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SEMIPRIMARY TAME NEARRINGS AND N-GROUPS

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Abstract: Semiprimary N-groups are those without non-zero submodules which are ring modules while a primary one is semiprimary and has no non-trivial disjoint submodules. Sections two to four introduce some useful theory needed later. Section five decomposes semiprimary N-groups in terms of primary ones. Six is about primary N-groups where there is no finiteness condition. Seven is the same except now $DCCR$ in N is assumed. Eight is about pointed N-groups (defined there). Essentially these are primary. Nine and ten prove the uniqueness of Fitting factors for certain perfect N-groups. This provides the uniqueness of certain semiprimary N-groups. The last section shows that in quite a general situation N being centerless means it is semiprimary.

1. Introduction

Throughout this paper nearrings are left distributive, zero-symmetric and have an identity. As is usual in nearring theory arbitrary groups are written additively. The set of non-zero elements of a group W is denoted by W^*. The additive convention also applies to N-groups. They will necessarily be taken as unitary. Most of the terminology we shall use is standard. The older convention of calling N-ideals of N-groups submodules is used. If V (if N) is an N-group (a nearring) and Γ a

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subset of V (of N) then $S(\Gamma)$ (then $R(\Gamma)$) will denote the submodule (right ideal) of V (of N) generated by Γ. When $\Gamma = \{w\}$ is a singleton subset $S(\Gamma)$ ($R(\Gamma)$) is denoted by $S(w)$ (by $R(w)$). A further notational convention regarding N-groups relates to elements that have been called distibutors. Here, if V is an N-group, w_i, $i = 1, 2$, in V and α in N, then $(w_1, w_2 : \alpha)$ is the element $(w_1 + w_2)\alpha - w_1\alpha - w_2\alpha$ of V. Something also needs to be said about certain special N-groups. A type 2 N-group is called minimal and the J_2 radical of a nearring N denoted by $J(N)$. Also an N-group which is a direct sum of N-isomorphic minimal N-groups is called homogeneous.

An N-subgroup (submodule) of an N-group is called a ring N-subgroup (submodule) if as an N-group it is a ring module. Here what is meant by a nonring N-subgroup (submodule) is clear. For nearrings we have ring and nonring right N-subgroups, right ideals and ideals. The definition of ring submodules allows the introduction of semiprimary N-groups. This is to a large extent what this paper is about. An N-group without non-zero ring submodules will be called semiprimary. Such N-groups are really quite general. A natural way to obtain such an N-group from any N-group V is to keep factoring out ring submodules (continued transfinitely if necessary) until a situation is reached where the N-homomorphic image V/Γ (Γ a submodule of V) has no non-zero ring submodules. Here Γ is in fact independent of the exact process used to factor out submodules. Γ is in its own right an important submodule of V (a type of radical). In this way we obtain the semiprimary N-group V/Γ.

What this paper does is study in reasonable depth semiprimary N-groups (thus nearrings). Much of it relates to certain tame situations (definitions covered below). Semiprimary N-groups will be seen in Sec. 5 to, in some sense, decompose into primary N-groups. These are defined in that section and much of this paper relates to such N-groups. The theory developed in this paper is not superficial. Indeed three sections that introduce necessary theory are required. This theory (introduced in Sections 2, 3 and 4) is of interest in its own right.

The definition of tame, n-tame ($n \geq 2$ an integer) and compatible N-groups is taken as well known (see Sec. 5 of [10]). Basic theory laid down in [10] is, on occasions, made use of here. Tame notions are really fairly general. Examples of tameness can be found in Sec. 5 of [10]. As is common in tame theory a nearring with a faithful tame (n-tame, compatible) N-group V is called tame (n-tame, compatible) or tame (n-tame, compatible) on V.

Three notions that will be of use to us are worth mentioning. They occur in [10]. In Sec. 6 of [10] centrality is considered. We shall have need to make use here of central sums and the fact that a submodule $\leq V_1 \oplus V_2$ (V_i, $i = 1, 2$, N-groups) intersecting the V_i trivially is central. The second notion is that of factors of an N-group. These are defined in Sec. 5 of [10]. Here we say two N-groups are coprime if no minimal factor (see [10]) of one is N-isomorphic to such a factor of the other. A top factor of an N-group V will be one of the form V/H (H a submodule of V). The third notion used here is the Fitting submodule of a 3-tame N-group V (see Sec. 33 of [10]). This is denoted by $F(V)$. The Fitting factor $V/F(V)$ will come into play later.

We finish this section by noting that the proof of 39.2 of [10] underlies that of one of this papers main theorems (see 10.3). In proving 10.3 explanation will be given as to how this is so.

2. Centralizers

There is a notion that has had a fairly immense impact on tame theory. It has been the subject of a number of investigations. It would be easy to say nothing more is to be said about this notion. However there are matters relating to it that are not all that well known. One of these will be the main theorem of this section. Th. 2.2 will be used again in Sec. 6.

The notion referred to above is that of a centralizer. Its very basic definition is available in all N-groups. Because perhaps, the fundamental nature of this definition has not been appreciated, it is now given. If V is an N-group and S a subset of V then an N-subgroup U of V will be said to centralize S if $(h + u)\alpha = h\alpha + u\alpha$ for all h in S, u in U and α in N. Since V is unitary this definition can be replaced by one of three others (interchange h and u on the left/right). The centralizer of S in V is now taken as the union of all N-subgroups of V centralizing S. This N-subset of V is denoted by $C_V(S)$. A well known result is the following:

Theorem 2.1. *If V is a 3-tame N-group and S a subset of V then $C_V(S)$ is an N-subgroup of V.*

This result is 31.1 of [10]. Indeed in Sec. 31 of [10] we have quite a number of results on centralizers. Here it is seen (see 31.4 of [10]) that a theorem like 2.1 holds with V replaced by a 3-tame nearring (the right N-subgroup obtained is a right ideal). However although this holds for 3-tame nearrings it is not necessarily true for N-homomorphic images of them. More information on centralizers can be gleaned from the above reference.

