgn}(p_{d-1}), \ldots, \mathrm{sgn}(p_1), -1))_d \geq 0. \tag{7}$$

Proof. Let us assume the contrary. In view of Lemma 3.1 we then have

$$p_0 = \sum_{i=2}^d s_{i-1} p_i - p_1,$$

where we set $s_i := \mathrm{sgn}(p_i)$ $(i = 1, \ldots, d)$. Therefore, we have

$$|p_1| + p_1 + \sum_{i=2}^d (|p_i| - s_{i-1} p_i) = 0,$$

which implies

$$s_1 \in \{0, -1\} \text{ and } s_i p_i = s_{i-1} p_i \quad (i = 2, \ldots, d)$$

and then $s_2, \ldots, s_d \in \{0, -1\}$ contradicting $s_d = 1$. $\Diamond$

Corollary 3.11. *Let $P = \sum_{i=0}^{d} p_i X^i \in \mathbb{Z}[X]$ be monic with $d \geq 3$, $p_1, ..., p_{d-1} \neq 0$ and $p_0 = \sum_{i=1}^{d} |p_i|$. If $(\mathrm{sgn}(p_{d-2}), ..., \mathrm{sgn}(p_1), -1, \delta) \in \mathcal{N}_P$ for $\delta \in \{0, 1\}$ then we have $\mathcal{F}_P \subset \mathcal{N}_P$.*

Proof. By definition we have $\mathcal{F}_P = \{(\mathrm{sgn}(p_{d-1}) \cdots \mathrm{sgn}(p_1)\,(-1))\}$ and Lemma 3.10 and our prerequisites yield

$$\tau_P(\mathrm{sgn}(p_{d-1}), \ldots, \mathrm{sgn}(p_1), -1) \in \mathcal{N}_P\,. \quad \Diamond$$

Remark 3.12. Note that in general we cannot abdicate the assumption $\mathcal{E}_P \subset \mathcal{N}_P$ in Lemma 3.8 above. For instance, the polynomial $P = X^4 + 2X^3 - 2X^2 - X + 6 \in \mathcal{S}_5$ fulfills $\mathcal{F}_P \subset \mathcal{N}_P$, but it is not a CNS polynomial, since $(1, 0, 1, 0) \in \mathcal{E}_P$ is a periodic element under τ_P.

The following lemma can be useful in verifying the prerequisites of Lemma 3.8.

Lemma 3.13. *Let $P \in \mathcal{S}_r$ with dominant constant term such that $p_{n_i} > 0$ $(i = 2, \ldots, r-1)$ and $d = \deg(P)$. Then the following statements hold.*

(i) *If $e \in E_0$ and $\|e\|_1 \leq r - 2$ then $e \in \mathcal{N}_P$.*

(ii) $(0, \delta^{d-1}) \mapsto (\delta^{d-1}, 0)$.

(iii) *If $n_2 = 1$ then $(\delta^{d-1}0)$, $(0, \delta^{d-1}) \in \mathcal{N}_P$.*

(iv) *If $(\delta^{d-1}, 0) \in \mathcal{N}_P$ then $(0, \delta^{d-1})$, $(-1, \delta^{d-1}) \in \mathcal{N}_P$.*

Proof. (i) We observe

$$(\tau_P(e))_d = 0,\ \tau_P(e) \in E_0,\ \|\tau_P(e)\|_1 \leq \|e\|_1$$

and use induction on the 1-norm of e.

(ii), (iii), (iv) This can immediately be checked by the definitions. $\Diamond$

Now we are in a position to extend a result by Akiyama and Rao [5, Th. 3.2].

