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DECOMPOSITIONS OF FREE PRODUCTS OF DIMONOIDS

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Abstract: In the author's preceding paper a free product of dimonoids was constructed and for such a dimonoid the investigation of the structure was started. In this paper we continue to study structural properties of free products of dimonoids. The main results are a characterization of basic types of diband and band decompositions of free products of dimonoids.

1. Introduction

Following J.-L. Loday [4], a *dimonoid* is a nonempty set equipped with two binary associative operations $\dashv$ and $\vdash$ satisfying the axioms $(x \dashv y) \dashv z = x \dashv (y \vdash z)$, $(x \vdash y) \dashv z = x \vdash (y \dashv z)$, $(x \dashv y) \vdash z = x \vdash (y \vdash z)$.

There exist numerous examples of dimonoids showing the consistency of axioms of a dimonoid. So, for example, an arbitrary semigroup S with operations defined by $x \dashv y = x(yf)$, $x \vdash y = (xf)y$ for all $x, y \in S$, where f is an idempotent endomorphism of S, forms a dimonoid. In [8] it was proved that a system of axioms of a dimonoid is independent.

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For further details and background see [4], [8], [9], [15]. Dialgebras, which are linear analogs of dimonoids, were studied by many authors (see, e.g., [1], [3], [4], [6]). A free dimonoid was constructed by J.-L. Loday [4] and applied to the study of free dialgebras and a cohomology of dialgebras. Decompositions of free dimonoids into dibands of subdimonoids have been studied in [9–11]. The variety theory of dimonoids was developed in [11–16].

In the author's preceding paper [17] a free product of dimonoids was constructed and for such a dimonoid the investigation of the structure was started. In this paper we continue to study structural properties of free products of dimonoids.

The paper is organized as follows. In Sec. 2 we give bands and dimonoid constructions which will be used in the paper. In Sec. 3 we describe decompositions of free products of dimonoids into relatively free dibands of subdimonoids. In Sec. 4 we describe decompositions of free products of dimonoids into relatively free bands of subdimonoids. The results from Sections 3 and 4 extend the corresponding results from [10] and [11]. In the final section we give a faithful representation of free products of left zero and right zero dimonoids.

The obtained results were announced in [18] and they can be applied in dialgebra theory for obtaining decomposition results.

2. Preliminaries

In this section we give bands and dimonoid constructions which will be used in the paper.

2.1. Recall the construction of a free product of dimonoids [17]. As usual, $\mathbb{N}$ denotes the set of all positive integers.

Let $Fr[S_i]_{i\in X}$ be the free product of arbitrary semigroups S_i, $i \in X$. For every $w \in Fr[S_i]_{i\in X}$ denote the first (respectively, last) letter of w by $w^{(0)}$ (respectively, $w^{(1)}$) and the length of w by l_w. Consider the set

$$G(S_i)_{i\in X} = \{(w,m) \in Fr[S_i]_{i\in X} \times \mathbb{N} \,|\, l_w \geq m\}.$$

For all $(w,m) \in G(S_i)_{i\in X}$ and $u \in Fr[S_i]_{i\in X}$ assume

$$f^u_{(w,m)} = \begin{cases} l_u + m, & l_{u^{(1)}w^{(0)}} = 2, \\ l_u + m - 1, & l_{u^{(1)}w^{(0)}} = 1. \end{cases} \tag{1}$$

For a given relation ρ on a dimonoid $(D, \dashv, \vdash)$, the congruence generated by ρ is the least congruence on $(D, \dashv, \vdash)$ containing ρ. It will be denoted by $\rho^\star$ and can be characterized as the intersection of all congruences on $(D, \dashv, \vdash)$ containing ρ.

Let $\{(D_i, \dashv_i, \vdash_i)\}_{i\in X}$ be a family of arbitrary pairwise disjoint dimonoids. Operations on $Fr[(D_i, \dashv_i)]_{i\in X}$ and $Fr[(D_i, \vdash_i)]_{i\in X}$ will be denoted by $\dashv$ and $\vdash$ respectively. For every $i \in X$ consider a relation

$$\theta_i = \{(a \vdash_i b, a \dashv_i b) \mid a, b \in D_i\}$$

on a dimonoid $(D_i, \dashv_i, \vdash_i)$. It is clear that operations of $(D_i, \dashv_i, \vdash_i)/\theta_i^\star$ coincide and it is a semigroup.

Let $\omega_1 = (x_1x_2...x_k...x_s, t), \omega_2 = (y_1y_2...y_k...y_p, r) \in G((D_i, \dashv_i))_{i\in X}$, where $x_1, x_2, ..., x_k, ..., x_s, y_1, y_2, ..., y_k, ..., y_p \in \bigcup_{i\in X} D_i$. Define a relation $\sim$ on $G((D_i, \dashv_i))_{i\in X}$ by

$$\omega_1 \sim \omega_2 \Leftrightarrow \begin{cases} s = p, t = r \text{ and } \ x_k\theta_{j_k}^\star y_k \text{ for all } 1 \le k \le s \text{ and some } j_k \in X, \\ \text{at that } x_t = y_r. \end{cases}$$

It is not hard to check that $\sim$ is an equivalence relation. Denote the equivalence class of $\sim$ containing an element $(w, m) \in G((D_i, \dashv_i))_{i\in X}$ by $[w, m]$ and the quotient set $G((D_i, \dashv_i))_{i\in X}/\sim$ by $G^\star((D_i, \dashv_i))_{i\in X}$.

Define operations $\dashv'$ and $\vdash'$ on $G^\star((D_i, \dashv_i))_{i\in X}$ by

$$[w_1, m_1] \dashv' [w_2, m_2] = [w_1 \dashv w_2,\ m_1],$$
$$[w_1, m_1] \vdash' [w_2, m_2] = [w_1 \vdash w_2,\ f^{w_1}_{(w_2,m_2)}]$$

for all $[w_1, m_1], [w_2, m_2] \in G^\star((D_i, \dashv_i))_{i\in X}$. The algebra $(G^\star((D_i,\dashv_i))_{i\in X}, \dashv', \vdash')$ will be denoted by $\breve{G}(D_i)_{i\in X}$.

Theorem 2.1 ([17], Th. 2.3). *$\breve{G}(D_i)_{i\in X}$ is the free product of dimonoids $(D_i, \dashv_i, \vdash_i)$, $i \in X$.*

2.2. Construct the free rectangular dimonoid [11]. A dimonoid $(D, \dashv, \vdash)$ is called a rectangular dimonoid or a rectangular diband (respectively, an idempotent dimonoid or a diband), if both semigroups $(D, \dashv)$ and $(D, \vdash)$ are rectangular bands (respectively, idempotent semigroups).

Let X be an arbitrary nonempty set and $X^3 = X \times X \times X$. Define operations $\dashv$ and $\vdash$ on X^3 by

$$(x_1, x_2, x_3) \dashv (y_1, y_2, y_3) = (x_1, x_2, y_3),$$
$$(x_1, x_2, x_3) \vdash (y_1, y_2, y_3) = (x_1, y_2, y_3)$$

for all $(x_1, x_2, x_3), (y_1, y_2, y_3) \in X^3$. Denote the algebra $(X^3, \dashv, \vdash)$ by $FRct(X)$.

Theorem 2.2 ([11], Th. 1). *$FRct(X)$ is the free rectangular dimonoid.*

2.3. Construct the free left zero and right zero dimonoid [11]. Let $X_{\ell z} = (X, \dashv)$, $X_{rz} = (X, \vdash)$, $X_{rb} = X_{\ell z} \times X_{rz}$ be a left zero semigroup, a right zero semigroup and a rectangular band, respectively. One can check that $X_{\ell z, rz} = (X, \dashv, \vdash)$ is a rectangular dimonoid. We call this dimonoid as a left zero and right zero dimonoid. We will call a left zero and right zero dimonoid also a left and right diband.

Lemma 2.3 ([11], Lemma 5). *Every left zero and right zero dimonoid is the free left zero and right zero dimonoid.*

2.4. Construct the free (left, right) normal band [5]. Let $B(X)$ be the semilattice of all nonempty finite subsets of X with respect to the operation of the set theoretical union and

$$B_{rb}(X) = \{((x, y), A) \in X_{rb} \times B(X) \,|\, x, y \in A\},$$
$$B_{\ell z}(X) = \{(x, A) \in X_{\ell z} \times B(X) \,|\, x \in A\},$$
$$B_{rz}(X) = \{(x, A) \in X_{rz} \times B(X) \,|\, x \in A\}.$$

It is clear that $B_{rb}(X)$, $B_{\ell z}(X)$, $B_{rz}(X)$ are subsemigroups of $X_{rb} \times B(X)$, $X_{\ell z} \times B(X)$, $X_{rz} \times B(X)$, respectively. By [5] $B_{rb}(X)$, $B_{\ell z}(X)$ and $B_{rz}(X)$ are the free normal band, the free left normal band and the free right normal band, respectively.

2.5. Construct the free $(\ell z, rb)$-dimonoid [11]. Let $(D, \dashv)$ be a left zero semigroup and $(D, \vdash)$ be a rectangular band. Then $(D, \dashv, \vdash)$ is a rectangular dimonoid. We call this rectangular dimonoid a $(\ell z, rb)$-dimonoid.

Let X be an arbitrary nonempty set. Define operations $\dashv$ and $\vdash$ on X^2 by

$$(x, y) \dashv (a, b) = (x, y), \quad (x, y) \vdash (a, b) = (x, b)$$

for all $(x, y), (a, b) \in X^2$. It is clear that $(X^2, \dashv)$ is a left zero semigroup, $(X^2, \vdash)$ is a rectangular band and $(X^2, \dashv, \vdash)$ is a $(\ell z, rb)$-dimonoid. We denote this dimonoid by $X_{\ell z, rb}$.

Lemma 2.4 ([11], Lemma 7). *$X_{\ell z, rb}$ is the free $(\ell z, rb)$-dimonoid.*

2.6. Construct the free (rb, rz)-dimonoid [11]. Let $(D, \dashv)$ be a rectangular band and $(D, \vdash)$ be a right zero semigroup. Then $(D, \dashv, \vdash)$ is a rectangular dimonoid. We call this rectangular dimonoid a (rb, rz)-dimonoid.

Let X be an arbitrary nonempty set. Define operations $\dashv$ and $\vdash$ on X^2 by

$$(x, y) \dashv (a, b) = (x, b), \quad (x, y) \vdash (a, b) = (a, b)$$

for all $(x, y), (a, b) \in X^2$. It is clear that $(X^2, \dashv)$ is a rectangular band, $(X^2, \vdash)$ is a right zero semigroup and $(X^2, \dashv, \vdash)$ is a (rb, rz)-dimonoid. We denote the obtained dimonoid by $X_{rb,rz}$.

Lemma 2.5 ([11], Lemma 6)**.** *$X_{rb,rz}$ is the free (rb, rz)-dimonoid.*

2.7. Construct the free normal diband [13]. A dimonoid is called a normal diband, if its both semigroups are normal bands.

