Mathematica Pannonica
25/1 (2014–2015), 93–103

(IR)RELEVANCE OF INTERACTIVITY IN FUZZY ARITHMETIC

Laura **Franzoi**

DMG, University of Trieste, via Valerio 12/1, 34128 Trieste, Italy

Andrea **Sgarro**

DMG, University of Trieste, via Valerio 12/1, 34128 Trieste, Italy

Dedicated to the memory of Professor Gyula I. Maurer (1927–2012)

Received: May 19, 2014

MSC 2000: 94 D 05, 03 B 52

Keywords: Fuzzy arithmetic, fuzzy numbers, possibility theory, non-interactivity.

Abstract: We put forward a criterion covering quite a large class of interactive continuous fuzzy numbers, which establishes when the interactivity between the two fuzzy operands is irrelevant to calculate the fuzzy result of a two-argument operation.

1. Introduction

Below we deal with a problem of fuzzy arithmetic for *interactive* fuzzy numbers; the interactivity between the two fuzzy operands of a two-argument operation will be expressed by means of a 2-dimensional *joint possibility distribution*, similarly to joint *probability* distributions as

E-mail addresses: laura.franzoi@gmail.com, sgarro@units.it

used in probability theory. Unlike *additive* probabilities, possibilities[1] are instead *maxitive*, cf. below Sec. 2 and the Appendix. This approach to fuzzy arithmetic is present in current literature, cf. e.g. [1, 2, 6, 7, 11]; cf. [3] for a precursor. In general, the fuzzy result $Z = X \circ Y$ of a generic operation $\circ$ between two fuzzy arguments X and Y depends on the form of their mutual interactivity (or "dependence", as one would say in probability theory), even when the "marginal distribution" of X and that of Y are constrained to remain fixed (the notions of joint and marginal distributions will be made rigorous in the next section).

The problem tackled in this paper is precisely assessing when knowledge of the interactivity is *irrelevant*, and the result of the operation $Z = X \circ Y$ depends *only* on their marginal distributions. In Sec. 3 we deal with overall, or *total*, irrelevance and put forward a criterion, while in Sec. 4 we cover notable cases of irrelevance with respect to specified families of joint distributions (*partial* irrelevance). The continuous fuzzy numbers introduced in Sec. 2 and dealt with in the criterion of Sec. 3 cover quite a wide class, with generously include *triangular* fuzzy numbers, those most often used in applications.

2. Interactivity and its (ir)relevance

In our approach, to deal with a *fuzzy* number X or a *fuzzy couple* XY what one needs to know[2] is the *possibility distributions* Π_X of X, or the *joint* possibility distribution Π_{XY} of XY. Cf. the Appendix for a quick reminder on possibilities, or cf. [4, 8]. Here we just recall:

$$\Pi_X(x) \in [0,1], \;\; \Pi_{XY}(x,y) \in [0,1]; \;\; \max_{x \in \mathbb{R}} \Pi_X(x) = \max_{x,y \in \mathbb{R}^2} \Pi_{XY}(x,y) = 1;$$

$$\Pi_X(E) \doteq \sup_{x \in E} \Pi_X(x)\,, \;\; E \subseteq \mathbb{R}; \qquad \Pi_{XY}(F) \doteq \sup_{x,y \in F} \Pi_{XY}(x,y)\,, \;\; F \subseteq \mathbb{R}^2.$$

We soon stress that we shall consider *only* distribution functions $\Pi_X(x)$ and $\Pi_{XY}(x,y)$ which are constrained as follows in order to ensure that

[1] Possibilities belong to *multi-valued logics*, but can be seen also as a special case of *upper probabilities*; as a general reference to possibility theory and to fuzzy set theory, used here quite sparsely, cf. e.g. [4, 8].

