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ON THE MULTIPLICATIVE GROUP GENERATED BY $\left\{\frac{[\sqrt{2}n]}{n} \mid n \in \mathbb{N}\right\}$. IV

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Dedicated to the memory of our friend, Dr. János Fehér

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Abstract: Assume that φ, ψ are completely additive functions mapping into G, where G is an Abelian topological group with the translation invariant metric ρ. Let $C \in G$. Assume that

$$\rho\left(\psi\left[\sqrt{2}n\right], \varphi(n) + C\right) \leq \varepsilon(n) \quad \text{and} \quad \sum_{n=2}^{\infty} \frac{\varepsilon(n) \log\log 2n}{n} < \infty,$$

where $\varepsilon(n) \downarrow 0$. Then $\varphi(n) = \psi(n), 2C = \varphi(2)$, and $\varphi(n)$ can be extended to $\mathbb{R}^+$ to be a continuous homomorphism into G.

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1. Introduction

1.1. Let G be an additively written Abelian topological group with the translation invariant metric ρ. A mapping $\varphi : \mathbb{N} \to G$ is called a completely additive function, if

$$\varphi(nm) = \varphi(n) + \varphi(m) \quad (n, n \in \mathbb{N}). \tag{1.1}$$

Let $\mathbb{Q}^+$, resp. $\mathbb{R}^+$ be the multiplicative groups of the positive rationals and the positive reals. We can extend the domain of φ to $\mathbb{Q}^+$ by the relation

$$\varphi\Big(\frac{n}{m}\Big) := \varphi(n) - \varphi(m),$$

uniquely. Then φ satisfies the relation

$$\varphi(rs) = \varphi(r) + \varphi(s) \quad (r, s \in \mathbb{Q}^+),$$

so $\varphi : \mathbb{Q}^+ \to G$ is a homomorphism. We shall say that φ is continuous at the point 1, if $r_\nu \in \mathbb{Q}^+$, $r_\nu \to 1$ implies that $\varphi(r_\nu) \to 0$.

Z. Daróczy and the first named author in [1] proved the next

Lemma 1. *Let G be an additively written closed Abelian topological group, $\varphi : \mathbb{Q}^+ \to G$ be a homomorphism that is continuous at the point* 1*. Then its domain can be extended onto $\mathbb{R}^+$ by the relation*

$$\varphi(\alpha) := \lim_{\substack{r_\nu \to \alpha \\ r_\nu \in \mathbb{Q}^+}} \varphi(r_\nu) \quad (\alpha \in \mathbb{R}^+)$$

uniquely. The so obtained mapping $\varphi : \mathbb{R}^+ \to G$ is a continuous homomorphism, consequently

$$\varphi(\alpha\beta) = \varphi(\alpha) + \varphi(\beta) \quad (\alpha, \beta \in \mathbb{R}^+). \tag{1.2}$$

In [2] the following theorem has been proved which is cited now as

Lemma 2. *Let $\varphi : \mathbb{N} \to G$ be completely additive such that*

$$\sum_{n=1}^{\infty} \frac{\rho(\varphi(n), \varphi(n+1))}{n} < \infty. \tag{1.3}$$

Then φ is a continuous $\mathbb{R}^+ \to G$ homomorphism.

1.2. We use the standard notation: $[x]$ = integer part of x,$\{x\}$ = = fractional part of x, $\| x \| = \min(\{x\}, 1 - \{x\})$.

In our paper [3] we formulated the following conjecture:

Conjecture 1. *If* α, β *are distinct real numbers at least one of which is irrational, then the multiplicative group* $\mathcal{F}_{\alpha,\beta}$ *generated by*

$$\left\{\xi_n = \frac{[\alpha n]}{[\beta n]} \;\middle|\; n \in \mathbb{N}\right\}$$

equals to $\mathbb{Q}^+$.

This conjecture is open, except the case $\alpha = \sqrt{2}$, $\beta = 1$ (proved in [3]).