Let $FRct(X)$ be the free rectangular dimonoid (see item 2.2) and

$$FND(X) = \{((x, y, z), A) \in FRct(X) \times B(X) \,|\, x, y, z \in A\}.$$

Theorem 2.6 ([13], Th. 2)**.** *$FND(X)$ is the free normal diband.*

2.8. Construct the free $(\ell n, n)$-diband [13]. A dimonoid $(D, \dashv, \vdash)$ is called a $(\ell n, n)$-diband, if $(D, \dashv)$ is a left normal band and $(D, \vdash)$ is a normal band.

Let $B_{\ell z,rb}(X) = \{((x, y), A) \in X_{\ell z,rb} \times B(X) \,|\, x, y \in A\}$.

Lemma 2.7 ([13], Lemma 6)**.** *$B_{\ell z,rb}(X)$ is the free $(\ell n, n)$-diband.*

2.9. Construct the free (n, rn)-diband [13]. A dimonoid $(D, \dashv, \vdash)$ is called a (n, rn)-diband, if $(D, \dashv)$ is a normal band and $(D, \vdash)$ is a right normal band.

Let $B_{rb,rz}(X) = \{((x, y), A) \in X_{rb,rz} \times B(X) \,|\, x, y \in A\}$.

Lemma 2.8 ([13], Lemma 7)**.** *$B_{rb,rz}(X)$ is the free (n, rn)-diband.*

2.10. Construct the free $(\ell n, rn)$-diband [13]. A dimonoid $(D, \dashv, \vdash)$ is called a $(\ell n, rn)$-diband, if $(D, \dashv)$ is a left normal band and $(D, \vdash)$ is a right normal band.

Let $B_{\ell z,rz}(X) = \{(x, A) \in X_{\ell z,rz} \times B(X) \,|\, x \in A\}$.

Lemma 2.9 ([13], Lemma 8)**.** *$B_{\ell z,rz}(X)$ is the free $(\ell n, rn)$-diband.*

2.11. In [7] L. M. Gluskin, B. M. Schein and L. N. Shevrin stated that the notion of a band of semigroups, which has been playing a very important role in the semigroup theory, can be naturally extended for any class of abstract algebras. So, the notion of a diband of subdimonoids for such a class of abstract algebras as dimonoids, was applied in [8–11], [13], [14], [17]. This notion is relevant for the study of structural properties of dimonoids. Recall this definition.

If $f : D_1 \to D_2$ is a homomorphism of dimonoids, then the corresponding congruence on D_1 will be denoted by Δ_f.

Let S be an arbitrary dimonoid, J be some idempotent dimonoid. Let

$$\alpha : S \to J : x \mapsto x\alpha$$

be a homomorphism. Then every class of the congruence Δ_α is a subdimonoid of the dimonoid S and the dimonoid S itself is a union of such dimonoids $S_\xi,\ \xi \in J$, that

$$x\alpha = \xi \Leftrightarrow x \in S_\xi = \Delta_\alpha^x = \{t \in S \mid (x,t) \in \Delta_\alpha\},$$

$$S_\xi \dashv S_\varepsilon \subseteq S_{\xi \dashv \varepsilon}, \quad S_\xi \vdash S_\varepsilon \subseteq S_{\xi \vdash \varepsilon},$$

$$\xi \neq \varepsilon \Rightarrow S_\xi \bigcap S_\varepsilon = \varnothing.$$

In this case we say that S is decomposable into a diband of subdimonoids (or S is a diband J of subdimonoids $S_\xi,\ \xi \in J$). If J is a band (= idempotent semigroup), then we say that S is a band J of subdimonoids $S_\xi,\ \xi \in J$. If J is a semilattice (=commutative band), then we say that S is a semilattice J of subdimonoids $S_\xi,\ \xi \in J$. If J is a left zero semigroup (respectively, right zero semigroup), then we say that S is a left band (respectively, right band) J of subdimonoids $S_\xi,\ \xi \in J$.

Note that the notion of a diband of subdimonoids generalizes the notion of a band of semigroups [2].

3. Diband decompositions of $\breve{G}(D_i)_{i\in X}$

In this section we give decompositions of the free product $\breve{G}(D_i)_{i\in X}$ of dimonoids $(D_i, \dashv_i, \vdash_i)$, $i \in X$, into relatively free dibands of subdimonoids.

For every $w = s_{\gamma_1} \ldots s_{\gamma_l} \ldots s_{\gamma_k} \in Fr[(D_i, \dashv_i)]_{i\in X}$ assume $\tilde{c}(w) = \bigcup_{l=1}^{k} \{s_{\gamma_l} j^*\}$, where

$$j^* : \bigcup_{i\in X} D_i \to X : a \mapsto i, \text{ if } a \in D_i,\ i \in X.$$

Let

$$H_{(a,b,c)} = \{[s_{\gamma_1}...s_{\gamma_l}...s_{\gamma_k}, m] \in \breve{G}(D_i)_{i\in X} \,|\, (s_{\gamma_1}j^*, s_{\gamma_m}j^*, s_{\gamma_k}j^*) = (a,b,c)\}$$

for $(a,b,c) \in FRct(X)$;

$$H_{(a,b]} = \{[s_{\gamma_1}...s_{\gamma_l}...s_{\gamma_k}, m] \in \breve{G}(D_i)_{i\in X} \,|\, (s_{\gamma_1}j^*, s_{\gamma_m}j^*) = (a,b)\}$$

for $(a,b) \in X_{\ell z,rb}$;

$$H_{[b,c)} = \{[s_{\gamma_1}...s_{\gamma_l}...s_{\gamma_k}, m] \in \breve{G}(D_i)_{i\in X} \,|\, (s_{\gamma_m}j^*, s_{\gamma_k}j^*) = (b,c)\}$$

for $(b,c) \in X_{rb,rz}$;

$$H_{(b]} = \{[s_{\gamma_1}...s_{\gamma_l}...s_{\gamma_k}, m] \in \breve{G}(D_i)_{i\in X} \,|\, s_{\gamma_m}j^* = b\}$$

for $b \in X_{\ell z,rz}$;

$$H^Y_{(a,b,c)} = \{[s_{\gamma_1}...s_{\gamma_l}...s_{\gamma_k}, m] \in \breve{G}(D_i)_{i\in X} \,|$$
$$((s_{\gamma_1}j^*, s_{\gamma_m}j^*, s_{\gamma_k}j^*), \tilde{c}(s_{\gamma_1}...s_{\gamma_l}...s_{\gamma_k})) = ((a,b,c),Y)\}$$

for $((a,b,c),Y) \in FND(X)$;

$$H^Y_{(a,b]} = \{[s_{\gamma_1}...s_{\gamma_l}...s_{\gamma_k}, m] \in \breve{G}(D_i)_{i\in X} \,|$$
$$((s_{\gamma_1}j^*, s_{\gamma_m}j^*), \tilde{c}(s_{\gamma_1}...s_{\gamma_l}...s_{\gamma_k})) = ((a,b),Y)\}$$

for $((a,b),Y) \in B_{\ell z,rb}(X)$;

$$H^Y_{[b,c)} = \{[s_{\gamma_1}...s_{\gamma_l}...s_{\gamma_k}, m] \in \breve{G}(D_i)_{i\in X} \,|$$
$$((s_{\gamma_m}j^*, s_{\gamma_k}j^*), \tilde{c}(s_{\gamma_1}...s_{\gamma_l}...s_{\gamma_k})) = ((b,c),Y)\}$$

for $((b,c),Y) \in B_{rb,rz}(X)$;

$$H^Y_{(b]} = \{[s_{\gamma_1}...s_{\gamma_l}...s_{\gamma_k}, m] \in \breve{G}(D_i)_{i\in X} \,|$$
$$(s_{\gamma_m}j^*, \tilde{c}(s_{\gamma_1}...s_{\gamma_l}...s_{\gamma_k})) = (b,Y)\}$$

for $(b,Y) \in B_{\ell z,rz}(X)$.

Further we will deal with diband decompositions (see item 2.11) of free products of dimonoids.

The following two structure theorems give decompositions of free products of dimonoids into relatively free dibands of subdimonoids.

Theorem 3.1. *Let $\breve{G}(D_i)_{i\in X}$ be the free product of dimonoids. Then*

(i) *$\breve{G}(D_i)_{i\in X}$ is a normal diband $FND(X)$ of subdimonoids $H^Y_{(a,b,c)}$, $((a,b,c),Y) \in FND(X)$;*

(ii) $\breve{G}(D_i)_{i\in X}$ *is a diband* $B_{\ell z,rb}(X)$ *of subdimonoids* $H^Y_{(a,b]}$, $((a,b),Y)\in$ $\in B_{\ell z,rb}(X)$. *Every dimonoid* $H^Y_{(a,b]}$, $((a,b),Y)\in B_{\ell z,rb}(X)$, *is a right band* Y_{rz} *of subdimonoids* $H^Y_{(a,b,c)}$, $c\in Y_{rz}$;

(iii) $\breve{G}(D_i)_{i\in X}$ *is a diband* $B_{rb,rz}(X)$ *of subdimonoids* $H^Y_{[b,c)}$, $((b,c),Y)\in$ $\in B_{rb,rz}(X)$. *Every dimonoid* $H^Y_{[b,c)}$, $((b,c),Y)\in B_{rb,rz}(X)$, *is a left band* $Y_{\ell z}$ *of subdimonoids* $H^Y_{(a,b,c)}$, $a\in Y_{\ell z}$;

(iv) $\breve{G}(D_i)_{i\in X}$ *is a diband* $B_{\ell z,rz}(X)$ *of subdimonoids* $H^Y_{(b]}$, $(b,Y)\in$ $\in B_{\ell z,rz}(X)$. *Every dimonoid* $H^Y_{(b]}$, $(b,Y)\in B_{\ell z,rz}(X)$, *is a rectangular band* Y_{rb} *of subdimonoids* $H^Y_{(a,b,c)}$, $(a,c)\in Y_{rb}$.