[2] Note that, to no risk of confusion, we are using the same symbol for distribution functions, $\Pi_X(x)$ or $\Pi_{XY}(x,y)$, whose arguments are (couples of) numbers, and the corresponding possibility distributions, Π_X and Π_{XY}, whose arguments are subsets.

any *sup*remum on a closed subset is actually achieved, and so is also a *max*imum:

a) *the function* $\Pi_X(x)$ *or* $\Pi_{XY}(x, y)$ *is upper semi-continuous*;

b) *for* $\alpha > 0$ *the* α-cuts $\{x \colon \Pi_X(x) \geq \alpha\}$ *or* $\{(x, y) \colon \Pi_{XY}(x, y) \geq \alpha\}$ *are* limited *subsets of* $\mathbb{R}$ *or* $\mathbb{R}^2$, *respectively.*

The definition of fuzzy numbers is not so well agreed upon in the literature, but the class we shall deal with is quite ample:

Definition 1. A *pseudo-triangular* fuzzy number X is described by a possibility distribution function $\Pi_X(x) : \mathbb{R} \to [0, 1]$ such that:

i) the function $\Pi_X(x)$ verifies a) and b) above;

ii) its *support* $\{x : \Pi_X(x) > 0\}$ is connected, possibly unlimited;

iii) the function $\Pi(x)$ is *unimodal*, i.e. the equation $\Pi_X(x) = 1$ admits of exactly *one* solution x_1;

iv) over its connected support, the function $\Pi(x)$ increases *strictly* from 0 to 1, and then decreases *strictly* from 1 to 0: on the support of X the infimum of $\Pi_X(x)$ is 0 both for $x \leq x_1$ and for $x \geq x_1$.

Note that set-convexity of the α-cuts, as often required, is actually implied by Def. 1. As an example, a *triangular fuzzy number* (a, b, c) as used to represent "*vaguely* b" can be described by a function $\Pi(x)$ which increases linearly from 0 to 1 on the interval $[a, b]$ and decreases linearly from 1 to 0 on the interval $[b, c]$, $a < b < c$. Our definition covers also *Gaussian fuzzy numbers* defined by a bell-shaped function, e.g. $\Pi(x) = e^{-x^2/2}$, $x \in \mathbb{R}$, whose support is *un*limited.

Given an operation $X \circ Y$ between two fuzzy numbers X and Y, $\circ : \mathbb{R}^2 \to \mathbb{R}$, the result $Z = X \circ Y$ is itself a fuzzy number[3], and so, to be able to deal with it, we need its possibility distribution Π_Z. Define the *counterimage* of $z \in \mathbb{R}$ as $\mathcal{C}_z \doteq \{(x, y) : \ x \circ y = z\}$. Since the two conditions $\{Z = z\}$ and $\{X = x, Y = y \text{ with } x \circ y = z\}$ are *equivalent* (they imply each other), the corresponding possibilities should be the same, and so we set:

$$\Pi_Z(z) \doteq \Pi_{XY}\{\mathcal{C}_z\} = \max_{x,y \in \mathcal{C}_z} \Pi_{XY}(x, y) \tag{1}$$

[3]Our results can be readily extended to operations whose domain is strictly included in $\mathbb{R}^2$; in this case the support $\{(x, y) : \Pi_{XY}(x, y) \neq 0\}$ is bound to be a subset of that domain.

without resorting explicitly to Zadeh's *extension principle*, as usually done in fuzzy arithmetic. Actually, to be allowed to write *max* rather than *sup*, also recalling *b*), we shall always assume that $\circ$ verifies the convenient property:

all counterimages $\mathcal{C}_z \doteq \{(x,y) : x \circ y = z\}$ *are* closed *subsets of* $\mathbb{R}^2$.

Remarkable operations are the two *projections* over the axes: given Π_{XY}, the *marginal* distribution Π_X, and so the behavior of the marginal fuzzy number X, cf. (1), is obtained by:

$$\Pi_X(x) \doteq \Pi_{XY}(\mathbb{R}_x) = \max_{y\in\mathbb{R}} \Pi_{XY}(x,y).$$

Here $\mathbb{R}_x \subset \mathbb{R}^2$ denotes the vertical line corresponding to abscissa x, which is a *closed* subset of $\mathbb{R}^2$. Analogously one defines the Y-marginal (the projection on the vertical axis, i.e. the Y-axis).