A result that is known (see 3.3.2 of [5]) but has not in any way much been circulated is the one that follows. As it would seem to be important, coverage is desirable.

Theorem 2.2. *If V is a 3-tame N-group and S a non-empty subset of V, then $C_V(S) = C_V(W)$ where W is the N-subgroup of V generated by S.*

Proof. Suppose it has been shown that for u in S, $uN \leq C_V(C_V(S))$. If this holds then by Th. 2.1, $W \leq C_V(C_V(S))$ and W centralizes $C_V(S)$. This means that $C_V(S)$ centralizes W and $C_V(W) \geq C_V(S)$. Since $C_V(W) \leq C_V(S)$, the result will follow.

Let α and β be in N and let $\gamma = (1+\alpha)\beta - \alpha\beta - \beta$. Now for w in $C_V(S)$, $(u+w)\gamma = u\gamma + w\gamma$. If v_i, $i = 1,2,3$, are any given elements of V then the 3-tame assumption implies that there exists λ_1 in N such that

$$(v_i + (u+w)\alpha)\beta - (u+w)\alpha\beta - v_i\beta = v_i\lambda_1.$$

Furthermore suppose λ_1 is chosen so that $v_1 = u+w$, $v_2 = u$ and $v_3 = w$. Now from the definition of $C_V(S)$ we have $(u+w)\lambda_1 = (u+w)\gamma = = u\lambda_1 + w\lambda_1$. Thus

$$\begin{aligned}(u+w)\gamma = (u + (u+w)\alpha)\beta - (u+w)\alpha\beta - u\beta + \\ + (w + (u+w)\alpha)\beta - (u+w)\alpha\beta - w\beta.\end{aligned}$$

Again with v_i, $i = 1,2,3$, in V we can find λ_2 in N such that

$$(u + v_i\alpha)\beta - v_i\alpha\beta - u\beta = v_i\lambda_2.$$

Furthermore suppose λ_2 is chosen so that $v_1 = u+w$, $v_2 = u$ and $v_3 = w$. Since $(u+w)\lambda_2 = u\lambda_2 + w\lambda_2$ we have

$$\begin{aligned}(u + (u+w)\alpha)\beta - (u+w)\alpha\beta - u\beta = (u + u\alpha)\beta - u\alpha\beta - u\beta + \\ + (u + w\alpha)\beta - w\alpha\beta - u\beta\end{aligned}$$

and a similar argument shows,

$$\begin{aligned}(w + (u+w)\alpha)\beta - (u+w)\alpha\beta - w\beta = (w + u\alpha)\beta - u\alpha\beta - w\beta + \\ + (w + w\alpha)\beta - w\alpha\beta - w\beta.\end{aligned}$$

We therefore conclude (see the expansion of $(u+w)\gamma$) that

$$(u+w)\gamma = u\gamma + (u+w\alpha)\beta - w\alpha\beta - u\beta + \\ + (w+u\alpha)\beta - u\alpha\beta - w\beta + w\gamma.$$

However, $w\alpha$ is in $C_V(S)$ and therefore $(u+w\alpha)\beta - w\alpha\beta - u\beta = 0$. Thus $(w+u\alpha)\beta = w\beta + u\alpha\beta$ for all w in $C_V(S)$ and α and β in N. It follows that $uN \le C_V(C_V(S))$ and the theorem is completely proved. $\Diamond$

3. Internal N-homomorphisms

Let N be a nearring. The category of N-groups is specified in the normal way. In this category the morphisms are N-homomorphisms between N-groups. In general it is not possible to say much about such maps. Even when the N-homomorphisms are N-endomorphisms this is the case. However it is a very pleasing fact that much more can be said when we are dealing with 2-tame N-groups. Here N-endomorphisms behave particularly well. But more is true. This desirable behavior extends to a wide range of maps. It is these maps this section is about.

From what is outlined above it might be expected that in tame theory N-endomorphisms play an important role. That this is the case is fairly well known. It stems from the fact that if on the 2-tame N-group V, μ is such a map then $1-\mu$ is also and V is a central sum of $V\mu$ and $V(1-\mu)$ (see [7] or [10]). What is not so well known is that a valuable generalization of the notion of an N-endomorphism exists that allows us to deduce this result and considerably more. This is the notion of an internal N-homomorphism.

An internal N-homomorphism is a triple (V, W, μ) where V is an N-group, W is an N-subgroup of V and μ is an N-homomorphism of W into V. The internal N-homomorphism (V, W, μ) is called n-tame (compatible) if V is n-tame (compatible). In this section it is 2-tame internal N-homomorphisms that will interest us.

Proposition 3.1. *If (V, W, μ) is a 2-tame internal N-homomorphism, then*

$$(w_1 + w_2\mu)\alpha - w_1\alpha = (w_1\mu + w_2\mu)\alpha - w_1\mu\alpha$$

for all w_i, $i = 1, 2$, in W and α in N.

The proof of this follows easily from the fact that there exists β in N such that $(w_1+w_2\mu)\alpha - w_1\alpha = w_2\mu\beta$ and $(w_1+w_2)\alpha - w_1\alpha = w_2\beta$. Also 3.1 has certain elementary consequences. One is that $w_1(1-\mu)$ additively centralizes $w_2\mu$ as may be seen on taking $\alpha = 1$. This implies (we further take $w_2 = -w_1$) that $w_1\mu$ additively centralizes w_1. The first deduction is readily seen to imply $(w_1 + w_2)(1 - \mu) = w_1 - w_1\mu + w_2 - w_2\mu$ so that $1 - \mu$ is a group homomorphism of W into V. However taking $w_2 = -w_1$ in 3.1 gives $w_1(1-\mu)\alpha - w_1\alpha = -w_1\alpha\mu$ so that $w_1(1-\mu)\alpha = w_1\alpha(1-\mu)$.