Proof. (i) Define a map $\eta_{FND}:\breve{G}(D_i)_{i\in X}\to FND(X)$ by

$$[s_{\gamma_1}...s_{\gamma_l}...s_{\gamma_k},m]\mapsto((s_{\gamma_1}j^*,s_{\gamma_m}j^*,s_{\gamma_k}j^*),\tilde{c}(s_{\gamma_1}...s_{\gamma_l}...s_{\gamma_k})),$$

$$[s_{\gamma_1}...s_{\gamma_l}...s_{\gamma_k},m]\in\breve{G}(D_i)_{i\in X}.$$

For arbitrary elements $[s_{\gamma_1}...s_{\gamma_l}...s_{\gamma_k},m]$, $[s_{\alpha_1}...s_{\alpha_l}...s_{\alpha_r},t]\in\breve{G}(D_i)_{i\in X}$, using (1), we obtain

$$\begin{aligned}
&([s_{\gamma_1}...s_{\gamma_l}...s_{\gamma_k},m]\dashv' [s_{\alpha_1}...s_{\alpha_l}...s_{\alpha_r},t])\eta_{FND}=\\
&=[s_{\gamma_1}...s_{\gamma_l}...s_{\gamma_{k-1}}(s_{\gamma_k}\dashv s_{\alpha_1})s_{\alpha_2}...s_{\alpha_l}...s_{\alpha_r},m]\eta_{FND}=\\
&=((s_{\gamma_1}j^*,s_{\gamma_m}j^*,s_{\alpha_r}j^*),\tilde{c}(s_{\gamma_1}...s_{\gamma_l}...s_{\gamma_{k-1}}(s_{\gamma_k}\dashv s_{\alpha_1})s_{\alpha_2}...s_{\alpha_l}...s_{\alpha_r}))=\\
&=((s_{\gamma_1}j^*,s_{\gamma_m}j^*,s_{\alpha_r}j^*),\tilde{c}(s_{\gamma_1}...s_{\gamma_l}...s_{\gamma_k})\cup\tilde{c}(s_{\alpha_1}...s_{\alpha_l}...s_{\alpha_r}))=\\
&=((s_{\gamma_1}j^*,s_{\gamma_m}j^*,s_{\gamma_k}j^*),\tilde{c}(s_{\gamma_1}...s_{\gamma_l}...s_{\gamma_k}))\dashv\\
&\qquad\dashv((s_{\alpha_1}j^*,s_{\alpha_t}j^*,s_{\alpha_r}j^*),\tilde{c}(s_{\alpha_1}...s_{\alpha_l}...s_{\alpha_r}))=\\
&=[s_{\gamma_1}...s_{\gamma_l}...s_{\gamma_k},m]\eta_{FND}\dashv[s_{\alpha_1}...s_{\alpha_l}...s_{\alpha_r},t]\eta_{FND},\\
&([s_{\gamma_1}...s_{\gamma_l}...s_{\gamma_k},m]\vdash' [s_{\alpha_1}...s_{\alpha_l}...s_{\alpha_r},t])\eta_{FND}=\\
&=[s_{\gamma_1}...s_{\gamma_l}...s_{\gamma_{k-1}}(s_{\gamma_k}\vdash s_{\alpha_1})s_{\alpha_2}...s_{\alpha_l}...s_{\alpha_r},f^{s_{\gamma_1}...s_{\gamma_l}...s_{\gamma_k}}_{(s_{\alpha_1}...s_{\alpha_l}...s_{\alpha_r},t)}]\eta_{FND}=\\
&=((s_{\gamma_1}j^*,s_{\alpha_t}j^*,s_{\alpha_r}j^*),\tilde{c}(s_{\gamma_1}...s_{\gamma_l}...s_{\gamma_{k-1}}(s_{\gamma_k}\vdash s_{\alpha_1})s_{\alpha_2}...s_{\alpha_l}...s_{\alpha_r}))=\\
&=((s_{\gamma_1}j^*,s_{\alpha_t}j^*,s_{\alpha_r}j^*),\tilde{c}(s_{\gamma_1}...s_{\gamma_l}...s_{\gamma_k})\cup\tilde{c}(s_{\alpha_1}...s_{\alpha_l}...s_{\alpha_r}))=\\
&=((s_{\gamma_1}j^*,s_{\gamma_m}j^*,s_{\gamma_k}j^*),\tilde{c}(s_{\gamma_1}...s_{\gamma_l}...s_{\gamma_k}))\vdash\\
&\qquad\vdash((s_{\alpha_1}j^*,s_{\alpha_t}j^*,s_{\alpha_r}j^*),\tilde{c}(s_{\alpha_1}...s_{\alpha_l}...s_{\alpha_r}))=\\
&=[s_{\gamma_1}...s_{\gamma_l}...s_{\gamma_k},m]\eta_{FND}\vdash[s_{\alpha_1}...s_{\alpha_l}...s_{\alpha_r},t]\eta_{FND}.
\end{aligned}$$

Thus, η_{FND} is a surjective homomorphism. It is clear that $H^Y_{(a,b,c)}$, $((a,b,c),Y)\in FND(X)$, is a class of $\Delta_{\eta_{FND}}$ which is a subdimonoid of $\breve{G}(D_i)_{i\in X}$. Hence $\breve{G}(D_i)_{i\in X}$ is a normal diband $FND(X)$ of subdimonoids $H^Y_{(a,b,c)}$, $((a,b,c),Y)\in FND(X)$.

(ii) Define a map $\eta^*_{\ell z,rb}:\breve{G}(D_i)_{i\in X}\to B_{\ell z,rb}(X)$ by

$$[s_{\gamma_1}...s_{\gamma_l}...s_{\gamma_k},m]\mapsto((s_{\gamma_1}j^*,s_{\gamma_m}j^*),\tilde{c}(s_{\gamma_1}...s_{\gamma_l}...s_{\gamma_k})),$$

$$[s_{\gamma_1}...s_{\gamma_l}...s_{\gamma_k},m]\in\breve{G}(D_i)_{i\in X}.$$

Similarly to the proof of (i), the fact that $\eta^*_{\ell z,rb}$ is a surjective homomorphism can be proved. Clearly, $H^Y_{(a,b]}$, $((a,b),Y)\in B_{\ell z,rb}(X)$, is a class of $\Delta_{\eta^*_{\ell z,rb}}$ which is a subdimonoid of $\breve{G}(D_i)_{i\in X}$. Hence $\breve{G}(D_i)_{i\in X}$ is a diband $B_{\ell z,rb}(X)$ of subdimonoids $H^Y_{(a,b]}$, $((a,b),Y)\in B_{\ell z,rb}(X)$. Moreover, it is not difficult to show that for every $((a,b),Y)\in B_{\ell z,rb}(X)$ the map

$$H^Y_{(a,b]}\to Y_{rz}:[s_{\gamma_1}...s_{\gamma_l}...s_{\gamma_k},m]\mapsto s_{\gamma_k}j^*$$

is a homomorphism. Hence $H^Y_{(a,b]}$ is a right band Y_{rz} of subdimonoids $H^Y_{(a,b,c)}$, $c\in Y_{rz}$.

(iii) Define a map $\eta^*_{rb,rz}:\breve{G}(D_i)_{i\in X}\to B_{rb,rz}(X)$ by

$$[s_{\gamma_1}...s_{\gamma_l}...s_{\gamma_k},m]\mapsto((s_{\gamma_m}j^*,s_{\gamma_k}j^*),\tilde{c}(s_{\gamma_1}...s_{\gamma_l}...s_{\gamma_k})),$$

$$[s_{\gamma_1}...s_{\gamma_l}...s_{\gamma_k},m]\in\breve{G}(D_i)_{i\in X}.$$

Similarly to (i), $\eta^*_{rb,rz}$ is a surjective homomorphism. It is evident that $H^Y_{[b,c)}$, $((b,c),Y)\in B_{rb,rz}(X)$, is a class of $\Delta_{\eta^*_{rb,rz}}$ which is a subdimonoid of $\breve{G}(D_i)_{i\in X}$. So, $\breve{G}(D_i)_{i\in X}$ is a diband $B_{rb,rz}(X)$ of subdimonoids $H^Y_{[b,c)}$, $((b,c),Y)\in B_{rb,rz}(X)$.

Moreover, one can show that for every $((b,c),Y)\in B_{rb,rz}(X)$ the map

$$H^Y_{[b,c)}\to Y_{\ell z}:[s_{\gamma_1}...s_{\gamma_l}...s_{\gamma_k},m]\mapsto s_{\gamma_1}j^*$$

is a homomorphism. Hence $H^Y_{[b,c)}$ is a left band $Y_{\ell z}$ of subdimonoids $H^Y_{(a,b,c)}$, $a\in Y_{\ell z}$.

(iv) Define a map $\eta^*_{\ell z,rz}:\breve{G}(D_i)_{i\in X}\to B_{\ell z,rz}(X)$ by

$$[s_{\gamma_1}...s_{\gamma_l}...s_{\gamma_k},m]\mapsto(s_{\gamma_m}j^*,\tilde{c}(s_{\gamma_1}...s_{\gamma_l}...s_{\gamma_k})),$$

$$[s_{\gamma_1}...s_{\gamma_l}...s_{\gamma_k},m]\in\breve{G}(D_i)_{i\in X}.$$

Similarly to (i), $\eta^*_{\ell z,rz}$ is a surjective homomorphism and $H^Y_{(b]}$, $(b,Y)\in$ $\in B_{\ell z,rz}(X)$, is a class of $\Delta_{\eta^*_{\ell z,rz}}$ which is a subdimonoid of $\breve{G}(D_i)_{i\in X}$.

Thus, $\breve{G}(D_i)_{i\in X}$ is a diband $B_{\ell z,rz}(X)$ of subdimonoids $H^Y_{(b]}$, $(b,Y)\in B_{\ell z,rz}(X)$. Now we shall prove the second part of (iv).

Let

$$\pi^Y_{(b]}: H^Y_{(b]}\to Y_{rb}: [s_{\gamma_1}...s_{\gamma_l}...s_{\gamma_k}, m]\mapsto (s_{\gamma_1}j^*, s_{\gamma_k}j^*)$$

for every $(b,Y)\in B_{\ell z,rz}(X)$. As

$$(w\circ u)^{(0)}j^* = w^{(0)}j^*, \quad (w\circ u)^{(1)}j^* = u^{(1)}j^*$$

for all $w,u\in Fr[(D_i,\dashv_i)]_{i\in X}$ and $\circ\in\{\dashv,\vdash\}$, then $\pi^Y_{(b]}$ is a homomorphism. From here $H^Y_{(b]}$ is a rectangular band Y_{rb} of subdimonoids $H^Y_{(a,b,c)}$, $(a,c)\in Y_{rb}$. $\Diamond$

Take $(a,b,c)\in FRct(X)$ (see item 2.2), $(b,c)\in X_{rb,rz}$ (see item 2.6) and $b\in X_{\ell z,rz}$ (see item 2.3). Let $\Omega^{(a,b,c)}(X)$ be the set of all finite subsets Y of X such that $a,b,c\in Y$; $\Omega^{(b,c)}(X)$ be the set of all finite subsets Y of X such that $b,c\in Y$; $\Omega^b(X)$ be the set of all finite subsets Y of X such that $b\in Y$. For every $f\in\{(a,b,c),(b,c),b\}$ assume $\Omega_f(X)$ be a semilattice defined on $\Omega^f(X)$ by the operation of the set theoretical union.