Theorem 3.14. *Let $P = \sum_{i=0}^{d} p_i X^i \in \mathbb{Z}[X]$ be a monic polynomial of degree $d \geq 2$ with positive dominant constant term. Then P is a CNS polynomial if one of the two following conditions is satisfied:*

(i) $p_1, \ldots, p_{d-1} > 0$,

(ii) *$p_i \geq 0$ for $i \in \{2, \ldots, d-1\}$ and*

$$-\sum_{i=2}^{d} p_i \leq p_1 \leq -1.$$

Proof. (i) If P satisfies the AP-condition then P is expansive by [4, Lemma 1] and the assertion is clear by [5, Th. 3.2]. Therefore, we assume $2P(0) = P(1)$. In view of Lemma 3.8 it suffices to show that

$$\mathcal{E}_P \cup \mathcal{F}_P \subset \mathcal{N}_P\,.$$

Note that $\mathcal{E}_P = \emptyset$ and $\mathcal{F}_P = \{(1\,\cdots\,1\,(-1))\}$. Obviously, we have

$$(1,\ldots,1,-1) \mapsto (1,\ldots,1,-1,\delta) \mapsto \cdots$$
$$\cdots \mapsto (-1,\delta^{d-1}) \mapsto \{(0^{d-1},1),\,(\delta^{d-1},0)\} \subset \mathcal{N}_P\,.$$

(ii) Clear by Th. 3.5. $\Diamond$

Remark 3.15. (i) Observe that in Th. 3.14 (i) the positivity condition on the coefficients cannot be relaxed. Indeed, the expansive polynomial $X^3 + X^2 + 2$ is not a CNS polynomial (e.g., see [10, Th. 3]).

(ii) Th. 3.14 (ii) is no longer valid if the linear term is replaced by a different negative coefficient. For instance, by Th. 3.5 we have $X^4 + aX^3 - 2X^2 + bX + c \notin \mathcal{C}$ for all $c \geq a + b + 3$.

We close this section by a characterization of certain CNS quadrinomials.

Theorem 3.16. *Let $P := X^d + aX^m + bX^n + c \in \mathcal{S}_4$ have a positive dominant constant term.*

(i) *Let $a < 0$. Then $P \in \mathcal{C}$ if and only if $a = -1$, $b > 0$ and m divides d.*

(ii) *Let $b < 0$. Then $P \in \mathcal{C}$ if and only if one of the following two conditions is satisfied.*

(a) *$a + b \geq 0$ and n divides m.*

(b) *$b = -1$, $a > 0$ and n divides d.*

Proof. In view of [10, Th. 1] we may assume that P is primitive. According to Jankauskas [17] we say that a polynomial $f(X)$ is primitive if it is not of the form $g(X^k)$ for some $k > 1$.

(i) If $P \in \mathcal{C}$ the we infer

$$b \geq -1 - a \geq 0$$

from Th. 2.2 and our assertion is clear by Th. 3.5. The same theorem implies the converse.

(ii) Let $P \in \mathcal{C}$. Assuming $a < 0$ Th. 2.2 yields the contradiction

$$-1 \leq a + b \leq -1 + b.$$

Thus we have $a > 0$.

If n divides m then n does not divide d by primitivity, and Th. 3.5 delivers $a + b \geq 0$. If n does not divide m then Th. 3.5 yields $1 + b \geq 0$, thus $b = -1$.

To prove the converse we first assume that (a) holds. Then clearly $a > 0$, and we are done by Th. 3.5. Finally, supposing that (b) holds an application of Th. 3.5 concludes the proof. $\Diamond$

Remark 3.17. Let P be as in Th. 3.16. If $ab = 0$ then the CNS property of P is characterized by [10, Theorems 1 and 3]. However, if $a, b > 0$ then Ex. 3.7 and the calculations in the last section suggest that the characterization of $P \in \mathcal{C}$ will be more difficult.

4. CNS polynomials of small degrees with dominant conditions

In this section we characterize CNS polynomials of small degrees with dominant constant term. Here our main tools are results by Akiyama and Rao [5] and Lemma 3.8. Several straightforward but tedious verifications are left to the reader or postponed to the final section.