Proposition 3.2. *If (V, W, μ) is a 2-tame internal N-homomorphism then so is $(V, W, 1 - \mu)$.*

If in Prop. 3.1, w_2 is taken as $-w_1 + w_2$ we obtain the fact that $(w_1(1 - \mu) + w_2\mu)\alpha - w_1\alpha = w_2\mu\alpha - w_1\mu\alpha$ which in conjunction with explanation above supplies the fact that the sum $W(1-\mu)+W\mu$ is central. Any w in W can be written as $w(1-\mu)+w\mu$ so that $W \leq W(1-\mu)+W\mu$ and $W + W\mu \leq W(1 - \mu) + W\mu$. However as $W(1 - \mu)$ is $\leq W + W\mu$ we have equality. It follows that:

Proposition 3.3. *If (V, W, μ) is a 2-tame internal N-homomorphism then $W + W\mu$ is a central sum of $W\mu$ and $W(1 - \mu)$.*

Propositions 3.1 to 3.3 cover known theory on N-endomorphisms of 2-tame N-groups. Here if V is such an N-group with N-endomorphism μ then 3.2 and 3.3 show $1 - \mu$ is an N-endomorphism on V and V is a central sum of $V\mu$ and $V(1 - \mu)$. This is just 1.2 of [7] or 6.6 of [10].

We finish this section with an application of internal N-homomorphisms (see 3.5). In order to do this a preliminary result (see 3.4) is needed. The application that is made is now motivated.

When V is a 2-tame N-group and μ an N-endomorphism of V onto V it is clear that $V(1 - \mu)$ is $\leq Z(V)$. So much for onto maps but one can ask how are things where μ is an N-isomorphism into V. It will turn out that $V(1 - \mu)$ is again $\leq Z(V)$. The proof of this stems from how central sums behave under internal N-homomorphisms. The proposition covering this states:

Proposition 3.4. *If in the 2-tame N-group V the sum $W_1 + W_2$ (W_i, $i = 1, 2$, submodules of V) is central and (V, W_i, μ_i) are internal N-homomorphisms, then the sum $W_1\mu_1 + W_2\mu_2$ is central.*

Proof. If it is shown $W_1\mu_1+W_2$ is a central sum then the same argument will give $W\mu_1 + W\mu_2$ as such a sum.

We have $(w_1 + w_2)\alpha - w_2\alpha - w_1\alpha = 0$ for all w_i, $i = 1, 2$, in W_i and α in N. $w_1\alpha$ and $(w_1 + w_2)\alpha - w_2\alpha$ are in W_1 so $[(w_1 + w_2)\alpha - w_2\alpha]\mu_1 - -w_1\mu_1\alpha = 0$. However, there exists β in N such that $(w_1+w_2)\alpha - w_2\alpha = = w_1\beta$ and $(w_1\mu_1 + w_2)\alpha - w_2\alpha = w_1\mu_1\beta$ so that it becomes possible to show $(w_1\mu_1+w_2)\alpha - w_2\alpha - w_1\mu_1\alpha = 0$ thereby implying the sum $W_1\mu_1+W_2$ is central. The proof of 3.4 is complete. $\Diamond$

It should be noted that if V is an N-group and W an N-subgroup of V then $(V, W, 1)$ (1 the identity map of W into V) is an internal N-homomorphism. In applications of 3.4 (only one made here) it tends to be the case that (V, W_2, μ_2) is $(V, W_2, 1)$.

Coming back to the question raised at the outset we have:

Corollary 3.5. *If V is a 2-tame N-group and μ an N-isomorphism of V into V then $V(1-\mu) \leq Z(V)$.*

Proof. Because the sum $V\mu + V(1-\mu)$ is central, $(V, V\mu, \mu^{-1})$ and $(V, V(1-\mu), 1)$ are 2-tame internal N-homomorphisms and by 3.4 the sum $V + V(1-\mu)$ is central, the corollary is proved. $\Diamond$

4. A 3-tame identity

In the last section it was seen that the 2-tame condition has consequences. Results that hold for 2-tame certainly hold for 3-tame. However the 3-tame assumption yields more. The simple step from 2 to 3-tame appears to be significant. This step would seem to supply us with greater freedom. Certain manipulation of algebraic expressions can take place. These manipulations supply valuable information. An example of this is the proof of 2.1 (found in [10]) that centralizers are N-subgroups. However, there is another important example. Quite straightforward manipulation of the 3-tame assumption supplies us with a useful identity.

Theorem 4.1. *If V is a 3-tame N-group then*

$$(v+u+w)\alpha = (v+u)\alpha - v\alpha + (v+w)\alpha$$

for all v in V, u in V, w in $C_V(u)$ and α in N.

Proof. For any given v_i, $i = 1, 2, 3$, in V we may find a γ of N such that $(v+v_i)\alpha - v\alpha = v_i\gamma$, $i = 1, 2, 3$. Let γ be chosen such that $v_1 = u + w$, $v_2 = u$ and $v_3 = w$. Since $w \in C_V(u)$ it follows that $(u+w)\gamma = u\gamma + w\gamma$. From the above definition of γ this means

$$(v+u+w)\alpha - v\alpha = (v+u)\alpha - v\alpha + (v+w)\alpha - v\alpha.$$

The theorem follows. $\Diamond$

An immediate corollary of 4.1 is the following:

Corollary 4.2. *If V is a 3-tame N-group and U a ring submodule of V then*

$$(v + u_1 + u_2)\alpha = (v + u_1)\alpha - v\alpha + (v + u_2)\alpha$$

for all v in V, u_i, $i = 1, 2$, in U and α in N.