Theorem 3.2. *Let $\breve{G}(D_i)_{i\in X}$ be the free product of dimonoids. Then*

(i) *$\breve{G}(D_i)_{i\in X}$ is a rectangular diband $FRct(X)$ of subdimonoids $H_{(a,b,c)}$, $(a,b,c)\in FRct(X)$. Every dimonoid $H_{(a,b,c)}$, $(a,b,c)\in FRct(X)$, is a semilattice $\Omega_{(a,b,c)}(X)$ of subdimonoids $H^Y_{(a,b,c)}$, $Y\in\Omega_{(a,b,c)}(X)$;*

(ii) *$\breve{G}(D_i)_{i\in X}$ is a diband $X_{\ell z,rb}$ of subdimonoids $H_{(a,b]}$, $(a,b)\in X_{\ell z,rb}$. Every dimonoid $H_{(a,b]}$, $(a,b)\in X_{\ell z,rb}$, is a semilattice $\Omega_{(a,b)}(X)$ of subdimonoids $H^Y_{(a,b]}$, $Y\in\Omega_{(a,b)}(X)$;*

(iii) *$\breve{G}(D_i)_{i\in X}$ is a diband $X_{rb,rz}$ of subdimonoids $H_{[b,c)}$, $(b,c)\in X_{rb,rz}$. Every dimonoid $H_{[b,c)}$, $(b,c)\in X_{rb,rz}$, is a semilattice $\Omega_{(b,c)}(X)$ of subdimonoids $H^Y_{[b,c)}$, $Y\in\Omega_{(b,c)}(X)$;*

(iv) *$\breve{G}(D_i)_{i\in X}$ is a left and right diband $X_{\ell z,rz}$ of subdimonoids $H_{(b]}$, $b\in X_{\ell z,rz}$. Every dimonoid $H_{(b]}$, $b\in X_{\ell z,rz}$, is a semilattice $\Omega_b(X)$ of subdimonoids $H^Y_{(b]}$, $Y\in\Omega_b(X)$.*

Proof. (i) Define a map $\eta_{FRct}:\breve{G}(D_i)_{i\in X}\to FRct(X)$ by

$[s_{\gamma_1}...s_{\gamma_l}...s_{\gamma_k}, m]\mapsto (s_{\gamma_1}j^*, s_{\gamma_m}j^*, s_{\gamma_k}j^*)$, $[s_{\gamma_1}...s_{\gamma_l}...s_{\gamma_k}, m]\in\breve{G}(D_i)_{i\in X}$.

Similarly to (i) from Th. 3.1, η_{FRct} is a surjective homomorphism. Then $H_{(a,b,c)}$, $(a,b,c) \in FRct(X)$, is a class of $\Delta_{\eta_{FRct}}$ which is a subdimonoid of $\breve{G}(D_i)_{i\in X}$. Hence $\breve{G}(D_i)_{i\in X}$ is a rectangular diband $FRct(X)$ of subdimonoids $H_{(a,b,c)}$, $(a,b,c) \in FRct(X)$. Besides, it is not difficult to show that for every $(a,b,c) \in FRct(X)$ the map

$$H_{(a,b,c)} \to \Omega_{(a,b,c)}(X) : [s_{\gamma_1}...s_{\gamma_l}...s_{\gamma_k}, m] \mapsto \tilde{c}(s_{\gamma_1}...s_{\gamma_l}...s_{\gamma_k})$$

is a homomorphism. Hence $H_{(a,b,c)}$ is a semilattice $\Omega_{(a,b,c)}(X)$ of subdimonoids $H^Y_{(a,b,c)}$, $Y \in \Omega_{(a,b,c)}(X)$.

(ii) Define a map $\eta_{\ell z,rb} : \breve{G}(D_i)_{i\in X} \to X_{\ell z,rb}$ by

$$[s_{\gamma_1}...s_{\gamma_l}...s_{\gamma_k}, m] \mapsto (s_{\gamma_1}j^*, s_{\gamma_m}j^*),\ [s_{\gamma_1}...s_{\gamma_l}...s_{\gamma_k}, m] \in \breve{G}(D_i)_{i\in X}.$$

Similarly to (i) from Th. 3.1, $\eta_{\ell z,rb}$ is a surjective homomorphism and $H_{(a,b]}$, $(a,b) \in X_{\ell z,rb}$, is a class of $\Delta_{\eta_{\ell z,rb}}$ which is a subdimonoid of $\breve{G}(D_i)_{i\in X}$. It means that $\breve{G}(D_i)_{i\in X}$ is a diband $X_{\ell z,rb}$ of subdimonoids $H_{(a,b]}$, $(a,b) \in X_{\ell z,rb}$. Moreover, we can show that for every $(a,b) \in X_{\ell z,rb}$ the map

$$H_{(a,b]} \to \Omega_{(a,b)}(X) : [s_{\gamma_1}...s_{\gamma_l}...s_{\gamma_k}, m] \mapsto \tilde{c}(s_{\gamma_1}...s_{\gamma_l}...s_{\gamma_k})$$

is a homomorphism. From here $H_{(a,b]}$ is a semilattice $\Omega_{(a,b)}(X)$ of subdimonoids $H^Y_{(a,b]}$, $Y \in \Omega_{(a,b)}(X)$.

(iii) Define a map $\eta_{rb,rz} : \breve{G}(D_i)_{i\in X} \to X_{rb,rz}$ by

$$[s_{\gamma_1}...s_{\gamma_l}...s_{\gamma_k}, m] \mapsto (s_{\gamma_m}j^*, s_{\gamma_k}j^*),\ [s_{\gamma_1}...s_{\gamma_l}...s_{\gamma_k}, m] \in \breve{G}(D_i)_{i\in X}.$$

Similarly to (i) from Th. 3.1, $\eta_{rb,rz}$ is a surjective homomorphism and $H_{[b,c)}$, $(b,c) \in X_{rb,rz}$, is a class of $\Delta_{\eta_{rb,rz}}$ which is a subdimonoid of $\breve{G}(D_i)_{i\in X}$. Hence $\breve{G}(D_i)_{i\in X}$ is a diband $X_{rb,rz}$ of subdimonoids $H_{[b,c)}$, $(b,c) \in X_{rb,rz}$. Moreover, it is not hard to verify that for every $(b,c) \in$ $\in X_{rb,rz}$ the map

$$H_{[b,c)} \to \Omega_{(b,c)}(X) : [s_{\gamma_1}...s_{\gamma_l}...s_{\gamma_k}, m] \mapsto \tilde{c}(s_{\gamma_1}...s_{\gamma_l}...s_{\gamma_k})$$

is a homomorphism. Then $H_{[b,c)}$ is a semilattice $\Omega_{(b,c)}(X)$ of subdimonoids $H^Y_{[b,c)}$, $Y \in \Omega_{(b,c)}(X)$.

(iv) Define a map $\eta_{\ell z,rz} : \breve{G}(D_i)_{i\in X} \to X_{\ell z,rz}$ by

$$[s_{\gamma_1}...s_{\gamma_l}...s_{\gamma_k}, m] \mapsto s_{\gamma_m}j^*,\ [s_{\gamma_1}...s_{\gamma_l}...s_{\gamma_k}, m] \in \breve{G}(D_i)_{i\in X}.$$

Similarly to (i) from Th. 3.1, $\eta_{\ell z,rz}$ is a surjective homomorphism. It is clear that $H_{(b]}$, $b \in X_{\ell z,rz}$, is a class of $\Delta_{\eta_{\ell z,rz}}$ which is a subdimonoid of $\breve{G}(D_i)_{i\in X}$. Hence $\breve{G}(D_i)_{i\in X}$ is a left and right diband $X_{\ell z,rz}$ of subdimonoids $H_{(b]}$, $b \in X_{\ell z,rz}$. Except this, one can verify that for every $b \in X_{\ell z,rz}$ the map

$$H_{(b]} \to \Omega_b(X) : [s_{\gamma_1}...s_{\gamma_l}...s_{\gamma_k}, m] \mapsto \tilde{c}(s_{\gamma_1}...s_{\gamma_l}...s_{\gamma_k})$$

is a homomorphism. Hence we obtain the last statement of (iv). $\Diamond$

Note that Thms. 3.1 and 3.2 extend the corresponding parts of Thms. 3–6 from [10] and [11].

The realization of Th. 3.2 may be shown by the following example.

Let us consider the dimonoid $\breve{G}(D_i)_{i\in X}$ when $\dashv_i = \vdash_i$ for all $i \in X$. First observe that if $\dashv_i = \vdash_i$ for all $i \in X$, then $\sim$ is the diagonal of $G((D_i, \dashv_i))_{i\in X}$ (see item 2.1) and $G((D_i, \dashv_i))_{i\in X}/\sim$ is identified with $G((D_i, \dashv_i))_{i\in X}$. It is clear that in this case $\dashv = \vdash$ and then operations $\dashv'$, $\vdash'$ on $G((D_i, \dashv_i))_{i\in X}$ defined by (2) and (3) take the form:

$$(w_1, m_1) \dashv' (w_2, m_2) = (w_1 \dashv w_2,\ m_1),$$
$$(w_1, m_1) \vdash' (w_2, m_2) = (w_1 \dashv w_2,\ f^{w_1}_{(w_2,m_2)}).$$

Let further $X = \{x, y\}$ and F^* be the free product of singleton dimonoids $\{x\}$ and $\{y\}$ in the variety of dimonoids. Then $FRct(X)$ consists of elements

$$(x, x, x), (x, x, y), (x, y, y), (x, y, x), (y, y, y), (y, y, x), (y, x, x), (y, x, y);$$

$$H_{(a,b,c)} = \{(s_{\gamma_1}...s_{\gamma_l}...s_{\gamma_k}, m) \in F^* \mid (s_{\gamma_1}, s_{\gamma_m}, s_{\gamma_k}) = (a, b, c)\}$$

for $(a, b, c) \in FRct(X)$;

$$\Omega^{(x,x,x)}(X) = \{\{x, y\}, \{x\}\}, \quad \Omega^{(y,y,y)}(X) = \{\{x, y\}, \{y\}\},$$
$$\Omega^{(x,x,y)}(X) = \Omega^{(x,y,y)}(X) = \Omega^{(x,y,x)}(X) = \Omega^{(y,y,x)}(X) =$$
$$= \Omega^{(y,x,x)}(X) = \Omega^{(y,x,y)}(X) = \{\{x, y\}\};$$

$$H^Y_{(a,b,c)} = \Big\{(s_{\gamma_1}...s_{\gamma_l}...s_{\gamma_k}, m) \in F^* \Big| \Big((s_{\gamma_1}, s_{\gamma_m}, s_{\gamma_k}), \bigcup_{l=1}^{k}\{s_{\gamma_l}\}\Big) = ((a, b, c), Y)\Big\}$$

for $Y \in \Omega_{(a,b,c)}(X)$, namely, the following subdimonoids are components of semilattice decompositions from (i): $H^{\{x,y\}}_{(x,x,x)}$, $H^{\{x\}}_{(x,x,x)}$, $H^{\{x,y\}}_{(y,y,y)}$, $H^{\{y\}}_{(y,y,y)}$, $H^{\{x,y\}}_{(x,x,y)}$, $H^{\{x,y\}}_{(x,y,y)}$, $H^{\{x,y\}}_{(x,y,x)}$, $H^{\{x,y\}}_{(y,y,x)}$, $H^{\{x,y\}}_{(y,x,x)}$, $H^{\{x,y\}}_{(y,x,y)}$.

Substituting obtained expressions to the statement (i) of Th. 3.2 we obtain decompositions of F^*. Similarly, the statements (ii) – (iv) of Th. 3.2 can be applied to the dimonoid F^*.

4. Band decompositions of $\breve{G}(D_i)_{i\in X}$

In this section we give decompositions of the free product $\breve{G}(D_i)_{i\in X}$ of dimonoids $(D_i, \dashv_i, \vdash_i)$, $i \in X$, into relatively free bands of subdimonoids.