Theorem 4.1. *Let $a, b, c \in \mathbb{Z}$ such that $c \geq 1+|a|+|b|$. Then X^3+aX^2+ $+bX+c$ is a CNS polynomial if and only if the following four conditions are satisfied.*

(i) $1 + a + b \geq 0,$
(ii) $a \geq 0,$
(iii) $a = 0 \implies b \leq c - 2,$
(iv) $b = 0 \implies a \leq c - 2.$

Proof. First assume that the polynomial P defined above is a CNS polynomial. The first property is clear by Th. 2.2 and the other three properties follow from Gilbert's conditions (see [3, Th. 3.1]).

Now we prove the converse and assume that all four conditions hold. If $c > 1 + |a| + |b|$ then we are done by [5, Th. 5.3]. Therefore, let $c = 1 + |a| + |b|$. Then we have $b \neq 0$ by (i) and (iv), and we distinguish two cases.

Case 1. $b \geq 1$.

Then we clearly have $a \geq 1$. If $a \leq b$ then we are done by [3, Th. 3.9]. Now, let $a > b$, hence $\mathcal{E} = \emptyset$, $\mathcal{F} = \{(1, 1, -1)\}$ and thus our assertion drops out from Lemma 3.8, since we have

$$(1,1,-1) \mapsto (1,-1,0) \mapsto (-1,0,1) \mapsto (0,1,1) \mapsto$$
$$\mapsto (1,1,0) \mapsto (1,0,0) \mapsto (0,0,0).$$

Case 2. $b \leq -1$.

If $a = 0$ then $b = -1$ then we are done by [3, Prop. 3.2] or [10, Th. 3], and if $a > 0$ then $b \geq -c + a + 1$ and we apply [3, Prop. 3.4]. $\Diamond$

Theorem 4.2. *Let $P = X^4 + aX^3 + bX^2 + cX + d$ have a positive dominant constant term. Then P is a CNS polynomial if and only if the following seven conditions are satisfied.*

(i) $1 + a + b + c \geq 0$,
(ii) $1 + a + b \geq 0$,
(iii) $a \geq -1$,
(iv) $b \geq -1$,
(v) $a = -1 \implies c \leq -2$,
(vi) $b = c = 0 \implies 0 \leq a \leq d - 2$,
(vii) $a = b = 0 \implies c \leq d - 2$.

Proof. If P defined above is a CNS polynomial then similarly as in the proof of [5, Th. 5.4] the above conditions immediately follow using Th. 2.2, Th. 3.2 and [10, Th. 3].

Now we assume that all seven conditions hold. If P satisfies the AP-condition then we are done by [5, Th. 5.4]. Therefore we suppose

$$d = 1 + |a| + |b| + |c|.$$

We note that $d \geq 2$ and prove $P \in \mathcal{C}$ using Lemma 3.8 and subdividing our considerations into several cases and subcases.

Case 1. $a < 0$.

Then $a = -1$, $c \leq -2$ and $b \geq 1$. We show that $\mathcal{E} \cup \{(-1, 1, -1, -1)\} \subset \subset \mathcal{N}$ by verifying the following statements:

(i) $(1, 1, 0, 0) \mapsto (1, 0, 0, 0) \in \mathcal{N}$.
(ii) $(0, 1, 1, 0) \mapsto (1, 1, 0, 0) \in \mathcal{N}$ by (i).
(iii) $(1, 1, 1, 0) \mapsto (1, 1, 0, 0) \in \mathcal{N}$ by (i).
(iv) $(1, 1, 1, 1) \mapsto (1, 1, 1, 0) \in \mathcal{N}$ by (iii).
(v) $(0, 1, 1, 1) \mapsto \{(1, 1, 1, 0), (1, 1, 1, 1)\} \subset \mathcal{N}$ by (iii) and (iv).
(vi) $(0, 1, 0, 1) \mapsto (1, 0, 1, 1) \mapsto \{(0, 1, 1, 0), (0, 1, 1, 1)\} \subset \mathcal{N}$ by (ii) and (v).
(vii) $(0, 0, 1, 1) \mapsto \{(0, 1, 1, 0), (0, 1, 1, 1)\} \subset \mathcal{N}$ by (ii) and (v).
(viii) $(1, 0, 0, 1) \mapsto (0, 0, 1, 1) \in \mathcal{N}$ by (vii).
(ix) $(0, 0, 0, 1) \mapsto (0, 0, 1, 1) \in \mathcal{N}$ again by (vii).