We mention three joint distributions which may "glue" together two given marginals Π_X and Π_Y. Below a wedge $\wedge$ denotes a minimum; we recall that *non-interactivity*, for which we use the notation $\Pi_{X\perp Y}$, is convincingly argued to be the possibilistic analogue of probabilistic independence, cf. e.g. [4, 8] and also the Appendix.

Non-interactivity: $\Pi_{X\perp Y}(x,y) \doteq \Pi_X(x) \wedge \Pi_Y(y)$.

Drastic interactivity: For x_1 and y_1 such that $\Pi_X(x_1) = \Pi_Y(y_1) = 1$ set:

$$\Pi_{XY}(x_1,y) = \Pi_Y(y),\ \Pi_{XY}(x,y_1) = \Pi_X(x),\ \text{else } \Pi_{XY}(x,y) = 0.$$

Deterministic equality: Assuming that X and Y are *equidistributed*, i.e. $\Pi_X(x) = \Pi_Y(x)$, set:

$$\Pi_{XY}(x,x) = \Pi_X(x) = \Pi_Y(x),\ \text{else}\ \ \Pi_{XY}(x,y) = 0.$$

Note that *distinct* fuzzy numbers $X \neq Y$ might have the very *same* possibility distribution function $\Pi_X(x) = \Pi_Y(x)$, but to deal with the *couple* XY one has to specify how the two mutually interact, which might vary from non-interactivity to deterministic equality, and only in the latter case one should feel entitled to write $X = Y$, and so also, say, $X \times Y = X \times X \doteq X^2$; cf. footnote 5.

Throughout, we shall fix two *marginal* distributions Π_X and Π_Y and deal *only* with *admissible* joint distributions Π_{XY}, i.e. with joint distributions which are bound to have these prescribed marginals. Below $\mathcal{F}$ is a family of *distinct* joint distributions Π_{XY} admissible with respect to Π_X and Π_Y. If the closed set $\mathcal{C} \doteq \mathcal{C}_z$ is one of the counterimages, assuming $\Pi_{XY}(\mathcal{C}) > 0$ for at least one $\Pi_{XY} \in \mathcal{F}$, we shall deal with the following problem:

Problem. $\mathcal{F}$-irrelevance on $\mathcal{C}$. *Given $\mathcal{C}$ and given a family $\mathcal{F}$ of* admissible *joint distributions Π_{XY}, find sufficient and/or necessary conditions such that $\Pi_{XY}(\mathcal{C})$ is constant over $\mathcal{F}$.*

3. Results for total irrelevance

In this section we deal with *total* irrelevance, i.e. with the case when $\mathcal{F}$ is made up of *all* the admissible distributions Π_{XY}; then the irrelevance requirement can be re-written as $\Pi_{XY}(\mathcal{C}) = \Pi_{X\perp Y}(\mathcal{C})$ whatever $\Pi_{XY} \in \mathcal{F}$ (use the fact that non-interactive possibilities are maximal, cf. the Appendix). So, to *disprove* total irrelevance, it is enough to exhibit just *one* admissible Π_{XY} such that $\Pi_{XY}(\mathcal{C}) < \Pi_{X\perp Y}(\mathcal{C})$, and the drastic distribution as defined in Sec. 2 often proves to be a convenient tool in such a case, cf. Prop. 2 in Sec. 4.

Given the closed set $\mathcal{C}$, thought of as one of the counterimages, we set:

$$\mu \doteq \Pi_{X\perp Y}(\mathcal{C}) \geq \Pi_{XY}(\mathcal{C}), \quad \Pi_{XY} \in \mathcal{F}.$$

Definition 2. A *μ-chunk* is made up of all the couples $(x, y) \in \mathbb{R}^2$ such that $\Pi_{X\perp Y}(x, y) = \mu$ with fixed abscissa x *(vertical chunk)* or with fixed ordinate y *(horizontal chunk)*.