Let

$H_{(a,c)} = \{[s_{\gamma_1}...s_{\gamma_l}...s_{\gamma_k}, m] \in \breve{G}(D_i)_{i\in X} \,|\, (s_{\gamma_1}j^*, s_{\gamma_k}j^*) = (a,c)\}$

for $(a,c) \in X_{rb}$;

$$H_{(a)} = \{[s_{\gamma_1}...s_{\gamma_l}...s_{\gamma_k}, m] \in \breve{G}(D_i)_{i\in X} \,|\, s_{\gamma_1}j^* = a\}$$

for $a \in X_{\ell z}$;

$$H_{[c]} = \{[s_{\gamma_1}...s_{\gamma_l}...s_{\gamma_k}, m] \in \breve{G}(D_i)_{i\in X} \,|\, s_{\gamma_k}j^* = c\}$$

for $c \in X_{rz}$;

$$H^Y_{(a,c)} = \{[s_{\gamma_1}...s_{\gamma_l}...s_{\gamma_k}, m] \in \breve{G}(D_i)_{i\in X} \,|$$
$$((s_{\gamma_1}j^*, s_{\gamma_k}j^*), \tilde{c}(s_{\gamma_1}...s_{\gamma_l}...s_{\gamma_k})) = ((a,c), Y)\}$$

for $((a,c), Y) \in B_{rb}(X)$;

$H^Y_{(a)} = \{[s_{\gamma_1}...s_{\gamma_l}...s_{\gamma_k}, m] \in \breve{G}(D_i)_{i\in X} \,|\, (s_{\gamma_1}j^*, \tilde{c}(s_{\gamma_1}...s_{\gamma_l}...s_{\gamma_k})) = (a, Y)\}$

for $(a, Y) \in B_{\ell z}(X)$;

$H^Y_{[c]} = \{[s_{\gamma_1}...s_{\gamma_l}...s_{\gamma_k}, m] \in \breve{G}(D_i)_{i\in X} \,|\, (s_{\gamma_k}j^*, \tilde{c}(s_{\gamma_1}...s_{\gamma_l}...s_{\gamma_k})) = (c, Y)\}$

for $(c, Y) \in B_{rz}(X)$.

The following two structure theorems give decompositions of free products of dimonoids into relatively free bands of subdimonoids.

Theorem 4.1. *Let $\breve{G}(D_i)_{i\in X}$ be the free product of dimonoids. Then*

(i) *$\breve{G}(D_i)_{i\in X}$ is a normal band $B_{rb}(X)$ of subdimonoids $H^Y_{(a,c)}$, $((a,c), Y) \in B_{rb}(X)$. Every dimonoid $H^Y_{(a,c)}$, $((a,c), Y) \in B_{rb}(X)$, is a left and right diband $Y_{\ell z, rz}$ of subdimonoids $H^Y_{(a,b,c)}$, $b \in Y_{\ell z, rz}$;*

(ii) *$\breve{G}(D_i)_{i\in X}$ is a left normal band $B_{\ell z}(X)$ of subdimonoids $H^Y_{(a)}$, $(a, Y) \in B_{\ell z}(X)$. Every dimonoid $H^Y_{(a)}$, $(a, Y) \in B_{\ell z}(X)$, is a diband $Y_{rb, rz}$ of subdimonoids $H^Y_{(a,b,c)}$, $(b, c) \in Y_{rb, rz}$;*

(iii) *$\breve{G}(D_i)_{i\in X}$ is a right normal band $B_{rz}(X)$ of subdimonoids $H^Y_{[c]}$, $(c, Y) \in B_{rz}(X)$. Every dimonoid $H^Y_{[c]}$, $(c, Y) \in B_{rz}(X)$, is a diband $Y_{\ell z, rb}$ of subdimonoids $H^Y_{(a,b,c)}$, $(a, b) \in Y_{\ell z, rb}$.*

Proof. (i) Define a map $\eta^*_{rb} : \breve{G}(D_i)_{i\in X} \to B_{rb}(X)$ by
$$[s_{\gamma_1}...s_{\gamma_l}...s_{\gamma_k}, m] \mapsto ((s_{\gamma_1}j^*, s_{\gamma_k}j^*), \tilde{c}(s_{\gamma_1}...s_{\gamma_l}...s_{\gamma_k})),$$
$$[s_{\gamma_1}...s_{\gamma_l}...s_{\gamma_k}, m] \in \breve{G}(D_i)_{i\in X}.$$
For arbitrary elements $[s_{\gamma_1}...s_{\gamma_l}...s_{\gamma_k}, m]$, $[s_{\alpha_1}...s_{\alpha_l}...s_{\alpha_r}, t] \in \breve{G}(D_i)_{i\in X}$ we obtain

$$\begin{aligned}
&([s_{\gamma_1}...s_{\gamma_l}...s_{\gamma_k}, m] \dashv' [s_{\alpha_1}...s_{\alpha_l}...s_{\alpha_r}, t])\eta^*_{rb} = \\
&= [s_{\gamma_1}...s_{\gamma_l}...s_{\gamma_{k-1}}(s_{\gamma_k} \dashv s_{\alpha_1})s_{\alpha_2}...s_{\alpha_l}...s_{\alpha_r}, m]\eta^*_{rb} = \\
&= ((s_{\gamma_1}j^*, s_{\alpha_r}j^*), \tilde{c}(s_{\gamma_1}...s_{\gamma_l}...s_{\gamma_{k-1}}(s_{\gamma_k} \dashv s_{\alpha_1})s_{\alpha_2}...s_{\alpha_l}...s_{\alpha_r})) = \\
&= ((s_{\gamma_1}j^*, s_{\alpha_r}j^*), \tilde{c}(s_{\gamma_1}...s_{\gamma_l}...s_{\gamma_k}) \cup \tilde{c}(s_{\alpha_1}...s_{\alpha_l}...s_{\alpha_r})) = \\
&= ((s_{\gamma_1}j^*, s_{\gamma_k}j^*), \tilde{c}(s_{\gamma_1}...s_{\gamma_l}...s_{\gamma_k}))((s_{\alpha_1}j^*, s_{\alpha_r}j^*), \tilde{c}(s_{\alpha_1}...s_{\alpha_l}...s_{\alpha_r})) = \\
&= [s_{\gamma_1}...s_{\gamma_l}...s_{\gamma_k}, m]\eta^*_{rb}[s_{\alpha_1}...s_{\alpha_l}...s_{\alpha_r}, t]\eta^*_{rb},
\end{aligned}$$

$$\begin{aligned}
&([s_{\gamma_1}...s_{\gamma_l}...s_{\gamma_k}, m] \vdash' [s_{\alpha_1}...s_{\alpha_l}...s_{\alpha_r}, t])\eta^*_{rb} = \\
&= [s_{\gamma_1}...s_{\gamma_l}...s_{\gamma_{k-1}}(s_{\gamma_k} \vdash s_{\alpha_1})s_{\alpha_2}...s_{\alpha_l}...s_{\alpha_r}, f^{s_{\gamma_1}...s_{\gamma_l}...s_{\gamma_k}}_{(s_{\alpha_1}...s_{\alpha_l}...s_{\alpha_r}, t)}]\eta^*_{rb} = \\
&= ((s_{\gamma_1}j^*, s_{\alpha_r}j^*), \tilde{c}(s_{\gamma_1}...s_{\gamma_l}...s_{\gamma_{k-1}}(s_{\gamma_k} \vdash s_{\alpha_1})s_{\alpha_2}...s_{\alpha_l}...s_{\alpha_r})) = \\
&= ((s_{\gamma_1}j^*, s_{\alpha_r}j^*), \tilde{c}(s_{\gamma_1}...s_{\gamma_l}...s_{\gamma_k}) \cup \tilde{c}(s_{\alpha_1}...s_{\alpha_l}...s_{\alpha_r})) = \\
&= ((s_{\gamma_1}j^*, s_{\gamma_k}j^*), \tilde{c}(s_{\gamma_1}...s_{\gamma_l}...s_{\gamma_k}))((s_{\alpha_1}j^*, s_{\alpha_r}j^*), \tilde{c}(s_{\alpha_1}...s_{\alpha_l}...s_{\alpha_r})) = \\
&= [s_{\gamma_1}...s_{\gamma_l}...s_{\gamma_k}, m]\eta^*_{rb}[s_{\alpha_1}...s_{\alpha_l}...s_{\alpha_r}, t]\eta^*_{rb}.
\end{aligned}$$

Consequently, η^*_{rb} is a surjective homomorphism. It is evident that $H^Y_{(a,c)}$, $((a, c), Y) \in B_{rb}(X)$, is a class of $\Delta_{\eta^*_{rb}}$ which is a subdimonoid of $\breve{G}(D_i)_{i\in X}$. Hence $\breve{G}(D_i)_{i\in X}$ is a normal band $B_{rb}(X)$ of subdimonoids $H^Y_{(a,c)}$, $((a, c), Y) \in B_{rb}(X)$. Moreover, one can show that for every $((a, c), Y) \in B_{rb}(X)$ the map
$$H^Y_{(a,c)} \to Y_{\ell z,rz} : [s_{\gamma_1}...s_{\gamma_l}...s_{\gamma_k}, m] \mapsto s_{\gamma_m}j^*$$
is a homomorphism. Hence $H^Y_{(a,c)}$ is a left and right diband $Y_{\ell z,rz}$ of subdimonoids $H^Y_{(a,b,c)}$, $b \in Y_{\ell z,rz}$.

(ii) Analysis similar that in the proof of (i) shows that a map
$$\eta^*_{\ell z} : \breve{G}(D_i)_{i\in X} \to B_{\ell z}(X),$$
defined by
$$[s_{\gamma_1}...s_{\gamma_l}...s_{\gamma_k}, m] \mapsto (s_{\gamma_1}j^*, \tilde{c}(s_{\gamma_1}...s_{\gamma_l}...s_{\gamma_k})),$$

$$[s_{\gamma_1}...s_{\gamma_l}...s_{\gamma_k}, m] \in \breve{G}(D_i)_{i\in X},$$

is a surjective homomorphism. From here $H^Y_{(a)}$, $(a, Y) \in B_{\ell z}(X)$, is a class of $\Delta_{\eta^*_{\ell z}}$ which is a subdimonoid of $\breve{G}(D_i)_{i\in X}$. Hence $\breve{G}(D_i)_{i\in X}$ is a left normal band $B_{\ell z}(X)$ of subdimonoids $H^Y_{(a)}$, $(a, Y) \in B_{\ell z}(X)$. Besides, we can verify that for every $(a, Y) \in B_{\ell z}(X)$ the map

$$H^Y_{(a)} \to Y_{rb,rz} : [s_{\gamma_1}...s_{\gamma_l}...s_{\gamma_k}, m] \mapsto (s_{\gamma_m} j^*, s_{\gamma_k} j^*)$$

is a homomorphism. Hence $H^Y_{(a)}$ is a diband $Y_{rb,rz}$ of subdimonoids $H^Y_{(a,b,c)}$, $(b, c) \in Y_{rb,rz}$.