The identity of 4.2 is rather useful. It will come into play in the proof of the main result of Sec. 8. There is however at least one other application of 4.1 that is needed. To motivate this consequence it is helpful to look at groups. In group theory it can be very important to decide when a subgroup is normal. There is one situation where a subgroup normal in a subgroup is in fact normal in the group. This occurs when $V = H + A$ (V a group, H a subgroup and A a normal abelian subgroup). Here $H \cap A$ is certainly normal in H. It is a valuable fact that it is also normal in V. The fact that something like this holds for 3-tame nearrings is also valuable. In the 3-tame result centralizers of subsets of such a nearring appear and the fact that they are right ideals is used (see Sec. 31 of [10]). The result is:

Theorem 4.3. *Suppose M_i, $i = 1, 2$, are right N-subgroups of the 3-tame nearring N and H is a submodule of M_1. If $C_N(H) \geq M_2$ then H is a submodule of $M_1 + R(M_2)$.*

Proof. Suppose V is a faithful 3-tame N-group. By 31.4 of [10] we have $C_N(H)$ is a right ideal of N. Thus $C_N(H) \geq R(M_2)$. It is clear from this that vH centralizers $vR(M_2)$ for all v in V. Thus by 4.1

$$(v\beta + v\eta_1 + v\eta)\alpha = (v\beta + v\eta_1)\alpha - v\beta\alpha + (v\beta + v\eta)\alpha$$

for all α in N, β in M_1, η in $R(M_2)$ and η_1 in H. As this is true for all v in V it follows that

$$(\beta + \eta_1 + \eta)\alpha = (\beta + \eta_1)\alpha - \beta\alpha + (\beta + \eta)\alpha$$

and because $\eta_1 + \eta = \eta + \eta_1$

$$(\beta + \eta + \eta_1)\alpha - (\beta + \eta)\alpha = (\beta + \eta_1)\alpha - \beta\alpha.$$

However $\beta + \eta$ is an arbitrary element of $M_1 + R(M_2)$ and $(\beta + \eta_1)\alpha - \beta\alpha$ is in H. Thus the above expression yields the fact that H is a submodule of $M_1 + R(M_2)$. The theorem is proved. $\diamondsuit$

With N a nearring, M a right N-subgroup of N and R a right ideal of N, it is certainly true that $M \cap R$ is a submodule of M. Thus an elementary corollary of 4.3 is:

Corollary 4.4. *Suppose N is a 3-tame nearring and R a ring right ideal of N. If $M + R = N$ then $M \cap R$ is a right ideal of N.*

5. Semiprimary subdirect decomposition

If N is a nearring then what it means for the N-group V to be semiprimary may be found in the introduction. Even in this generality a substantial theorem about such N-groups can be given. In order to provide this result primary N-groups (defined shortly) are introduced. First however a characterization of semiprimary is given. This depends on the notion of an *sp*-system.

An *sp*-system for the N-group V, is a subset P of V such that, for any w in P there exists w_i, $i = 1, 2$, in $S(w)$ and α in N with $(w_1, w_2 : \alpha)$ in P.

Proposition 5.1. *The N-group V is semiprimary if and only if V^* is an sp-system of V.*

Proof. Clearly for any $w \neq 0$ in V, no $S(w)$ is such that $(w_1, w_2 : \alpha) = 0$ for all w_i, $i = 1, 2$, in $S(w)$ and α in N. Thus if V is semiprimary V^* is an *sp*-system.

On the other hand if V^* is an *sp*-systen but $U \neq \{0\}$ a ring submodule of V then for any w in U^* we obtain the contradiction that $(w_1, w_2 : \alpha) = 0$ for all w_i, $i = 1, 2$, in $S(w)$ and α in N. The proposition holds. $\diamondsuit$

The goal of this section is to obtain a subdirect decomposition of semiprimary N-groups into simpler N-groups (viz. primary ones). These are now defined but in order to do this the notion of uniformity is first introduced. A submodule U of an N-group V will be called uniform in V if no two non-zero submodules of V contained in U have zero intersection. When V is uniform in V we call V uniform. A primary N-group is now taken as one which is both semiprimary and uniform.

A notion similar to *sp*-systems exists in the case of primary N-groups. These *p*-systems (primary systems) are of very real use in proving this section's theorem. In a sense, the definition follows that which is used in prime nearring and ring situations (e.g. look at page 65 of [4]). Indeed in a number of respects our decomposing semiprimary N-groups into primary ones is similar to what is presented in [4] for semiprime nearrings.

A subset P of an N-group V will be called a *p*-system if for all u_i, $i = 1, 2$, in P there exists w_i, $i = 1, 2$, in $S(u_i)$ and α in N with $(w_1, w_2 : \alpha)$ in P. A manner in which *p*-systems impinge on present developments is given in the next lemma.

Lemma 5.2. *If V is an N-group and $P \subseteq V^*$ a p-system, then there exists a submodule U of V such that $U \cap P = \emptyset$ and V/U is primary.*

Proof. It follows from Zorn's lemma that there exists a submodule U of V maximal for avoiding P entirely. We shall show V/U is primary. Let $H > U$ be a submodule of V. For h_i, $i = 1, 2$, in $H \cap P$ there will exist v_i, $i = 1, 2$, in $S(h_i)$ and α in N such that $(v_1, v_2 : \alpha)$ is in P. But this element of P is also in H and cannot be in U. Thus H/U is nonring and V is semiprimary.

Now if H_i, $i = 1, 2$, are submodules of V properly containing U then for u_i, $i = 1, 2$, in $H_i \cap P$ there must exist w_i, $i = 1, 2$, in $S(u_i)$ and β in N such that $(w_1, w_2 : \beta)$ is in P. Because this element of P is in $H_1 \cap H_2$ it cannot be the case that $H_1 \cap H_2 = U$. Thus $H_1 \cap H_2 > U$ and V/U is uniform. The lemma is completely proved. $\Diamond$

The next lemma, which shows *sp*-systems give rise to *p*-systems, is a key step in verifying the main result of this section (see Th. 5.4).