Recalling Def. 1, μ-chunks are closed segments, at most four, parallel to one of the axes. To deal with total irrelevance the following condition is of interest:

Condition $\aleph$. *$\mathcal{C}$ contains an entire μ-chunk, whether horizontal or vertical.*

As $\Pi_{XY} \in \mathcal{F}$ is bound to have the proper marginals, the sufficiency of $\aleph$ is obvious; we stress that, to prove sufficiency, of Def. 1 we need only i), which is enough to ensure that μ-chunks are compact sets (closed and limited), and so the sufficiency of $\aleph$ holds for much larger classes of fuzzy numbers than the pseudo-triangular ones.

Theorem 1. *When both X and Y are pseudo-triangular, condition $\aleph$ is necessary and sufficient to have total irrelevance over $\mathcal{C}$, and so $\aleph$ is an irrelevance criterion.*

Proof. We can move directly to necessity with $0 < \mu \doteq \Pi_{X\perp Y}(\mathcal{C}) < 1$ (the case $\mu = 1$ is trivial). Assuming that $\aleph$ is violated, we shall modify $\Pi_{X\perp Y}$ by lowering or keeping equal its values to obtain a joint Π_{XY}, which will turn out to be admissible, even if $\Pi_{XY}(\mathcal{C}) < \mu$. The equations $\Pi_X(x) = \mu$ and $\Pi_Y(y) = \mu$ have at most four solutions (twice each), and at least one solution, either for $\Pi_X(x)$ or $\Pi_Y(y)$; recall that $\Pi_X(x)$ and $\Pi_Y(y)$ verify *iv*) in Def. 1, and that $\Pi_{X\perp Y}(\mathcal{C}) = \mu$ implies that either $\Pi_X(x) = \mu$ or $\Pi_Y(y) = \mu$ for $(x, y) \in \mathcal{C}$. If the solutions are four distinct, the μ-chunks form a full rectangle in $\mathbb{R}^2$, else one to three sides are lacking. Assume e.g. $\Pi_Y(y^*) = \mu$ with $\Pi_Y(y)$ increasing in y^*. Think of the horizontal μ-chunk corresponding to y^*, which is a segment parallel to the X-axis, by assumption not entirely included in $\mathcal{C}$. $\mathcal{C}$ being closed, one can take a closed disc of radius $\epsilon > 0$ centered on the μ-chunk, which does not intersect $\mathcal{C}$; actually, one can take the disc center (x^*, y^*) in the interior of the μ-chunk. In case reducing ϵ, one can assume that $y^* + \epsilon$ lies where $\Pi_Y(y)$ is still increasing. Clearly, values of $\Pi_{X\perp Y}$ on the segment $\mathcal{S} \doteq [x^*, y^* \mp \epsilon]$ parallel to the Y-axis ensure correct projections on the Y-axis interval $[y^* \mp \epsilon]$, which corresponds to $\Pi_Y(y)$-values between $\mu - \delta_1$ and $\mu + \delta_2$, say, with δ_1 and δ_2 strictly positive; values of $\Pi_{X\perp Y}$ on the segment $\mathcal{S}$ will *not* be tampered with in the sequel. (Notice that y^* can be a point of discontinuity for $\Pi_Y(y)$, in which case the supremum of $\Pi_Y(y)$ for $y < y^*$ is $\nu < \mu$). Repeat the same on all the μ-chunks to obtain one to four segments perpendicular to the respective μ-chunks and not to be tampered with. Consider now the union $\mathcal{D}$ of the open discs centered in the μ-chunks in points which belong to $\mathcal{C}$ and with radius $\eta > 0$; choose η small enough so as not to intersect any of the segments not to be tampered with (recall that these are perpendicular to the respective μ-chunks and that each of their middle points is the center of a disc outside $\mathcal{C}$). Obtain an upper semicontinuous Π_{XY} from $\Pi_{X\perp Y}$ by lowering to zero inside $\mathcal{D}$ all values of $\Pi_{X\perp Y}$ belonging to $]\mu - \gamma, \mu + \gamma[$, with $\gamma > 0$ small enough so as to ensure that the correct projections around μ are taken care of by the one to four segments not to be tampered with. One has $\Pi_{XY}(\mathcal{C}) \leq \mu - \gamma$. $\Diamond$