(x) $(1,1,0,1) \mapsto (1,0,1,1) \in \mathcal{N}$ by (vi). Observe that we now have shown $\mathcal{E} \subset \mathcal{N}$.

(xi) $(1,-1,-1,0) \mapsto (-1,-1,0,1) \mapsto (-1,0,1,1) \mapsto$
$\mapsto \{(0,1,1,0),(0,1,1,1)\} \subset \mathcal{N}$ by (ii) and (v).

(xii) $(1,-1,-1,1) \mapsto (-1,-1,1,1) \mapsto (-1,1,1,0) \mapsto$
$\mapsto \{(1,1,0,0),(1,1,0,1)\} \subset \mathcal{N}$ by (i) and (x).

(xiii) $(-1,1,-1,-1) \mapsto \{(1,-1,-1,0),(1,-1,-1,1)\} \subset \mathcal{N}$ by (xi) and (xii).

Case 2. $a = 0$.

Then we have $b \geq -1$ and $c \neq 0$.

Case 2.1. $b < 0$.

Then we have $b = -1$ and $c > 0$ and Th. 3.5 yields our assertion.

Case 2.2. $b = 0$.

Then we are done by [10, Th. 3].

Case 2.3. $b > 0$.

For $c < 0$ our claim is clear by Th. 3.5 and for $c > 0$ by Lemma 5.1 below.

Case 3. $a > 0$.

Case 3.1. $b < 0$.

Then we have $b = -1$ and $a + c \geq 0$.

Case 3.1.1. $c < 0$.

We verify $\mathcal{E} \subset \mathcal{N}$ and then

$$(-1,-1,-1,0) \mapsto (-1,-1,0,1) \mapsto (-1,0,1,1) \mapsto (0,1,1,1) \in \mathcal{N}$$

and

$$\begin{aligned}(-1,-1,-1,1) \mapsto (-1,-1,1,1) \mapsto (-1,1,1,1) \mapsto \\ \mapsto \{(1,1,1,0),(1,1,1,1)\} \subset \mathcal{N}\end{aligned}$$

which imply $\mathcal{F} = \{(1,-1,-1,-1)\} \subset \mathcal{N}$.

Case 3.1.2. $c \geq 0$.

Now Th. 3.5 yields our assertion.

Case 3.2. $b = 0$.

Case 3.2.1. $c < 0$.

Then we have $a + c \geq -1$ and our claim follows from Th. 3.5.

Case 3.2.2. $c = 0$.

We are done by [10, Th. 3].

Case 3.2.3. $c > 0$.

We apply Lemma 5.1.

Case 3.3. $b > 0$.

Our claim follows from Th. 3.5 (for $c < 0$), Lemma 5.2 (for $c = 0$) and Th. 3.14 (for $c > 0$). $\diamondsuit$

Theorem 4.3. *Let $P = X^5 + aX^4 + bX^3 + cX^2 + dX + e \in \mathbb{Z}[X]$ have a positive dominant constant term. Then P is a CNS polynomial if and only if the following eleven conditions are satisfied.*