Lemma 5.3. *Let P_1 be an sp-system of the N-group V and u be in P_1. There exists a p-system $P \subseteq P_1$ of V with u in P.*

Proof. Since u is in P_1 there exists u_{1i}, $i = 1, 2$, in $S(u)$ and α_1 in N such that $(u_{11}, u_{12} : \alpha_1)$ is in P_1. Similarly there exists u_{2i}, $i = 1, 2$, in $S(h_1)$ (here $h_1 = (u_{11}, u_{12} : \alpha_1)$) and α_2 in N such that $(u_{21}, u_{22} : \alpha_2)$ is in P_1. Continuing in this way we construct a sequence

$$u,\ (u_{11}, u_{12} : \alpha_1)(= h_1),\ (u_{21}, u_{22} : \alpha_2), \ldots$$

of elements of P_1. Furthermore $S(u) \geq S(h_1) \geq S(h_2) \geq \ldots$, where $h_2 = (u_{21}, u_{22} : \alpha_2)$, etc. Take $P = \{u, h_1, h_2, \ldots\}$. If b_i, $i = 1, 2$, are in P then from above $S(b_1)$ and $S(b_2)$ contain $S(h_m)$ for some integer $m \geq 1$. Thus $S(b_1)$ and $S(b_2)$ contain $u_{m+1,1}$ and $u_{m+1,2}$ and h_{m+1} is in P. This means P is a *p*-system and the proof is complete. $\Diamond$

The main result of this section is now easily established.

Theorem 5.4. *A semiprimary N-group is a subdirect product of primary N-groups.*

Proof. Let V be a semiprimary N-group. By 5.1, V^* is an *sp*-system. Lemma 5.3 implies that for each v in V^* there exists a *p*-system $P_v \subseteq V^*$ containing v. Thus by 5.2 there exists a submodule U_v of V such that V/U_v is primary and v is not in U_v. Because the intersection over all v in V^* of the U_v is $\{0\}$ the theorem holds. $\Diamond$

The goal of this section has been achieved. Semiprimary N-groups are very general and consequently subdirect components deserve attention. In the next two sections considerations focus on these primary N-groups. It is certain tame primary N-groups that are considered.

6. Primary N-groups

This section is concerned with tame primary N-groups. In it situations not encompassed by chain conditions are considered. In a fairly distinct way it divides into two parts. The first is where the primary N-group is compatible. Here theory developed in [5] and [9] is available. Indeed the compatible case will draw from these sources and consist of discussion. It will cover the topological insights compatibility supplies us with. As far as this paper is concerned it is the only time we move away from an algebraic approach.

The second part of this section consists of developing certain algebraic features 2 and 3-tame primary N-groups have. The two results here are of a general nature. It is in the next section that we move into what assuming $DCCR$ will mean. Here, as indicated above, no chain conditions of any kind are used.

Associated with a primary compatible N-group is the Z^t topology (see [9]). This arises naturally from zero sets of subsets of N. Indeed the closed subsets of Z^t are taken as all left translations of such zero subsets. There is nothing skewed in this definition as all right translations give the same topology.

The sort of questions that can arise for compatible primary N-groups are along the lines of what happens when conditions are imposed on Z^t? A question that can be asked here is how rudimentary can Z^t be? If V is a compatible primary N-group, $\{0\}$ is the zero set of 1. This means every v in V gives rise to the closed subset $\{v\}$. V is also closed (it is the zero set of 0). So what can be said about V when the only proper closed subsets are singletons? Here if v_i, $i = 1, 2$, are distinct non-zero elements of V, $(0 : v_1) = (0 : V) = (0 : v_2)$ and results from Sec. 3 and the fact that $(V, +)$ is not a C_2, supply a contradiction. This is not spelled out here but the interested reader may investigate how 3.3 means the non-trivial N-isomorphism of v_1N onto v_2N implies V cannot be primary. What has been shown is that the only V without non-zero proper closed subsets other than singletons has to be $\{0\}$.

The above example is perhaps not of all that much interest. However if we suppose that rather than no authentic closed subsets we have no proper non-zero closed N-subgroups, something very real unfolds. Indeed in this case if V is faithful then N is a prime nonring (prime means there are no non-zero ideals A and B with $A.B = \{0\}$). Furthermore a feature of compatible prime nonrings is that they have faithful primary N-groups. Here these N-groups have no proper non-zero closed N-subgroups. These facts are fully covered in 3.8.3 of [5].

Let V be a compatible primary N-group. In a situation above the case of V having no authentic closed subsets was looked at. The opposite extreme is where every subset of V is closed. This is equivalent to V being discrete. Here by 4.3 of [9] we have a rather nice characterization when V is faithful. What 4.3 of [9] supplies us with is the fact that V is discrete if and only if N has a minimal right ideal. Discreteness is a property that has been looked into further in the case of N being primitive on V. Indeed (see 6.3.1 of [5]) the primitive compatible nonring N has a minimal right ideal and is simple if and only if every finitely generated right ideal is a direct summand.

Let V be a faithful compatible primary N-group. Disconnections of V are often associated with direct decomposition of N. A direct decomposition of N is an expression $R_1 \oplus R_2$ of N as a direct sum of two non-zero right ideals R_i, $i = 1, 2$. When R_i, $i = 1, 2$, are the only two such non-zero right ideals, N is said to have singular direct decomposition. The fact that disconnections in V are often associated with direct decomposition in N is evidenced by Th. 11.1 of [9]. This tells us that if V is connected and N has direct decomposition, then V is a minimal N-group and the direct decomposition is singular. Furthermore in this case there is an isomorphism of V onto the additive group of the reals. It has been seen that connectedness together with direct decomposition can only arise in quite special cases. Indeed if V were also to be locally compact, then it could be topologically viewed as the reals and N as a subnearring of the nearring of all continuous functions on the reals (see 11.1 of [9]).