Let $\mathcal{A}^*, \mathcal{M}^*$ be the sets of real valued completely additive, respectively complex valued completely multiplicative functions. In [5] we proved

Theorem A. *Let* $\varepsilon(n) \downarrow 0$ *arbitrarily. Assume that*

$$\sum_{n=1}^{\infty} \frac{\varepsilon(n) \log\log 2n}{n} < \infty. \tag{1.4}$$

Let $f, g \in \mathcal{M}^*$, $C \in \mathbb{C}$, $|f(n)| = |g(n)| = 1$ $(n \in \mathbb{N})$ *and*

$$\left|g\big([\sqrt{2}n]\big) - Cf(n)\right| \le \varepsilon(n) \quad (n \in \mathbb{N}).$$

Then $f(n) = g(n) = n^{i\tau}$ $(\tau \in \mathbb{R})$, *where* $C = \big(\sqrt{2}\big)^{i\tau}$.

Hopefully this assertion remains true if (1.4) holds without the $\log\log 2n$ for every $n \in \mathbb{N}$.

The above conjecture is equivalent to the next one.

Conjecture 2. *If* α, β *are distinct real numbers at least one of which is irrational, then*

$$f \in \mathcal{A}^*, \quad f([\alpha n]) - f([\beta n]) \equiv 0 \pmod 1 \quad (n \in \mathbb{N})$$

implies that $f(n) \equiv 0 \pmod 1$ $(n \in \mathbb{N})$.

Our aim is to prove

Theorem. *Let* φ, ψ *be completely additive functions mapping into* G, *where* G *is an Abelian topological group with the translation invariant metric* ρ. *Let* $\varepsilon(n)$ *be as in Th. A. Assume that*

$$\rho\Big(\psi\big([\sqrt{2}n]\big), \varphi(n) + C\Big) \le \varepsilon(n). \tag{1.5}$$

Then $\varphi(n) = \psi(n)$ $(n \in \mathbb{N})$ *and* $\varphi(2) = 2C$, *furthermore* φ *is a continuous homomorphism,* $\varphi : \mathbb{R}^+ \to G$.

First we prove

Lemma 3. *Let $\varphi : \mathbb{N} \to G$, $\psi : \mathbb{N} \to G$ be completely additive functions such that*

$$\sum_{n=1}^{\infty} \frac{\rho\Big(\psi\big([\sqrt{2}n]\big), \varphi(n) + C\Big)}{n} < \infty. \tag{1.6}$$

Then $\psi(n) = \varphi(n)$ $(n \in \mathbb{N})$, furthermore

$$\varphi(2) = 2C. \tag{1.7}$$

2. Auxiliary results

Let

$$J_1 = \Big\{ n \mid \big\{\sqrt{2}n\big\} < \frac{1}{\sqrt{2}} \Big\} \quad \text{and} \quad J_2 = \Big\{ n \mid \big\{\sqrt{2}n\big\} > \frac{1}{\sqrt{2}} \Big\}. \tag{2.1}$$

Then $\mathbb{N} = J_1 \cup J_2$.

Let

$$a_n := \frac{\Big[\sqrt{2}[\sqrt{2}n]\Big]}{n}. \tag{2.2}$$

We proved in [3] (Lemma 2) that

$$a_n = \begin{cases} \frac{2n-1}{n} & \text{if} \quad n \in J_1, \\ \frac{2(n-1)}{n} & \text{if} \quad n \in J_2. \end{cases} \tag{2.3}$$

For some $N \in \mathbb{N}$, let

$$\mathcal{B}_0 = N \quad \text{and} \quad \mathcal{B}_j = 2\mathcal{B}_{j-1} - 1 \quad \text{for all} \quad j \in \mathbb{N}.$$

In [3] (Lemma 3) we proved that either $N \in J_2$, or there is a positive k, for which $\mathcal{B}_k \in J_2$. Let $T(N) := k + 1$, where k is the smallest positive integer for which $\mathcal{B}_k \in J_2$. We proved that

$$\frac{2^{k+1}(N-1)}{N} = \frac{2(\mathcal{B}_k - 1)}{N} = \Big(\prod_{j=0}^{k-1} a_{\mathcal{B}_j}\Big) a_{\mathcal{B}_k}, \tag{2.4}$$

furthermore that

$$T(N) \le \frac{1}{\log 2} \cdot \log \frac{1}{\big\|\sqrt{2}(N-1)\big\|} + c_1, \tag{2.5}$$

c_1 is an explicit constant.