(i) $1 + a + b + c + d \geq 0$,
(ii) $a + c \geq 0$,
(iii) $a < 0 \implies a = -1,\ b \geq 1,$ *and* $d \leq -2$,
(iv) $b < 0$ *and* $a + d \geq 0 \implies b = -1$ *and* $c \leq -2$,
(v) $b < 0$ *and* $a + d < 0 \implies a \geq 0$ *and* $a + b \geq 0$,
(vi) $a = b = c = 0 \implies -1 \leq d \leq e - 2$,
(vii) $a = b = d = 0 \implies 0 \leq c \leq e - 2$,
(viii) $a = c = d = 0 \implies 0 \leq b \leq e - 2$,
(ix) $b = c = d = 0 \implies 0 \leq a \leq e - 2$,
(x) $a = c = 0$ *and* $b > 0 \implies d = 0$ *or* $e > 1 + b + |d|$,
(xi) $a > 0,\ b > 0,$ *and* $c = d = 0 \implies e > 1 + a + b$.

Proof. Let P be a CNS polynomial. Proceeding analogously as in the proof of [5, Th. 5.6] the above conditions are immediately verified using Th. 2.2, Th. 3.2, [10, Th. 3] and Ex. 3.7.

Now we assume that all indicated conditions hold. If P satisfies the AP-condition then we are done by [5, Th. 5.6]. Thus we suppose

$$e = 1 + |a| + |b| + |c| + |d|.$$

The proof is completed analogously as in Th. 4.2; therefore, we leave most of the rather involved but straightforward verifications to the reader. Note that $e \geq 2$ and that the first five conditions imply

$$a + b \geq -1.$$

Case 1. $a < 0$.

Then $a = -1$, $b \geq 1$, $d \leq -2$ and $c \geq 1$. We show $\mathcal{E} \subset \mathcal{N}$ by successively checking the following elements:

$$(0,1,1,0,0),(\delta,1,1,1,0),(0,0,1,1,0),(1,1,1,1,1),$$
$$(0,1,1,1,1),(0,0,1,1,1),(\delta,0,0,1,1),$$
$$(\delta,0,0,0,1),(0,1,0,0,0),(1,0,0,1,0),(\delta,1,0,0,1),(1,0,1,1,\delta).$$

Then we convince ourselves that $(1,1,1,-1,\delta) \in \mathcal{N}$ and infer $\mathcal{F} \subset \mathcal{N}$ from Cor. 3.11. An application of Lemma 3.8 concludes the proof of Case 1.

Case 2. $a = 0$.

Then we have $c \geq 0$ and $b \geq 0$.

Case 2.1. $b = 0$.

Case 2.1.1. $c = 0$.

Then we are done by [10, Th. 3].

Case 2.1.2. $c > 0$.

We apply Th. 3.5 (for $d < 0$), [10, Th. 3] (for $d = 0$) and Lemma 5.1 (for $d > 0$).

Case 2.2. $b > 0$.

If $c = 0$ then we are done by [10, Th. 3], and for $c > 0$ we apply Th. 3.5 (for $d < 0$), Lemma 5.2 (for $d = 0$) and Lemmas 3.8 and 3.13 (for $d > 0$).

Case 3. $a > 0$.

Case 3.1. $b < 0$.

Case 3.1.1. $c < 0$.

Case 3.1.1.1. $d < 0$.

First, we observe that $a + d < 0$ implies $a + b > 0$ and that $a + d \geq 0$ implies $b = -1$ and $c \leq -2$. In both subcases tedious, but straightforward applications of Lemma 3.8 prove our assertion.

Case 3.1.1.2. $d \geq 0$.

Then we have $b = -1$, $c \leq -2$ and $a + d \geq 2$. Again we check that the prerequisites of Lemma 3.8 are satisfied.

Case 3.1.2. $c = 0$.

Then we have $a+d < 0$, hence $a+d \leq -2$ and consequently $d \leq -3$. The proof of this subcase is accomplished by Lemma 3.8.

Case 3.1.3. $c > 0$.

Then we have $a+d<0$, $a+b\geq 0$ and $d\leq -2$, and we again infer our assertion from Lemma 3.8.

Case 3.2. $b=0$.