So far in this section compatible primary N-groups have occupied our attention. The important feature in this has been the Z^t topology. By placing conditions on Z^t a number of interesting consequences have been seen to come about. Things now take a change. When the primary

N-group is 2 or 3-tame we must fall back on algebra. Two theorems are proved. The first is of a completely general nature. It is a characterization of 3-tame primary N-groups. The second supplies us with information about annihilators of elements of a 2-tame primary N-group.

Theorem 6.1. *The 3-tame N-group V is primary if and only if $C_V(u) = \{0\}$ for each non-zero element u of V.*

Proof. Suppose $C_V(u) = \{0\}$ for all $u \neq 0$ in V. A non-zero ring module W of V cannot exist since for $u \neq 0$ in W, $C_V(u) \neq \{0\}$. Thus V is semiprimary. Also if W_i, $i = 1, 2$, are non-zero submodules of V with $W_1 \cap W_2 = \{0\}$, then with $u \neq 0$ in W_1 the contradiction that $C_V(u) \geq W_2$ is obtained. Thus V is uniform and therefore primary.

If V is primary and $u \neq 0$ in V, then by 2.2, $C_V(u) = C_V(uN)$ and $C_V(uN) \cap uN$ is clearly a ring module. Since $uN \neq \{0\}$ this can only mean $C_V(uN) = \{0\}$. The theorem therefore holds. $\Diamond$

Now for the theorem that supplies information about annihilators.

Theorem 6.2. *Let V be a 2-tame primary N-group. If v_i, $i = 1, 2$, are non-zero elements of V such that $(0 : v_1) \leq (0 : v_2)$ then $v_1 = v_2$.*

Proof. Suppose v_i, $i = 1, 2$, are distinct. It is clear that there exists an internal N-homomorphism (V, v_1N, μ) of V mapping v_1N onto v_2N and having $v_1\mu = v_2$. From results of Sec. 3 $(V, v_1N, 1 - \mu)$ is an internal N-homomorphism of V and $v_1N + v_1N\mu$ $(= v_1N + v_2N)$ is a central sum of $v_1N\mu$ and $v_1N(1-\mu)$. Because V is semiprimary the intersection of these two submodules is zero. However because V is uniform $v_1N(1-\mu) = \{0\}$ $(v_1N\mu = v_2N$ which is non-zero). It follows that $v_1(1 - \mu) = 0$ and because $v_1\mu = v_2$ the contradiction that $v_1 = v_2$ is arrived at. The theorem is proved. $\Diamond$

A question can be asked about 6.2. It makes sense to inquire whether the condition of that theorem implies V is primary. This is not the case but with a slight restriction a converse holds. Indeed this holds without the 2-tame assumption. The restriction is that our N-group V has no ring submodules of exponent two. Thus it is not difficult to show such an N-group V is primary if no inclusions $(0 : v_1) \leq (0 : v_2)$ $(v_i$, $i = 1, 2$, distinct non-zero elements of $V)$ occur.

7. Primary N-groups (N with $DCCR$)

In this section faithful 2-tame primary N-groups (N with $DCCR$) are studied. Three results will be given. The first is substantial. It supplies a one-one correspondence between non-zero elements of the N-group and minimal right ideals of N. The second is a corollary of this. It states that the N-group is finite. The third relates together any two such N-groups. They are proved to be N-bijective (defined later).

Before stating and proving the results just mentioned some notation and definitions are given. They are of a general nature but are very useful in the situation outlined above. First if N is a nearring, then $mr(N)$ will denote the set of all minimal right ideals of N. Furthermore, in situations where V is an N-group and R is in $mr(N)$, $s(R)$ will denote the set of all v in V^* such that $vR \neq \{0\}$. This subset of V^* is called the support of R. With V any given N-group a relationship between V^* and $mr(N)$ (i.e. a subset of $(V^*, mr(N))$) that will be important to us is the support relationship. It is defined as all (v, R) in $(V^*, mr(N))$ where $vR \neq \{0\}$ and denoted by $\Psi(V^*, mr(N))$. In applications where V and N are given, this is abbreviated to Ψ.

We are about to supply a fundamental theorem on Ψ in the case of V being primary and N (with $DCCR$) 2-tame on V. However before doing so a situation where centralizers arise differently is very briefly looked at. If N is a nearring and R in $mr(N)$ is nonring then right ideals R_i, $i = 1, 2$, of N such that $R \cap R_i = \{0\}$ centralize R and are such that $R \cap (R_1 + R_2) = \{0\}$. This means the unique right ideal $K_N(R)$ of N maximal for the property that $R \cap K_N(R) = \{0\}$ centralizers R and is the sum of all right ideals that do so. In quite a number of settings maximal complements of this type (unique or otherwise) have proved to be important (see [2] and [3]). As far as the next theorem goes $K_N(R)$ will be viewed as a centralizer. In fact if N is 3-tame $K_N(R) = C_N(R)$ by 31.4 of [10]. Indeed even if N is only tame this is the case. Here a right N-subgroup M of N centralizing R is such that $(v\rho + v\beta)\alpha - v\beta\alpha - v\rho\alpha = 0$, for all v in a faithful tame N-group V, ρ in R, β in $R(M)$ (remember $vR(M) = vM$) and α in N. Consequently $R(M)$ centralizers R and is $\leq K_N(R)$. However this is also true for $K_N(R)$ so that the union of all M as above must be $K_N(R)$. Thus even when N is tame $C_N(R) = K_N(R)$.

Theorem 7.1. *Suppose V is a primary N-group where N has $DCCR$ and is 2-tame on V. If R is in $mr(N)$ and v in $s(R)$, then R is nonring* *and $C_N(R) = (0 : v)$. Also the relationship Ψ is a one-one map of V^* onto $mr(N)$.*

Proof. It is an elementary matter to show V has a unique minimal submodule H, H is nonring and every element of $mr(N)$ is N-isomorphic to H. Thus the element R of $mr(N)$ is nonring and $C_N(R)$ exists. The faithfulness of V ensures there is a v in V^* such that $vR \neq \{0\}$. As the sum $R + C_N(R)$ is direct $v[R + C_N(R)]$ is a central sum of vR and $vC_N(R)$. The uniqueness and nature of H implies $vC_N(R) = \{0\}$ and $C_N(R) \leq (0 : v)$. However it is clear $(0 : v) \cap R = \{0\}$ so the maximality of $C_N(R)$ ensures $(0 : v) = C_N(R)$. The first part of 7.1 is proved.