(iii) Define a map $\eta^*_{rz} : \breve{G}(D_i)_{i\in X} \to B_{rz}(X)$ by

$$[s_{\gamma_1}...s_{\gamma_l}...s_{\gamma_k}, m] \mapsto (s_{\gamma_k} j^*, \tilde{c}(s_{\gamma_1}...s_{\gamma_l}...s_{\gamma_k})),$$

$$[s_{\gamma_1}...s_{\gamma_l}...s_{\gamma_k}, m] \in \breve{G}(D_i)_{i\in X}.$$

Similarly to the proof of (i), η^*_{rz} is a surjective homomorphism. It is evident that $H^Y_{[c]}$, $(c, Y) \in B_{rz}(X)$, is a class of $\Delta_{\eta^*_{rz}}$ which is a subdimonoid of $\breve{G}(D_i)_{i\in X}$. Thus, $\breve{G}(D_i)_{i\in X}$ is a right normal band $B_{rz}(X)$ of subdimonoids $H^Y_{[c]}$, $(c, Y) \in B_{rz}(X)$. Except this, it is obvious that for every $(c, Y) \in B_{rz}(X)$ the map

$$H^Y_{[c]} \to Y_{\ell z,rb} : [s_{\gamma_1}...s_{\gamma_l}...s_{\gamma_k}, m] \mapsto (s_{\gamma_1} j^*, s_{\gamma_m} j^*)$$

is a homomorphism. Hence $H^Y_{[c]}$ is a diband $Y_{\ell z,rb}$ of subdimonoids $H^Y_{(a,b,c)}$, $(a, b) \in Y_{\ell z,rb}$. $\Diamond$

Theorem 4.2. *Let $\breve{G}(D_i)_{i\in X}$ be the free product of dimonoids. Then*

(i) *$\breve{G}(D_i)_{i\in X}$ is a rectangular band X_{rb} of subdimonoids $H_{(a,c)}$, $(a, c) \in$ $\in X_{rb}$. Every dimonoid $H_{(a,c)}$, $(a, c) \in X_{rb}$, is a semilattice $\Omega_{(a,c)}(X)$ of subdimonoids $H^Y_{(a,c)}$, $Y \in \Omega_{(a,c)}(X)$;*

(ii) *$\breve{G}(D_i)_{i\in X}$ is a left band $X_{\ell z}$ of subdimonoids $H_{(a)}$, $a \in X_{\ell z}$. Every dimonoid $H_{(a)}$, $a \in X_{\ell z}$, is a semilattice $\Omega_a(X)$ of subdimonoids $H^Y_{(a)}$, $Y \in \Omega_a(X)$;*

(iii) *$\breve{G}(D_i)_{i\in X}$ is a right band X_{rz} of subdimonoids $H_{[c]}$, $c \in X_{rz}$. Every dimonoid $H_{[c]}$, $c \in X_{rz}$, is a semilattice $\Omega_c(X)$ of subdimonoids $H^Y_{[c]}$, $Y \in \Omega_c(X)$.*

Proof. (i) Define a map $\eta_{rb} : \breve{G}(D_i)_{i\in X} \to X_{rb}$ by

$$[s_{\gamma_1}...s_{\gamma_l}...s_{\gamma_k}, m] \mapsto (s_{\gamma_1} j^*, s_{\gamma_k} j^*),\ [s_{\gamma_1}...s_{\gamma_l}...s_{\gamma_k}, m] \in \breve{G}(D_i)_{i\in X}.$$

Similarly to the proof of (i) from Th. 4.1, η_{rb} is a surjective homomorphism. Obviously, $H_{(a,c)}$, $(a,c)\in X_{rb}$, is a class of $\Delta_{\eta_{rb}}$ which is a subdimonoid of $\breve{G}(D_i)_{i\in X}$. Hence $\breve{G}(D_i)_{i\in X}$ is a rectangular band X_{rb} of subdimonoids $H_{(a,c)}$, $(a,c)\in X_{rb}$. Moreover, it is clear that for every $(a,c)\in X_{rb}$ the map

$$H_{(a,c)}\to\Omega_{(a,c)}(X):[s_{\gamma_1}...s_{\gamma_l}...s_{\gamma_k},m]\mapsto\tilde{c}(s_{\gamma_1}...s_{\gamma_l}...s_{\gamma_k})$$

is a homomorphism. So, $H_{(a,c)}$ is a semilattice $\Omega_{(a,c)}(X)$ of subdimonoids $H^Y_{(a,c)}$, $Y\in\Omega_{(a,c)}(X)$.

(ii) Define a map $\eta_{\ell z}:\breve{G}(D_i)_{i\in X}\to X_{\ell z}$ by

$$[s_{\gamma_1}...s_{\gamma_l}...s_{\gamma_k},m]\mapsto s_{\gamma_1}j^*,\ \ [s_{\gamma_1}...s_{\gamma_l}...s_{\gamma_k},m]\in\breve{G}(D_i)_{i\in X}.$$

Similarly to the proof of (i) from Th. 4.1, $\eta_{\ell z}$ is a surjective homomorphism. It is evident that $H_{(a)}$, $a\in X_{\ell z}$, is a class of $\Delta_{\eta_{\ell z}}$ which is a subdimonoid of $\breve{G}(D_i)_{i\in X}$. Hence $\breve{G}(D_i)_{i\in X}$ is a left band $X_{\ell z}$ of subdimonoids $H_{(a)}$, $a\in X_{\ell z}$. Evidently, for every $a\in X_{\ell z}$ the map

$$H_{(a)}\to\Omega_a(X):[s_{\gamma_1}...s_{\gamma_l}...s_{\gamma_k},m]\mapsto\tilde{c}(s_{\gamma_1}...s_{\gamma_l}...s_{\gamma_k})$$

is a homomorphism. Thus, $H_{(a)}$ is a semilattice $\Omega_a(X)$ of subdimonoids $H^Y_{(a)}$, $Y\in\Omega_a(X)$.

(iii) Define a map $\eta_{rz}:\breve{G}(D_i)_{i\in X}\to X_{rz}$ by

$$[s_{\gamma_1}...s_{\gamma_l}...s_{\gamma_k},m]\mapsto s_{\gamma_k}j^*,\ \ [s_{\gamma_1}...s_{\gamma_l}...s_{\gamma_k},m]\in\breve{G}(D_i)_{i\in X}.$$

Similarly to the proof of (i) from Th. 4.1, η_{rz} is a surjective homomorphism. Then $H_{[c]}$, $c\in X_{rz}$, is a class of $\Delta_{\eta_{rz}}$ which is a subdimonoid of $\breve{G}(D_i)_{i\in X}$. So, $\breve{G}(D_i)_{i\in X}$ is a right band X_{rz} of subdimonoids $H_{[c]}$, $c\in X_{rz}$. One can check that for every $c\in X_{rz}$ the map

$$H_{[c]}\to\Omega_c(X):[s_{\gamma_1}...s_{\gamma_l}...s_{\gamma_k},m]\mapsto\tilde{c}(s_{\gamma_1}...s_{\gamma_l}...s_{\gamma_k})$$

is a homomorphism. Hence we obtain the last statement of (iii). $\Diamond$

Note that Thms. 4.1 and 4.2 extend the corresponding parts of Thms. 7–9 from [10] and [11]. The semilattice decomposition of free products of dimonoids was described in [17].

5. Free products of left zero and right zero dimonoids

In this section we give a faithful representation of free products of left zero and right zero dimonoids.

Let $\{(D_i,\dashv_i,\vdash_i)\}_{i\in X}$ be a family of arbitrary pairwise disjoint left zero and right zero dimonoids (see item 2.3). Denote by $F[(D_i,\dashv_i)]_{i\in X}$

the free product of semigroups $(D_i, \dashv_i), i \in X$, and singleton semigroups $\{i\}, i \in X$.

Let

$$\begin{aligned} R(D_i)_{i\in X} &= \{(s_{\gamma_1}...s_{\gamma_l}...s_{\gamma_k}, m) \in F[(D_i, \dashv_i)]_{i\in X} \times \mathbb{N} \,|\, k \geq m\}, \\ R^\star(D_i)_{i\in X} &= \{(s_{\gamma_1}...s_{\gamma_l}...s_{\gamma_k}, m) \in R(D_i)_{i\in X} \,| \\ &\qquad s_{\gamma_l} \in \cup_{i\in X} D_i \Leftrightarrow l = m, 1 \leq l \leq k\}, \end{aligned}$$

$$\mu : (\cup_{i\in X} D_i) \cup X \to (\cup_{i\in X} D_i) \cup X : a \mapsto a\mu = \begin{cases} i, & a \in D_i, \\ a, & a \in X. \end{cases}$$

If $k = 1$, then the sequences $s_{\gamma_1}...s_{\gamma_l}...s_{\gamma_{k-1}}$, $s_{\gamma_2}...s_{\gamma_l}...s_{\gamma_k}$ will be regarded empty.

Define operations $\dashv$ and $\vdash$ on $R^\star(D_i)_{i\in X}$ by

$$(s_{\gamma_1}...s_{\gamma_l}...s_{\gamma_k}, m) \dashv (s_{\alpha_1}...s_{\alpha_l}...s_{\alpha_r}, t) =$$
$$= \begin{cases} (s_{\gamma_1}...s_{\gamma_l}...s_{\gamma_k} s_{\alpha_1}\mu...s_{\alpha_l}\mu...s_{\alpha_r}\mu, m)\,, & s_{\gamma_k}\mu \neq s_{\alpha_1}\mu, \\ (s_{\gamma_1}...s_{\gamma_l}...s_{\gamma_k} s_{\alpha_2}\mu...s_{\alpha_l}\mu...s_{\alpha_r}\mu, m)\,, & s_{\gamma_k}\mu = s_{\alpha_1}\mu, \end{cases}$$

$$(s_{\gamma_1}...s_{\gamma_l}...s_{\gamma_k}, m) \vdash (s_{\alpha_1}...s_{\alpha_l}...s_{\alpha_r}, t) =$$
$$= \begin{cases} (s_{\gamma_1}\mu...s_{\gamma_l}\mu...s_{\gamma_k}\mu s_{\alpha_1}...s_{\alpha_l}...s_{\alpha_r}, k + t)\,, & s_{\gamma_k}\mu \neq s_{\alpha_1}\mu, \\ \big(s_{\gamma_1}\mu...s_{\gamma_l}\mu...s_{\gamma_{k-1}}\mu s_{\alpha_1}...s_{\alpha_l}...s_{\alpha_r}, k + t - 1\big)\,, & s_{\gamma_k}\mu = s_{\alpha_1}\mu \end{cases}$$

for all $(s_{\gamma_1}...s_{\gamma_l}...s_{\gamma_k}, m), (s_{\alpha_1}...s_{\alpha_l}...s_{\alpha_r}, t) \in R^\star(D_i)_{i\in X}$. The algebra $(R^\star(D_i)_{i\in X}, \dashv, \vdash)$ will be denoted by $\breve{R}(D_i)_{i\in X}$.

Theorem 5.1. *The free product $\breve{G}(D_i)_{i\in X}$ of left zero and right zero dimonoids $(D_i, \dashv_i, \vdash_i)$, $i \in X$, is isomorphic to the dimonoid $\breve{R}(D_i)_{i\in X}$.*

Proof. Observe that $\mu^2 = \mu$ and show that $\breve{R}(D_i)_{i\in X}$ is a dimonoid.