3. Proof of Lemma 3

Let $k \in \mathbb{N}$, $\mathcal{T}_k = \left\{n \in \mathbb{N} \;\middle|\; \{\sqrt{2}n\} < \frac{1}{k}\right\}$. Then $[\sqrt{2}kn] = k[\sqrt{2}n]$ and so

$$\rho\Big(\psi[\sqrt{2}kn], \varphi(kn) + C\Big) = \rho\Big(\psi(k) + \psi([\sqrt{2}n]), \varphi(n) + \varphi(k) + C\Big) =$$
$$= \rho\Big(\psi([\sqrt{2}n]), \varphi(n) + \varphi(k) - \psi(k) + C\Big),$$

$$\sum_{n\in\mathcal{T}_k} \frac{\rho\Big(\psi([\sqrt{2}n]), \varphi(n) + \varphi(k) - \psi(k) + C\Big)}{n} < \infty.$$

Hence, and from (1.6) we obtain that

$$\sum_{n\in\mathcal{T}_k} \frac{\rho\Big(0, \varphi(k) - \psi(k)\Big)}{n} < \infty,$$

which by

$$\sum_{n\in\mathcal{T}_k} \frac{1}{n} = \infty$$

implies that $\varphi(k) = \psi(k)$.

Now we prove (1.7).

Let $\Theta_1 = \{\sqrt{2}m\}$, $\Theta_2 = \{\sqrt{2}\cdot 2m\}$, $\Theta_3 = \{\sqrt{2}(2m-1)\}$. If

$$\Theta_1 \in \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}} + 1\right)\right) := U,$$

then

$$\Theta_2 = 2\Theta_1 - 1 < \frac{1}{\sqrt{2}},\ 0 < \Theta_3 = \Theta_2 - \left(\sqrt{2} - 1\right) < \frac{1}{\sqrt{2}},$$

and so $m \in J_2, 2m-1 \in J_1, 2m \in J_1$. Such integers m for which $\Theta_1 \in U$ can be found, since $\{\sqrt{2}m\}$ is dense in $[0,1)$. Then there exist positive integers k_1, k_2 such that

$$\begin{cases} 2^\ell(2m-2)+1 \in \begin{cases} J_1 & \text{for } \ell < k_1, \\ J_2 & \text{for } \ell = k_1, \end{cases} \\ \\ 2^\ell(2m-1)+1 \in \begin{cases} J_1 & \text{for } \ell < k_2, \\ J_2 & \text{for } \ell = k_2. \end{cases} \end{cases}$$

Then there exists a suitable $\delta > 0$ such that if $|\{\sqrt{2}N\} - \Theta_1| < \delta$, then

$$\begin{cases} a_N = \frac{2(N-1)}{N}, \qquad N \in J_2, \\ 2^\ell(2N-2)+1 \in \begin{cases} J_1 & \text{for } \ell < k_1, \\ J_2 & \text{for } \ell = k_1, \end{cases} \\ 2^\ell(2N-1)+1 \in \begin{cases} J_1 & \text{for } \ell < k_2, \\ J_2 & \text{for } \ell = k_2. \end{cases} \end{cases}$$

Let S_k be the set of the positive integers n , for which k is the smallest nonnegative integer for which $\mathcal{B}_k \in J_2$. Let $d_n = \frac{[\sqrt{2}n]}{n}$ and $a_n = d_n \cdot d_{[\sqrt{2}n]}$. Since ρ is translation invariant,

$$\rho\Big(\psi[\sqrt{2}n], \varphi(n) + C\Big) = \rho\Big(\varphi(d_n) - C, 0\Big),$$

furthermore from the triangle inequality we obtain that

$$\rho\Big(\varphi(a_n) - 2C, 0\Big) \le \rho\Big(\varphi(d_n) - C, 0\Big) + \rho\Big(\varphi(d_{[\sqrt{2}n]}) - C, 0\Big).$$