Case 3.2.1. $c<0$.

Then we have $a+d\geq 0$. Our proof is accomplished by Lemma 3.8 (for $d<0$) and by Th. 3.5 (for $d\geq 0$).

Case 3.2.2. $c=0$.

Then we have $a+d\geq -1$, and we apply Th. 3.5 (for $d<0$), [10, Th. 3] (for $d=0$) and Lemma 5.1 (for $d>0$).

Case 3.2.3. $c>0$.

We exploit Th. 3.5 (for $d<0$), Lemma 5.2 (for $d=0$) and Lemmas 3.8 and 3.13 (for $d>0$).

Case 3.3. $b>0$.

Case 3.3.1. $c<0$.

Then we have $d\geq -(a+b)$, and our can be settled by Lemma 3.8 (for $d<0$) and by Th. 3.5 (for $d\geq 0$).

Case 3.3.2. $c=0$.

If $d<0$ then we apply Th. 3.5, and if $d\geq 0$ then we observe $d>0$, and the proof is accomplished by Lemma 3.8.

Case 3.3.3. $c>0$.

We exploit Th. 3.5 (for $d<0$), Lemmas 3.8 and 3.13 (for $d=0$) and Th. 3.14 (for $d>0$). $\Diamond$

5. Proofs of auxiliary results

In this section we study the CNS property of low degree integer quadrinomials

$$P:=X^d+aX^m+bX^n+c \qquad (d>m>n>0)$$

whose coefficients enjoy the following properties:

$$a,b>0 \quad \text{and} \quad c=1+a+b.$$

We keep in mind that the set $\mathcal{E}_P$ is empty, and we subdivide our results according to the least positive degree of a non-vanishing term.

Lemma 5.1. *Let $4\leq d\leq 5$ and $2\leq m\leq d-1$. Then X^d+aX^m+ $+bX+c\in\mathcal{C}$ if and only if $(d,m)\neq(5,3)$.*

Proof. We have
$$\mathcal{F}_P = \{(\varepsilon^{d-m-1}\, w\, (-1)) \in \mathbb{Z}^d \ : \ w \in \Pi_P\}$$
with
$$\Pi_P = \{(1\, \varepsilon^{m-2}\, 1)\}.$$
In view of $\mathcal{E}_P = \emptyset$ and Lemma 3.8 we show that $\mathcal{F}_P \subset \mathcal{N}_P$ exactly in the cases mentioned above. The proof consists of a long sequence of elementary verifications.

Case 1. $d = 4$.

Case 1.1. $m = 2$.

We have $\Pi_P = \{(1\ 1)\}$ and $\mathcal{F}_P = \{(\varepsilon\, 1\, 1\ -1)\} \subset \mathcal{N}_P$ by successive verification of the subsequent statements:

(i) $(-1,1,1,0) \mapsto (1,1,0,0)$, hence using Lemma 3.13 we find $(-1,1,1,0) \in \mathcal{N}$.

(ii) $(1,-1,0,\delta) \mapsto (-1,0,\delta,0) \mapsto \{(0,0,0,1),(0,1,0,0)\} \subset \mathcal{N}$ again by Lemma 3.13.

(iii) $(1,1,-1,0) \mapsto (1,-1,0,\delta) \in \mathcal{N}$ by (ii).

(iv) If $b-a \geq 0$ then $(1,1,-1,1) \in \mathcal{N}$, because $(1,1,-1,1) \mapsto \mapsto (1,-1,1,0) \mapsto\mapsto (-1,1,0,0) \mapsto (1,0,0,1) \in \mathcal{N}$ by Lemma 3.13.

(v) $(-1,1,1,-1) \in \mathcal{N}$ by (iii) and (iv), since
$$(1,-1,1,-1) \to \begin{cases} (1,1,-1,0) & (a-b \geq -1), \\ (1,1,-1,1) & (a-b < -1). \end{cases}$$

(vi) $(1,-1,1,1) \mapsto (-1,1,1,-1) \in \mathcal{N}$ by (v).