Let

$$(s_{\gamma_1} \dots s_{\gamma_l} \dots s_{\gamma_k}, m), (s_{\alpha_1} \dots s_{\alpha_l} \dots s_{\alpha_r}, t),$$

$$(s_{\beta_1} \dots s_{\beta_l} \dots s_{\beta_g}, f) \in \breve{R}(D_i)_{i\in X}.$$

Consider the following four cases.

Case 1: $s_{\gamma_k}\mu = s_{\alpha_1}\mu$, $s_{\alpha_r}\mu = s_{\beta_1}\mu$. Then

$$\begin{aligned} &((s_{\gamma_1} \dots s_{\gamma_l} \dots s_{\gamma_k}, m) \dashv (s_{\alpha_1} \dots s_{\alpha_l} \dots s_{\alpha_r}, t)) \dashv (s_{\beta_1} \dots s_{\beta_l} \dots s_{\beta_g}, f) = \\ &= (s_{\gamma_1} \dots s_{\gamma_l} \dots s_{\gamma_k} s_{\alpha_2}\mu \dots s_{\alpha_l}\mu \dots s_{\alpha_r}\mu, m) \dashv (s_{\beta_1} \dots s_{\beta_l} \dots s_{\beta_g}, f) = \\ &= (s_{\gamma_1} \dots s_{\gamma_l} \dots s_{\gamma_k} s_{\alpha_2}\mu \dots s_{\alpha_l}\mu \dots s_{\alpha_r}\mu s_{\beta_2}\mu \dots s_{\beta_l}\mu \dots s_{\beta_g}\mu, m) = \end{aligned}$$

$$= (s_{\gamma_1} \dots s_{\gamma_l} \dots s_{\gamma_k} s_{\alpha_2}\mu \dots s_{\alpha_l}\mu \dots s_{\alpha_r}\mu s_{\beta_2}\mu^2 \dots s_{\beta_l}\mu^2 \dots s_{\beta_g}\mu^2, m) =$$
$$= (s_{\gamma_1} \dots s_{\gamma_l} \dots s_{\gamma_k}, m) \dashv (s_{\alpha_1} \dots s_{\alpha_l} \dots s_{\alpha_r} s_{\beta_2}\mu \dots s_{\beta_l}\mu \dots s_{\beta_g}\mu, t) =$$
$$= (s_{\gamma_1} \dots s_{\gamma_l} \dots s_{\gamma_k}, m) \dashv ((s_{\alpha_1} \dots s_{\alpha_l} \dots s_{\alpha_r}, t) \dashv (s_{\beta_1} \dots s_{\beta_l} \dots s_{\beta_g}, f)),$$
$$(s_{\gamma_1} \dots s_{\gamma_l} \dots s_{\gamma_k}, m) \dashv ((s_{\alpha_1} \dots s_{\alpha_l} \dots s_{\alpha_r}, t) \vdash (s_{\beta_1} \dots s_{\beta_l} \dots s_{\beta_g}, f)) =$$
$$= (s_{\gamma_1} ... s_{\gamma_l} ... s_{\gamma_k}, m) \dashv (s_{\alpha_1}\mu ... s_{\alpha_l}\mu ... s_{\alpha_{r-1}}\mu s_{\beta_1} ... s_{\beta_l} ... s_{\beta_g}, r + f - 1) =$$
$$= (s_{\gamma_1} \dots s_{\gamma_l} \dots s_{\gamma_k} s_{\alpha_2}\mu^2 \dots s_{\alpha_l}\mu^2 \dots s_{\alpha_{r-1}}\mu^2 s_{\beta_1}\mu \dots s_{\beta_l}\mu \dots s_{\beta_g}\mu, m) =$$
$$= (s_{\gamma_1} \dots s_{\gamma_l} \dots s_{\gamma_k} s_{\alpha_2}\mu \dots s_{\alpha_l}\mu \dots s_{\alpha_{r-1}}\mu s_{\alpha_r}\mu s_{\beta_2}\mu \dots s_{\beta_l}\mu \dots s_{\beta_g}\mu, m),$$
$$((s_{\gamma_1} \dots s_{\gamma_l} \dots s_{\gamma_k}, m) \vdash (s_{\alpha_1} \dots s_{\alpha_l} \dots s_{\alpha_r}, t)) \dashv (s_{\beta_1} \dots s_{\beta_l} \dots s_{\beta_g}, f) =$$
$$= (s_{\gamma_1}\mu ... s_{\gamma_l}\mu ... s_{\gamma_{k-1}}\mu s_{\alpha_1} ... s_{\alpha_l} ... s_{\alpha_r}, k + t - 1) \dashv (s_{\beta_1} ... s_{\beta_l} ... s_{\beta_g}, f) =$$
$$= (s_{\gamma_1}\mu ... s_{\gamma_l}\mu ... s_{\gamma_{k-1}}\mu s_{\alpha_1} ... s_{\alpha_l} ... s_{\alpha_r} s_{\beta_2}\mu ... s_{\beta_l}\mu ... s_{\beta_g}\mu, k + t - 1) =$$
$$= (s_{\gamma_1} \dots s_{\gamma_l} \dots s_{\gamma_k}, m) \vdash (s_{\alpha_1} \dots s_{\alpha_l} \dots s_{\alpha_r} s_{\beta_2}\mu \dots s_{\beta_l}\mu \dots s_{\beta_g}\mu, t) =$$
$$= (s_{\gamma_1} \dots s_{\gamma_l} \dots s_{\gamma_k}, m) \vdash ((s_{\alpha_1} \dots s_{\alpha_l} \dots s_{\alpha_r}, t) \dashv (s_{\beta_1} \dots s_{\beta_l} \dots s_{\beta_g}, f)).$$

Case 2: $s_{\gamma_k}\mu \neq s_{\alpha_1}\mu$, $s_{\alpha_r}\mu = s_{\beta_1}\mu$. Then

$$((s_{\gamma_1} \dots s_{\gamma_l} \dots s_{\gamma_k}, m) \dashv (s_{\alpha_1} \dots s_{\alpha_l} \dots s_{\alpha_r}, t)) \dashv (s_{\beta_1} \dots s_{\beta_l} \dots s_{\beta_g}, f) =$$
$$= (s_{\gamma_1} \dots s_{\gamma_l} \dots s_{\gamma_k} s_{\alpha_1}\mu \dots s_{\alpha_l}\mu \dots s_{\alpha_r}\mu, m) \dashv (s_{\beta_1} \dots s_{\beta_l} \dots s_{\beta_g}, f) =$$
$$= (s_{\gamma_1} \dots s_{\gamma_l} \dots s_{\gamma_k} s_{\alpha_1}\mu \dots s_{\alpha_l}\mu \dots s_{\alpha_r}\mu s_{\beta_2}\mu \dots s_{\beta_l}\mu \dots s_{\beta_g}\mu, m) =$$
$$= (s_{\gamma_1} \dots s_{\gamma_l} \dots s_{\gamma_k} s_{\alpha_1}\mu \dots s_{\alpha_l}\mu \dots s_{\alpha_r}\mu s_{\beta_2}\mu^2 \dots s_{\beta_l}\mu^2 \dots s_{\beta_g}\mu^2, m) =$$
$$= (s_{\gamma_1} \dots s_{\gamma_l} \dots s_{\gamma_k}, m) \dashv (s_{\alpha_1} \dots s_{\alpha_l} \dots s_{\alpha_r} s_{\beta_2}\mu \dots s_{\beta_l}\mu \dots s_{\beta_g}\mu, t) =$$
$$= (s_{\gamma_1} \dots s_{\gamma_l} \dots s_{\gamma_k}, m) \dashv ((s_{\alpha_1} \dots s_{\alpha_l} \dots s_{\alpha_r}, t) \dashv (s_{\beta_1} \dots s_{\beta_l} \dots s_{\beta_g}, f)),$$
$$(s_{\gamma_1} \dots s_{\gamma_l} \dots s_{\gamma_k}, m) \dashv ((s_{\alpha_1} \dots s_{\alpha_l} \dots s_{\alpha_r}, t) \vdash (s_{\beta_1} \dots s_{\beta_l} \dots s_{\beta_g}, f)) =$$
$$= (s_{\gamma_1} \dots s_{\gamma_l} ... s_{\gamma_k}, m) \dashv (s_{\alpha_1}\mu ... s_{\alpha_l}\mu ... s_{\alpha_{r-1}}\mu s_{\beta_1} ... s_{\beta_l} ... s_{\beta_g}, r + f - 1) =$$
$$= (s_{\gamma_1} \dots s_{\gamma_l} \dots s_{\gamma_k} s_{\alpha_1}\mu^2 \dots s_{\alpha_l}\mu^2 \dots s_{\alpha_{r-1}}\mu^2 s_{\beta_1}\mu \dots s_{\beta_l}\mu \dots s_{\beta_g}\mu, m) =$$
$$= (s_{\gamma_1} \dots s_{\gamma_l} \dots s_{\gamma_k} s_{\alpha_1}\mu \dots s_{\alpha_l}\mu \dots s_{\alpha_{r-1}}\mu s_{\alpha_r}\mu s_{\beta_2}\mu \dots s_{\beta_l}\mu \dots s_{\beta_g}\mu, m),$$
$$((s_{\gamma_1} \dots s_{\gamma_l} \dots s_{\gamma_k}, m) \vdash (s_{\alpha_1} \dots s_{\alpha_l} \dots s_{\alpha_r}, t)) \dashv (s_{\beta_1} \dots s_{\beta_l} \dots s_{\beta_g}, f) =$$
$$= (s_{\gamma_1}\mu \dots s_{\gamma_l}\mu \dots s_{\gamma_k}\mu s_{\alpha_1} \dots s_{\alpha_l} \dots s_{\alpha_r}, k + t) \dashv (s_{\beta_1} \dots s_{\beta_l} \dots s_{\beta_g}, f) =$$
$$= (s_{\gamma_1}\mu \dots s_{\gamma_l}\mu \dots s_{\gamma_k}\mu s_{\alpha_1} \dots s_{\alpha_l} \dots s_{\alpha_r} s_{\beta_2}\mu \dots s_{\beta_l}\mu \dots s_{\beta_g}\mu, k + t) =$$
$$= (s_{\gamma_1} \dots s_{\gamma_l} \dots s_{\gamma_k}, m) \vdash (s_{\alpha_1} \dots s_{\alpha_l} \dots s_{\alpha_r} s_{\beta_2}\mu \dots s_{\beta_l}\mu \dots s_{\beta_g}\mu, t) =$$
$$= (s_{\gamma_1} \dots s_{\gamma_l} \dots s_{\gamma_k}, m) \vdash ((s_{\alpha_1} \dots s_{\alpha_l} \dots s_{\alpha_r}, t) \dashv (s_{\beta_1} \dots s_{\beta_l} \dots s_{\beta_g}, f)).$$