The proof of necessity can be readily extended to slightly larger classes of fuzzy numbers, e.g. allowing that the increasing (the decreasing) part in *iv*), Def. 1, is lacking, and/or allowing that the infimum in *iv*) is strictly positive for $x \leq x_1$ and/or for $x \geq x_1$. We recall that in [11] $\aleph$ was proved to be a criterion in case of fuzzy numbers with a *finite* support, with no further constraints as those in Def. 1.

To have *total* irrelevance, the quite demanding condition $\aleph$ should hold true for *all* counterimages, and so our criterion is exacting, indeed, as had been argued already in [11] on the basis there of purely combinatorial arguments; thus, *total* irrelevance is only a *limit situation*, which will not hold for "interesting" operations[4] $x \circ y$. In itself, however, the notion of irrelevance is quite flexible, since the family $\mathcal{F}$ can be properly restricted; cf. next section, and more specifically Prop. 1. In Sec. 5, which is devoted to open problems, we comment upon the fact that this flexibility might be made good use of in $\top$-norm extensions of fuzzy arithmetic, for which cf. e.g. [4, 5, 9]. In Sec. 4, Prop. 2, we shall put forward an unassuming condition on $x \circ y$ which is enough to rule out *total* irrelevance for a lot of "interesting" two-argument operations, starting with addition and multiplication, and so covering cases which are quite familiar in current literature.

4. Notable cases of partial irrelevance

As $\aleph$ is so severe a demand, one rather moves to *partial* irrelevance. Here we shall restrict ourselves[5] to a very special situation when $\mathcal{F}$ has only *two* distributions Π_1 and Π_2 with $\Pi_1 < \Pi_2$ in the *poset* defined in the Appendix. Note that $\mathcal{F}$-irrelevance for $\mathcal{F} = \{\Pi_1, \Pi_2\}$ implies irrelevance for any superset of $\mathcal{F}$ including only joint distributions which are intermediate between Π_1 and Π_2, cf. the Appendix, Prop. 3. Clearly, to have this sort of partial irrelevance, whenever $\Pi_2(\mathcal{C}_z) = \mu > 0$ there must be in $\mathcal{C}_z$ a couple (x, y) such that $x \circ y = z$ and $\Pi_1(x, y) = \mu$. If X and Y are equidistributed, $\Pi_X(x) = \Pi_Y(x)$, and Π_1 is deterministic equality, there must be an x such that $x \circ x = z$, i.e. $(x, x) \in \mathcal{C}_z$, and $\Pi_1(x, x) = \Pi_X(x) = \Pi_Y(x) = \mu$ (recall that in the case of deterministic equality $\Pi_1(x, y) > 0$ implies $x = y$). Now, in more traditional fuzzy arithmetic, interactivity is not considered (in practice only *non*-interactivity is allowed), but even so there is at least one situation one

[4]Or at least interesting on the Cartesian product of the supports of X and Y. As a (necessarily trivial) example of total irrelevance whatever X and Y fix $\alpha \in \mathbb{R}$ and take $x \circ y = x$ for $x \leq \alpha$, else $x \circ y = y$; note that the result z depends in turn on only one of the arguments.