(vii) $(1,1,-1,1) \mapsto (1,-1,1,\delta) \in \mathcal{N}$ by (i) and (vi).

(viii) $(\varepsilon,1,1,-1) \mapsto (1,1,-1,\delta) \in \mathcal{N}$ by (iii) and (vii).

Case 1.2. $m = 3$.

We have $\Pi_P = \{(1\,\varepsilon\, 1)\}$ and $\mathcal{F}_P = \{(1\,\varepsilon\, 1\ -1)\} \subset \mathcal{N}_P$ by arguing similarly as above:

(i) $(1,-1,1,0) \mapsto (-1,1,0,\delta \mapsto (1,0,\delta,0) \in \mathcal{N}$ by Lemma 3.13.

(ii) $(1,-1,1,-1) \mapsto (-1,1,-1,1) \mapsto (1,-1,1,0) \in \mathcal{N}$ by (i).

(iii) $(1,-1,0,0) \mapsto (-1,0,0,\delta) \mapsto \{(0,0,0,1),(0,0,1,0)\} \subset \mathcal{N}$ again by Lemma 3.13.

(iv) $(1,\varepsilon,1,-1) \mapsto (\varepsilon,1,-1,\delta) \mapsto \{(1,-1,0,0),(1,-1,1,-\delta')\} \subset \mathcal{N}$ by (i), (ii) and (iii).

Case 2. $d = 5$.

Case 2.1. $m = 2$.

We have $\Pi_P = \{(1\,1)\}$ and $\mathcal{F}_P = \{(\varepsilon^2\,1\,1\ -1)\} \subset \mathcal{N}_P$ analogously as before:

(i) For $(\delta, \delta') \neq (1, 1)$ we have $(1, 1, -1, \delta, \delta') \mapsto (1, -1, \delta, \delta', 0) \mapsto$ $\mapsto (-1, \delta, \delta', 0, 0) \mapsto (\delta, \delta', 0, 0, 1)$, hence using Lemma 3.13 we find $(1, 1, -1, \delta, \delta') \in \mathcal{N}$.

(ii) $(-1, 1, 1, -1, 0) \mapsto (1, 1, -1, 0, 1) \in \mathcal{N}$ by (i).

(iii) If $a - b \leq -1$ then $(-1, 1, 1, -1, 1) \in \mathcal{N}$, because $(-1, 1, 1, -1, 1) \mapsto$ $\mapsto (1, 1, -1, 1, 0) \in \mathcal{N}$ by (i).

(iv) $(1, -1, 1, 1, -1) \in \mathcal{N}$ by (ii) and (iii), since

$$(1, -1, 1, 1, -1) \to \begin{cases} (-1, 1, 1, -1, 0) & (a - b \geq -1), \\ (-1, 1, 1, -1, 1) & (a - b < -1). \end{cases}$$

(v) $(1, 1, -1, 1, 1) \mapsto (1, -1, 1, 1, -1) \in \mathcal{N}$ by (iv).

(vi) $(\varepsilon', \varepsilon,\ 1\,1\,-1) \mapsto (\varepsilon\,1\,1\,-1, \delta) \mapsto \{(1, 1, -1, 1, 0, \delta'), (1, 1, -1, 1, 1, \delta')\} \subset$ $\subset \mathcal{N}$ by (i) and (v).

Case 2.2. $m = 3$.

Then $P \notin \mathcal{C}$ by Ex. 3.7.

Case 2.3. $m = 4$.

We have $\Pi_P = \{(1\,\varepsilon^2\,1)\}$ and $\mathcal{F}_P = \{(1\,\varepsilon^2\,1\ -1)\} \subset \mathcal{N}_P$ by successive verification of the subsequent statements:

(i) $(1, -1, \delta^3) \mapsto (-1, \delta^4) \mapsto \{(0^4, 1), (\delta^4, 0)\} \subset \mathcal{N}$ by Lemma 3.13.