Case 3: $s_{\gamma_k}\mu = s_{\alpha_1}\mu$, $s_{\alpha_r}\mu \neq s_{\beta_1}\mu$. Then

$$((s_{\gamma_1} \dots s_{\gamma_l} \dots s_{\gamma_k}, m) \dashv (s_{\alpha_1} \dots s_{\alpha_l} \dots s_{\alpha_r}, t)) \dashv (s_{\beta_1} \dots s_{\beta_l} \dots s_{\beta_g}, f) =$$

$$= (s_{\gamma_1} \dots s_{\gamma_l} \dots s_{\gamma_k} s_{\alpha_2}\mu \dots s_{\alpha_l}\mu \dots s_{\alpha_r}\mu, m) \dashv (s_{\beta_1} \dots s_{\beta_l} \dots s_{\beta_g}, f) =$$
$$= (s_{\gamma_1} \dots s_{\gamma_l} \dots s_{\gamma_k} s_{\alpha_2}\mu \dots s_{\alpha_l}\mu \dots s_{\alpha_r}\mu s_{\beta_1}\mu \dots s_{\beta_l}\mu \dots s_{\beta_g}\mu, m) =$$
$$= (s_{\gamma_1} \dots s_{\gamma_l} \dots s_{\gamma_k} s_{\alpha_2}\mu \dots s_{\alpha_l}\mu \dots s_{\alpha_r}\mu s_{\beta_1}\mu^2 \dots s_{\beta_l}\mu^2 \dots s_{\beta_g}\mu^2, m) =$$
$$= (s_{\gamma_1} \dots s_{\gamma_l} \dots s_{\gamma_k}, m) \dashv (s_{\alpha_1} \dots s_{\alpha_l} \dots s_{\alpha_r} s_{\beta_1}\mu \dots s_{\beta_l}\mu \dots s_{\beta_g}\mu, t) =$$
$$= (s_{\gamma_1} \dots s_{\gamma_l} \dots s_{\gamma_k}, m) \dashv ((s_{\alpha_1} \dots s_{\alpha_l} \dots s_{\alpha_r}, t) \dashv (s_{\beta_1} \dots s_{\beta_l} \dots s_{\beta_g}, f)),$$
$$(s_{\gamma_1} \dots s_{\gamma_l} \dots s_{\gamma_k}, m) \dashv ((s_{\alpha_1} \dots s_{\alpha_l} \dots s_{\alpha_r}, t) \vdash (s_{\beta_1} \dots s_{\beta_l} \dots s_{\beta_g}, f)) =$$
$$= (s_{\gamma_1} \dots s_{\gamma_l} \dots s_{\gamma_k}, m) \dashv (s_{\alpha_1}\mu \dots s_{\alpha_l}\mu \dots s_{\alpha_r}\mu s_{\beta_1} \dots s_{\beta_l} \dots s_{\beta_g}, r + f) =$$
$$= (s_{\gamma_1} \dots s_{\gamma_l} \dots s_{\gamma_k} s_{\alpha_2}\mu^2 \dots s_{\alpha_l}\mu^2 \dots s_{\alpha_r}\mu^2 s_{\beta_1}\mu \dots s_{\beta_l}\mu \dots s_{\beta_g}\mu, m) =$$
$$= (s_{\gamma_1} \dots s_{\gamma_l} \dots s_{\gamma_k} s_{\alpha_2}\mu \dots s_{\alpha_l}\mu \dots s_{\alpha_r}\mu s_{\beta_1}\mu \dots s_{\beta_l}\mu \dots s_{\beta_g}\mu, m),$$
$$((s_{\gamma_1} \dots s_{\gamma_l} \dots s_{\gamma_k}, m) \vdash (s_{\alpha_1} \dots s_{\alpha_l} \dots s_{\alpha_r}, t)) \dashv (s_{\beta_1} \dots s_{\beta_l} \dots s_{\beta_g}, f) =$$
$$= (s_{\gamma_1}\mu \dots s_{\gamma_l}\mu \dots s_{\gamma_{k-1}}\mu s_{\alpha_1} \dots s_{\alpha_l} \dots s_{\alpha_r}, k + t - 1) \dashv (s_{\beta_1} \dots s_{\beta_l} \dots s_{\beta_g}, f) =$$
$$= (s_{\gamma_1}\mu \dots s_{\gamma_l}\mu \dots s_{\gamma_{k-1}}\mu s_{\alpha_1} \dots s_{\alpha_l} \dots s_{\alpha_r} s_{\beta_1}\mu \dots s_{\beta_l}\mu \dots s_{\beta_g}\mu, k + t - 1) =$$
$$= (s_{\gamma_1} \dots s_{\gamma_l} \dots s_{\gamma_k}, m) \vdash (s_{\alpha_1} \dots s_{\alpha_l} \dots s_{\alpha_r} s_{\beta_1}\mu \dots s_{\beta_l}\mu \dots s_{\beta_g}\mu, t) =$$
$$= (s_{\gamma_1} \dots s_{\gamma_l} \dots s_{\gamma_k}, m) \vdash ((s_{\alpha_1} \dots s_{\alpha_l} \dots s_{\alpha_r}, t) \dashv (s_{\beta_1} \dots s_{\beta_l} \dots s_{\beta_g}, f)).$$

Case 4: $s_{\gamma_k}\mu \neq s_{\alpha_1}\mu$, $s_{\alpha_r}\mu \neq s_{\beta_1}\mu$. Then

$$((s_{\gamma_1} \dots s_{\gamma_l} \dots s_{\gamma_k}, m) \dashv (s_{\alpha_1} \dots s_{\alpha_l} \dots s_{\alpha_r}, t)) \dashv (s_{\beta_1} \dots s_{\beta_l} \dots s_{\beta_g}, f) =$$
$$= (s_{\gamma_1} \dots s_{\gamma_l} \dots s_{\gamma_k} s_{\alpha_1}\mu \dots s_{\alpha_l}\mu \dots s_{\alpha_r}\mu, m) \dashv (s_{\beta_1} \dots s_{\beta_l} \dots s_{\beta_g}, f) =$$
$$= (s_{\gamma_1} \dots s_{\gamma_l} \dots s_{\gamma_k} s_{\alpha_1}\mu \dots s_{\alpha_l}\mu \dots s_{\alpha_r}\mu s_{\beta_1}\mu \dots s_{\beta_l}\mu \dots s_{\beta_g}\mu, m) =$$
$$= (s_{\gamma_1} \dots s_{\gamma_l} \dots s_{\gamma_k} s_{\alpha_1}\mu \dots s_{\alpha_l}\mu \dots s_{\alpha_r}\mu s_{\beta_1}\mu^2 \dots s_{\beta_l}\mu^2 \dots s_{\beta_g}\mu^2, m) =$$
$$= (s_{\gamma_1} \dots s_{\gamma_l} \dots s_{\gamma_k}, m) \dashv (s_{\alpha_1} \dots s_{\alpha_l} \dots s_{\alpha_r} s_{\beta_1}\mu \dots s_{\beta_l}\mu \dots s_{\beta_g}\mu, t) =$$
$$= (s_{\gamma_1} \dots s_{\gamma_l} \dots s_{\gamma_k}, m) \dashv ((s_{\alpha_1} \dots s_{\alpha_l} \dots s_{\alpha_r}, t) \dashv (s_{\beta_1} \dots s_{\beta_l} \dots s_{\beta_g}, f)),$$
$$(s_{\gamma_1} \dots s_{\gamma_l} \dots s_{\gamma_k}, m) \dashv ((s_{\alpha_1} \dots s_{\alpha_l} \dots s_{\alpha_r}, t) \vdash (s_{\beta_1} \dots s_{\beta_l} \dots s_{\beta_g}, f)) =$$
$$= (s_{\gamma_1} \dots s_{\gamma_l} \dots s_{\gamma_k}, m) \dashv (s_{\alpha_1}\mu \dots s_{\alpha_l}\mu \dots s_{\alpha_r}\mu s_{\beta_1} \dots s_{\beta_l} \dots s_{\beta_g}, r + f) =$$
$$= (s_{\gamma_1} \dots s_{\gamma_l} \dots s_{\gamma_k} s_{\alpha_1}\mu^2 \dots s_{\alpha_l}\mu^2 \dots s_{\alpha_r}\mu^2 s_{\beta_1}\mu \dots s_{\beta_l}\mu \dots s_{\beta_g}\mu, m) =$$
$$= (s_{\gamma_1} \dots s_{\gamma_l} \dots s_{\gamma_k} s_{\alpha_1}\mu \dots s_{\alpha_l}\mu \dots s_{\alpha_r}\mu s_{\beta_1}\mu \dots s_{\beta_l}\mu \dots s_{\beta_g}\mu, m),$$
$$((s_{\gamma_1} \dots s_{\gamma_l} \dots s_{\gamma_k}, m) \vdash (s_{\alpha_1} \dots s_{\alpha_l} \dots s_{\alpha_r}, t)) \dashv (s_{\beta_1} \dots s_{\beta_l} \dots s_{\beta_g}, f) =$$
$$= (s_{\gamma_1}\mu \dots s_{\gamma_l}\mu \dots s_{\gamma_k}\mu s_{\alpha_1} \dots s_{\alpha_l} \dots s_{\alpha_r}, k + t) \dashv (s_{\beta_1} \dots s_{\beta_l} \dots s_{\beta_g}, f) =$$
$$= (s_{\gamma_1}\mu \dots s_{\gamma_l}\mu \dots s_{\gamma_k}\mu s_{\alpha_1} \dots s_{\alpha_l} \dots s_{\alpha_r} s_{\beta_1}\mu \dots s_{\beta_l}\mu \dots s_{\beta_g}\mu, k + t) =$$
$$= (s_{\gamma_1} \dots s_{\gamma_l} \dots s_{\gamma_k}, m) \vdash (s_{\alpha_1} \dots s_{\alpha_l} \dots s_{\alpha_r} s_{\beta_1}\mu \dots s_{\beta_l}\mu \dots s_{\beta_g}\mu, t) =$$
$$= (s_{\gamma_1} \dots s_{\gamma_l} \dots s_{\gamma_k}, m) \vdash ((s_{\alpha_1} \dots s_{\alpha_l} \dots s_{\alpha_r}, t) \dashv (s_{\beta_1} \dots s_{\beta_l} \dots s_{\beta_g}, f)).$$

Comparing these expressions we conclude that the associativity of the operation $\dashv$ and two axioms of a dimonoid hold.

Similarly, the associativity of the operation $\vdash$ and the remaining axiom of a dimonoid can be checked. So, $\breve{R}(D_i)_{i\in X}$ is a dimonoid.

It is clear that $\theta_i^\star = D_i \times D_i$ for every $(D_i, \dashv_i, \vdash_i)$, $i \in X$, (see item 2.1) and so, $(D_i, \dashv_i, \vdash_i)/\theta_i^\star$ is a singleton dimonoid. Using this fact and the notation from Sec. 3, one can prove that the map

$$\breve{G}(D_i)_{i\in X} \to \breve{R}(D_i)_{i\in X} :$$

$$[s_{\gamma_1}...s_{\gamma_l}...s_{\gamma_k}, m] \mapsto (s_{\gamma_1}j^*...s_{\gamma_{m-1}}j^* s_{\gamma_m} s_{\gamma_{m+1}}j^*...s_{\gamma_k}j^*, m)$$

is an isomorphism. $\Diamond$

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