[5]Were a substantial possibilistic equivalent of probabilistic *copulas* [10] available, one would have a tool to enhance the significance of partial irrelevance. One may rather resort to the related notion of a $\top$-norm, cf. e.g. [4], which has instead no possibilistic drawback; cf. the open problems of Sec. 5.

should not overlook[6], i.e. when X and Y are equidistributed, Π_1 is deterministic equality and $\Pi_2 \doteq \Pi_{X\perp Y}$ is non-interactivity. A straightforward result, well-known but difficult to trace back in the literature also because sometimes left rather implicit, amounts to stating that a remarkable case of partial irrelevance for deterministic equality vs. non-interactivity is sum and product of triangular numbers, the latter operation limited to the case of an all-positive[7] support. For self-readability, we now prove this "folk-theorem", actually in larger generality. Below $\simeq$ denotes equidistribution, rather than deterministic equality; *order-preserving*, e.g. non-decreasing, means the following: $x < y$ implies that for all u one has $x \circ u \le y \circ u$ and $u \circ x \le u \circ y$.

Proposition 1. *Take X and Y non-interactive and equidistributed, with $\Pi_X(x) = \Pi_Y(x)$ a concave (convex-cap) function over its connected support. Take $\circ$ order-preserving such that $f(x) = x \circ x \doteq x^{(2)}$ is a continuous function of its argument. One has $X \circ Y \simeq X \circ X \doteq X^{(2)}$.*

Proof. If $\circ$ is e.g. non-decreasing, so is the function $f(x)$. If $u < v$, $u \circ v = z$, since $\circ$ is order-preserving one has $u^{(2)} \le u \circ v \le v^{(2)}$ and so, by continuity of f, there is a value $x = \alpha u + (1-\alpha)v$ between u and v $(0 \le \alpha \le 1)$ such that $x^{(2)} = u \circ v = z$; then by concavity $\Pi_X(x) \ge \alpha \Pi_X(u) + (1-\alpha)\Pi_X(v) \ge \Pi_X(u) \wedge \Pi_X(v) \doteq \Pi_{X\perp Y}(u, v)$. $\Diamond$

The result applies literally to *addition*, and also to the *positive product* (usual product for x, $y \ge 0$, else 0), which actually coincides with ordinary *multiplication* when the support is all-positive. Observe that if the support is instead "mixed" (there are $x < 0$ and $x' > 0$ of positive marginal possibility), partial irrelevance w.r. to deterministic equality vs. non-interactivity does *not* hold for multiplications, since $z \doteq x \times x' < 0$ has positive possibility with non-interactivity, while $X \times X \doteq X^2$ is a square and so its support is non-negative.

If Prop. 1 is not really novel, the reader may appreciate the general framework we are now able to set it in. Consider X and Y triangular and equidistributed on a, b, c, $0 \le a < b < c$. One has partial irrelevance with respect to deterministic equality vs. non-interactivity both for sum and (positive) product, but one does *not* have total irrelevance. Take e.g. a *drastic* distribution as in Sec. 2: in this case the support of $X + Y$

---

[6]Else, one ends up writing $X \times X \neq X^2$, as it does happen, to mean simply that equidistribution $\Pi_X = \Pi_Y$ does not imply $X \times Y = X^2$.

[7]Or all-negative, just use the obvious equality $X \times Y = (-X) \times (-Y)$.

would end at $b + c$ rather than $2c$, while the support of the (positive) product would end at $b \times c$ rather than c^2.

The latter argument can be generalized, so as to offer a convenient way to rule out partial irrelevance with respect to *drastic interactivity* vs. *non-interactivity*, and so *a fortiori* to rule out total irrelevance. Below x_1 and y_1 are the two marginal *modes* as in Def. 1, i.e. $\Pi_X(x_1) = 1$ and $\Pi_Y(y_1) = 1$, while $\overline{x}$ and $\overline{y}$ are the right endpoints, assumed *finite*, of the closures of the supports of X and Y, i.e. $\overline{x} \doteq \sup\{x : \Pi_X(x) > 0\}$, $\overline{y} \doteq \sup\{y : \Pi_Y(y) > 0\}$, $\overline{x}$, $\overline{y} < +\infty$; clearly, $x_1 < \overline{x}$ and $y_1 < \overline{y}$.