If (2.4) holds, then

$$\rho\left(\varphi\left(\frac{2^{k+1}(N-1)}{N}\right) - 2(k+1)C, 0\right) \le$$

$$\le \sum_{j=0}^{\infty} \rho\big(\varphi(d_{\mathcal{B}_j}) - C, 0\big) + \sum_{j=0}^{\infty} \rho\left(\varphi(d_{[\sqrt{2}n]}) - C, 0\right).$$

Let $D := \varphi(2) - 2C$.

Let $\mathcal{H}_{k_1,k_2}$ be the set of those N, for which $N \in S_0$, $2N-1 \in S_{k_1}$, $2N \in S_{k_2}$. If $\mathcal{H}_{k_1,k_2}$ is not empty, then

$$\sum_{N \in \mathcal{H}_{k_1,k_2}} \frac{1}{N} = \infty.$$

We have

$$\sum_{N \in S_0} \frac{\rho\Big(\varphi\Big(\frac{N-1}{N}\Big) + D, 0\Big)}{N} < \infty,$$

$$\sum_{2N-1 \in S_{k_1}} \frac{\rho\Big(\varphi\Big(\frac{2N-2}{2N-1}\Big) + (k_1+1)D, 0\Big)}{2N-1} < \infty,$$

$$\sum_{2N\in S_{k_2}} \frac{\rho\Big(\varphi\Big(\frac{2N-1}{2N}\Big)+(k_2+1)D,0\Big)}{2N}<\infty.$$

By using the triangle inequality we obtain that

$$\rho\Big((k_1+k_2+1)D,0\Big)\sum_{N\in\mathcal{H}_{k_1,k_2}}\frac{1}{N}<\infty,$$

therefore

$$(k_1+k_2+1)D=0.$$

One can compute easily that if $m=2$, then $k_1=2, k_2=1$ and if $m=14$, then $k_1=k_2=1$. Consequently $D=0$.

The proof of Lemma 3 is complete. ◊

4. Proof of Theorem

Due to Lemma 3 we may assume that $\psi(n)=\varphi(n)$, and that $\varphi(2)=2C$.

Let $\mathcal{L}_M := [2^M, 2^{M+1})$, $k_M=\log 2M$. Let $N\in\mathcal{L}_M$, $T(N)=k$ $(\le k_M)$. Then

$$\rho\left(\varphi\left(\frac{N-1}{N}\right),0\right)\le\sum_{j=0}^{\infty}\rho\left(\varphi(d_{\mathcal{B}_j}),0\right)+\sum_{j=0}^{\infty}\rho\left(\varphi\big(d_{[\sqrt{2}n]}\big),0\right),$$

and so

$$\rho\left(\varphi\left(\frac{N-1}{N}\right),0\right)\le 2(k+1)\varepsilon(N)\le 3k_M\varepsilon(2^M).$$

Let us estimate the number of those $N\in\mathcal{L}_M$ for which $T(N)\ge k_M$.

From (2.5) we obtain that

$$\big\|\sqrt{2}(N-1)\big\|\le 2^{c_1}2^{-k_M}.$$

Thus, the size of those integers $N\in\mathcal{L}_M$ for which $T(N)\ge k_M$ is less than

$$c_2 2^M 2^{-k_M}+O(M).$$

The $O(M)$ term comes from the estimate of the discrepancy of $\{\sqrt{2}n \mid n\in\mathcal{L}_M\}$, which is bounded by $O(M)$. See Th. 3.4 in [6].

Collecting our results we obtain that

$$\sum_{N\in\mathbb{N},N\ge 2}\frac{\rho\Big(\varphi\Big(\frac{N-1}{N}\Big),0\Big)}{N}<\infty,$$

which, by Lemma 2 implies the theorem. ♢

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