(ii) $(1, 1, -1, 1, -1) \mapsto (1, -1, 1, -1, \delta) \mapsto (-1, 1, -1, \delta^2) \mapsto (1, -1, \delta^3) \in \mathcal{N}$ by (i).

(iii) $(\delta, 1, -1, 1, 0) \mapsto (1, -1, \delta^3) \in \mathcal{N}$ by (i).

(iv) $(1, -1, \delta, 1, -1) \mapsto (-1, \delta, 1, -1, 1) \mapsto (\delta, 1, -1, 1, 0) \in \mathcal{N}$ by (iii).

(v) $(\varepsilon, 1, -1, \delta^2) \mapsto \{(1, -1, 1, 1, -1), (1, -1, \delta^3)\} \subset \mathcal{N}$ by (i) and (iv).

(vi) $(1, \varepsilon^2, 1, -1) \mapsto (\varepsilon^2, 1, -1, \delta) \mapsto \{(\varepsilon, 1, -1, \delta^2), (1, 1, -1, 1, -1)\} \subset \mathcal{N}$ by (ii) and (v). $\Diamond$

Lemma 5.2. *If $4 \leq d \leq 5$ and $3 \leq m \leq d - 1$ then $X^d + aX^m + bX^2 +$ $+ c \in \mathcal{C}$.*

Proof. In view of $\mathcal{E}_P = \emptyset$ and Lemma 3.8 we aim at showing $\mathcal{F}_P \subset$ $\subset \mathcal{N}_P$. Similarly as in Lemma 5.1 the proof consists of several elementary verifications. In the following we only indicate the succession of elements to be checked.

Case 1. $d = 4$.

Then we have $m = 3$, $\mathcal{F}_P = \{(1\,1\,\varepsilon\,-1)\}$ and check the following elements:

$$(1,1,0,-1),(\delta^3,0),(1,-\delta',-1,\delta),(1,1,-1,1),(1,1,\varepsilon,-1).$$

Case 2. $d = 5$.

We have $\mathcal{F}_P = \{(\varepsilon^{4-m}\,1\,\varepsilon^{m-3}\,1\,\varepsilon\,-1)\}$.

Case 2.1. $m = 3$.

Then we have $\mathcal{F}_P = \{(\varepsilon'\,1\,1\,\varepsilon\,-1)\}$ and check the following elements:

$$(\delta,1,1,0-1),(\delta^3,0,0),(\delta^4,0),(-1,0,0,\delta,1),(\delta',0,0,\delta,1),$$
$$(-1,\delta^4),(-1,1,1,\delta,-1),(1,1,\varepsilon,-1,\delta),(1,1,1,-1,-1),$$
$$(1,1,1,1,-1),(0,1,1,\varepsilon,-1),(-1,1,1,\varepsilon,-1).$$

Case 2.2. $m = 4$.

First we verify that the following elements belong to $\mathcal{N}_P$:

$$(1,0,-1,0,\delta),(1,0,1,0,-1),(1,1,1,0,-1),(\delta^3,0,\delta),$$
$$(\delta^4,0),(0,\delta',0,\delta,1),$$
$$(1,0,-1,\delta_2),(-1,\delta^4),(1,0,1,\delta,1),(1,1,1,-1,-1),$$
$$(1,-1,1,0,-1),(-1,1,1,-1,0),(\delta'',-1,\delta',\delta).$$

Now, if $a - b > 1$ then we easily check that $(-1,1,1,-1,1) \in \mathcal{N}_P$. Then we successively show that the following elements also belong to $\mathcal{N}_P$: $(1,-1,1,-1),(1,-1,1,\delta,-1),(\varepsilon,-1,\delta^3),(\varepsilon',1,\varepsilon,-1,\delta),(1,\varepsilon',1,\varepsilon,-1)$. $\Diamond$

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