Proposition 2. *Assume that the operation $\circ$ is continuous, non-decreasing and* strictly[8] *increasing for $x > \tilde{x}$, $y > \tilde{y}$ (possibly $\tilde{x}$ and/or $\tilde{y}$ are $-\infty$). If $\tilde{x} < \overline{x}$ and $\tilde{y} < \overline{y}$ there cannot be irrelevance with respect to drastic interactivity vs. non-interactivity.*

Proof. The right endpoint of the closure of the support of $Z = X \circ Y$ under drastic interactivity is the maximum between $\sup\{z : z = x \circ y_1, \Pi_X(x) \neq 0\} = \overline{x} \circ y_1$ and $\sup\{z : z = x_1 \circ y, \Pi_Y(y) \neq 0\} = x_1 \overline{y}$. Under our assumptions this maximum is strictly less than the right endpoint of the closure of the support of $Z = X \circ Y$ under non-interactivity, which is $\sup\{z : z = x \circ y, \Pi_X(x) \neq 0, \Pi_Y(y) \neq 0\} = \overline{x} \circ \overline{y}$. $\Diamond$

5. Open problems

Readers may have noticed that the three examples of joint distributions in Sec. 2 are all describable in terms of $\top$-norms (actually, the term *drastic* is a loanword; for $\top$-norms cf. e.g. [4]). As stressed by an anonymous referee, $\top$-norms give one the chance of considering relevant families $\mathcal{F}$ of joint distributions, while in the paper we have restricted our attention to two limit cases only, in Sec. 4 the very special case when $\mathcal{F}$ is made up of *two* distributions, deterministic equality and non-interactivity, in practice the two distributions taken into account, more or less explicitly, in more traditional approaches to fuzzy arithmetic, and the "too generous" case when $\mathcal{F}$ contains everything admissible, cf. Sec. 3.

[8] I.e. $\tilde{x} < x \leq x'$ and $\tilde{y} < y \leq y'$ imply $x \circ y \leq x' \circ y'$ with equality only for $x = x'$ and $y = y'$; note that we need the continuity of $\circ$ only in $(x_1, \overline{y})$, $(\overline{x}, y_1)$ and $(\overline{x}, \overline{y})$. Generalizations of Prop. 2 are at hand, e.g. for $\Pi_X = \Pi_Y$ our argument can be used to prove that, with the given assumptions, there cannot be partial irrelevance for drastic interactivity vs. deterministic equality.

In the latter situation, it is no wonder that the irrelevance criterion we obtain is so exacting; instead, by properly circumscribing $\mathcal{F}$ to suitable families obtained by $\top$-norms, possibly bound to verify convenient regularity assumptions, and by suitably defining what one means by a fuzzy number as we tried to do in Def. 1, one might get at significant criteria for $\mathcal{F}$-irrelevance in $\top$-norm extensions (cf. e.g. [4, 5, 9]) of fuzzy arithmetic.

6. Appendix: possibilities

1. Let $\mathcal{U}$ be a non-void crisp set, finite or infinite. Let $\Pi(x)$ be a function on $\mathcal{U}$ with range $[0,1]$ such that the value 1 is taken on at least once. This is enough to define a *possibility distribution* $\Pi : 2^{\mathcal{U}} \to [0,1]$ by simply setting $\Pi(E) = \sup_{x\in E} \Pi(x)$; to no risk of confusion, we are using the same symbol Π both for the function $\Pi(x)$ and its extension $\Pi(E)$ on subsets $E \subseteq \mathcal{U}$.