In proving Ψ is a one-one correspondence the first step is to show Ψ is a function of V^* into $mr(N)$. Thus it must be shown that for each w in V^* there is a unique R_1 in $mr(N)$ such that $wR_1 \neq \{0\}$. To show R_1 exists suppose that for all R_2 in $mr(N)$, $wR_2 = \{0\}$. This has the consequence that $wN.(H : V) = w\ soc(N) = \{0\}$, wN is a tame $N/soc(N)$-group and wN has minimal factors N-isomorphic to those of V/H (V/H is a faithful tame $N/soc(N)$-group). As $wN \neq \{0\}$ it contains H and V/H has a minimal factor N-isomorphic to H contrary to 14.2 of [10]. Thus there is an R_1 in $mr(N)$ with $wR_1 \neq \{0\}$. For Ψ to be the function indicated it must be true that for given w, R_1 is unique. If $R_3 \neq R_1$ in $mr(N)$ were such that $wR_3 \neq \{0\}$, then the sum $R_1 + R_3$ is direct and $w[R_1 + R_3]$ is a central sum of wR_1 and wR_3. The uniqueness and nature of H makes this impossible. R_1 has been shown to be unique.

In the above it was seen Ψ is a function of V^* into $mr(N)$. We may therefore write $w\Psi$ ($w \in V^*$) for the unique K in $mr(N)$ with $wK \neq \{0\}$. It must be shown $V^*\Psi = mr(N)$ and if $w_1\Psi = w_2\Psi$ (w_i, $i = 1, 2$, in V^*) then $w_1 = w_2$. The fact that $V^*\Psi = mr(N)$ follows from the fact that V is faithful (for given K_1 in $mr(N)$ there certainly exists u in V^* with $uK_1 \neq \{0\}$). If u_1 and u_2 are in V^* and $u_1K_2 = u_2K_2 \neq \{0\}$ (K_2 in $mr(N)$), then by the first part of the theorem $(0 : u_1) = (0 : u_2)$ and there is an N-isomorphism μ of u_1N onto u_2N with $u_1\mu = u_2$. By 3.3 this means $u_1N + u_2N$ is a central sum of u_2N and $u_1(1-\mu)N$. Because V is primary and $u_2N \neq \{0\}$ the only possibility is that $u_1(1-\mu) = 0$ and $u_1 = u_2$. The theorem is proved. $\Diamond$

We now move towards obtaining our second result. It is a corollary of 7.1. Clearly $soc(N) = \oplus R_i$ (a finite direct sum over $i = 1, \ldots, n$, of elements R_i of $mr(N)$). If $\{R_1, R_2, \ldots, R_n\} \subset mr(N)$ then there is an R

in $mr(N)$ contained in $H_1 \oplus H_2$ (H_i, $i = 1, 2$, right ideals of N contained in $soc(N)$) with $R \not\leq H_i$, $i = 1, 2$. As this would imply R is a ring module (see 6.4 of [10]) it follows that $\{R_1, \ldots, R_n\} = mr(N)$ and $mr(N)$ is finite. By 7.1 it must follow that V is finite. In this argument if N were not faithful it would be possible to work within $N/(0 : V)$.

Corollary 7.2. *A primary 2-tame N-group (N with $DCCR$) is finite.*

This section is about primary N-groups but 7.2 also holds for semiprimary ones. This follows readily enough from that result. Such an N-group (V say) has a finite collection of minimal submodules (all nonring) each with a unique submodule maximal for avoiding it. The intersection of these avoiding submodules is $\{0\}$ so that V is embedded into a finite direct sum of N-groups of the form V/H (H such a maximal avoider). As the V/H are primary 7.2 gives the result.

Th. 7.1 and its corollary are substantial advances into the study of primary 2-tame N-groups (N with $DCCR$). Because there is such a definite relationship (viz. Ψ) between V and N (see 7.1) one might ask whether V is determined by N? This may not be the case but it is very near to being so. The proximity of V's to each other is expressed by N-bijectivity. This is now defined.

If V_i, $i = 1, 2$, are N-groups then a one-one map γ of V_1 onto V_2 is called an N-bijection if $(v_1\alpha)\gamma = (v_1\gamma)\alpha$ for all v_1 in V_1 and α in N. The inverse of such a γ is an N-bijection of V_2 onto V_1. If such a γ exists, then V_1 and V_2 are said to be N-bijective. Clearly if V_1 (N-bijective to V_2) is cyclic then V_1 and V_2 are N-isomorphic.

With the above definition behind us we have:

Theorem 7.3. *If V_i, $i = 1, 2$, are faithful primary 2-tame N-groups (N with $DCCR$), then the map taking the zero of V_1 to that of V_2 and being $\Psi_1\Psi_2^{-1}$ on V_1^*, is the unique N-bijection of V_1 onto V_2 (here Ψ_i, $i = 1, 2$, is the map Ψ of 7.1 with V replaced by V_i).*

Proof. Let γ be the map of V_1 onto V_2 specified in the statement of the theorem. $0\gamma = 0$ and γ maps V_1^* bijectively onto $mr(N)$ and then bijectively onto V_2^*. Thus γ is a one-one map of V_1 onto V_2.