2. In this appendix we find it convenient to generalize fuzzy numbers to *fuzzy attributes* X by relinquishing the requirement that the "universe" $\mathcal{U}$ is bound to be numeric, $\mathbb{R}$ or $\mathbb{R}^2$. Let $f : \mathcal{U} \to \mathcal{V}$ be a deterministic function from $\mathcal{U}$ to $\mathcal{V}$, and let X be a fuzzy attribute on $\mathcal{U}$. If the fuzzy attribute Y over $\mathcal{V}$ is defined by $Y \doteq f(X)$, let us compute its distribution Π_Y. Fix $y \in \mathcal{V}$: now, the two conditions $\{X = x \text{ with } f(x) = y\}$ and $\{Y \doteq f(X) = y\}$ imply each other, and so they must have the same possibility. This gives $\Pi_Y(y) = \Pi_X(\{x : f(x) = y\}) = \sup_{x:\, f(x)=y} \Pi_X(x)$. If $\mathcal{U}$ is the Cartesian square $\mathbb{R}^2$, $\mathcal{V} = \mathbb{R}$ and $f = \circ$, one re-finds (1).

3. Given $\mathcal{U}$, a *poset*, or *p*artially *o*rdered *set*, is soon obtained over the space of all possibility distributions over $\mathcal{U}$ by setting $\Pi_1 \le \Pi_2$ if and only if $\Pi_1(x) \le \Pi_2(x)$ $\forall x \in \mathcal{U}$, and so $\forall E \subseteq \mathcal{U}$. As soon proved, in the family $\mathcal{F}$ of all joint distributions with fixed marginals Π_X and Π_Y, non-interactivity $\Pi_{X\perp Y}$ is the single maximum element in the poset, while, assuming equidistribution $\Pi_X = \Pi_Y$, deterministic equality is a minimal element; such is also the drastic distribution assuming that $\Pi_X(x) = 1$ and $\Pi_Y(y) = 1$ have only one solution each. Actually, it is precisely the fact that non-interactive possibilities are maximal which leads one to maintain that non-interactivity is an adequate analogue of probabilistic independence, cf. [4, 8]. Since $\Pi_{X_1} < \Pi_{X_2}$ clearly implies $\Pi_{f(X_1)}! \le \Pi_{f(X_2)}$, one has:

Proposition 3. *If* $\Pi_{X_1} < \Pi_{X_2} < \Pi_{X_3}$ *and* $\Pi_{f(X_1)} = \Pi_{f(X_3)}$, *then* $\Pi_{f(X_1)} = \Pi_{f(X_2)} = \Pi_{f(X_3)}$.

References

[1] CARLSSON, C., FULLÉR, R. and MAJLENDER, P.: Additions of completely correlated fuzzy numbers, *IEEE International Conference on Fuzzy Systems, 2004*, 535–539.

[2] COROIANU, L. and FULLÉR, R.: On multiplication of interactive fuzzy numbers, *IEEE International Symposium on Intelligent Systems and Informatics, SISY 2013*, 181–185.

[3] DUBOIS, D. and PRADE, H.: Additions of interactive fuzzy numbers, *IEEE Transactions on Automatic Control*, Vol. AC-26, No. 4 (1981), 926–936.

[4] *Fundamentals of Fuzzy Sets*, ed. by Dubois D. Prade H., Kluwer Academic Publishers, 2000.

[5] FULLÉR, R. and KERESZTFALVI, T.: *t*-norm based addition of fuzzy intervals. *Fuzzy Sets and Systems* **51** (1992), 155–159.

[6] FULLÉR, R. and MAJLENDER, P.: On interactive fuzzy numbers, *Fuzzy Sets and Systems* **143** (3) (2004), 353–369.

[7] FULLÉR, R. and MAJLENDER, P.: On addition of interactive fuzzy numbers, *Acta Polytechnica Hungarica* (2) (2005), 59–73.

[8] KLIR, G. J. and FOLGER, T. A.: *Fuzzy Sets, Uncertainty, and Information*, Prentice Hall Int, 1988.

[9] MESIAR, R.: Triangular-norm-based addition of fuzzy intervals, *Fuzzy Sets and Systems* **91** (1997), 231–237.

[10] NELSEN, R.: *An Introduction to Copulas*, Springer Verlag, 2006.

[11] SGARRO, A. and FRANZOI, L.: Fuzzy arithmetic for fuzzy n-poles, *Proceedings of IPMU 2012* (2) (2012), 1–8.