Given any R in $mr(N)$ there exists v in V_1^* such that $v\Psi_1 = R$ and v is the unique element of V_1^* such that $(0 : v) = C_N(R)$ (see 7.1). It is clear that in this any v of V_1^* can occur. Now $R\Psi_2^{-1} = v\Psi_1\Psi_2^{-1} = v\gamma$ so that $v\gamma\Psi_2 = R$ and $v\gamma$ is the unique element of V_2^* such that $(0 : v\gamma) = (0 : v)$ (see 7.1). Thus vN is N-isomorphic by μ to $v\gamma N$ and $v\mu = v\gamma$. Non-zero

$v\alpha$ (α in N) is mapped by μ to $v\gamma\alpha$. Thus $(0 : v\alpha) = (0 : v\gamma\alpha)$ and from what has just been proved $v\alpha\gamma = v\gamma\alpha$. When $v = 0$ or $v\alpha = 0$ this is also true (for $v \neq 0$ but $v\alpha = 0$ $0 = v\alpha\mu = v\gamma\alpha$). It has been shown γ is an N-bijection.

If γ' is an N-bijection of V_1 onto V_2 distinct from γ then $\gamma'\gamma^{-1}$ is a non-identity N-bijection of V_1 onto V_1. Thus $\gamma'\gamma^{-1}$ restricted to some cyclic N-subgroup v_1N ($v_1 \in V_1$) of V_1 is a non-identity N-isomorphism μ of v_1N into V_1. By 3.3, $v_1N + v_1\mu N$ is the central sum of $v_1\mu N$ and $v_1(1-\mu)N$. Certainly $v_1\mu N \neq \{0\}$ but $v_1(1-\mu)N \neq \{0\}$ since $v_1\mu \neq v_1$. By the semiprimary nature of V_1 the sum $v_1\mu N + v_1(1-\mu)N$ is direct. Because V_1 is uniform we have a contradiction. Thus γ' cannot exist and the theorem is proved. $\diamondsuit$

8. Pointed 3-tame *N*-groups

The motivation for this section comes partly from Th. 7.1. That theorem amongst other things showed that for a faithful primary 2-tame N-group V (N with $DCCR$) the support of an element of $mr(N)$ consisted of a single element of V^*. It therefore would seem natural to ask what it means when V is an N-group, for the support of each element of $mr(N)$ to consist of a single element of V^*? A definition along these lines helps terminology. Indeed such an N-group will be called pointed. In this section we shall show that a pointed 3-tame N-group (N with $DCCR$) is faithful and nearly always primary. Before stating the theorem involved we note that V being pointed is equivalent to Ψ^{-1} (Ψ the relationship of the last section) being a function.

The theorem that this section is all about is now stated.

Theorem 8.1. *Suppose V is a 3-tame N-group where N has $DCCR$. V is pointed if and only if it is faithful and is either primary or a cyclic group C_2 of order two.*

Establishing 8.1 is not that easy. The body of the proof will be carried out in a series of propositions and lemmas.

Proposition 8.2. *If V is a pointed N-group and N has $DCCR$ then V is faithful.*

Proof. If $(0 : V)$ were $\neq \{0\}$ then it would contain a minimal right ideal R. Since V is pointed the contradiction that $vR \neq \{0\}$ for some v in V is obtained. 8.2 holds. $\diamondsuit$

The next result plays quite a substantial part in the verification of 8.1.

Lemma 8.3. *Suppose V is a 3-tame N-group. If there is an R in $mr(N)$ with $s(R)$ consisting of a single element, then a ring submodule U of V must have order ≤ 2.*

Proof. Suppose U has order > 2 and v is the unique element of V^* such that $vR \neq \{0\}$. Consider the elements of $v+U$. Here v is the unique element of $v+U$ such that $vR \neq \{0\}$. Thus for some $v' \neq v$ in $v+U$, $v'R = \{0\}$. Consider those elements u of U for which $(v'+u)R = \{0\}$. Call this subset of U, H. We shall show H is a subgroup of U. First 0 is in H as $v'R = \{0\}$. Also if h_1 and h_2 are in H then by 4.2

$$(v'+h_1+h_2)\rho = (v'+h_1)\rho - v'\rho + (v'+h_2)\rho$$

for all ρ in R and because $v'\rho = (v'+h_1)\rho = (v'+h_2)\rho = 0$ it must follow that $(v'+h_1+h_2)\rho = 0$. This means h_1+h_2 is in H. Also

$$(v'+h_1-h_1)\rho = (v'+h_1)\rho - v'\rho + (v'-h_1)\rho$$

and since $v'\rho = (v'+h_1)\rho = 0$ we see $-h_1$ is in H. It has been shown H is a subgroup of U. It is a proper subgroup of U since $v'+U = v+U$ contains v and $vR \neq \{0\}$. Now $(v'+U)\backslash(v'+H) = v'+(U\backslash H)$ and every v_1 of this set is such that $v_1R \neq \{0\}$. Thus $|U\backslash H| = 1$ and there is a unique $\overline{u}$ in U not in H. As $\overline{u}+h$ ($h \neq 0$ in H) is in $U\backslash H$ we see $H = \{0\}$ and $|U| = 2$ contrary to the choice of U. The lemma has been proved. $\Diamond$

The above result is rather nice. It tells us that if for at least one R in $mr(N)$, $s(R)$ is a singleton, then it is very like V is semiprimary. What is required however is a method of reducing minimal submodule types of V. A lemma is of help in doing this.

Lemma 8.4. *Suppose V is a faithful 2-tame N-group where N has DCCR. If U is a nonring minimal submodule of V, then there exists R in $mr(N)$ with $s(R) \cap U \neq \emptyset$.*

Proof. By 14.2 of [10], V/U and U are coprime. It follows from 13.3 of [10] that $(U:V)+(0:U) = N$. Thus $1 = e_1 + e_2$ where e_1 is in $(U:V)$ and e_2 in $(0:U)$. Now $e_1^r = e_1\ mod(0:U)$ for $r = 1, 2, \ldots$, and $(U:V)$ (not contained in $(0:U)$) cannot be nil and is non-nilpotent. However $(U:V) \leq (soc(V):V) = soc(N)$ and is a sum of minimal right ideals of N. Let R be a non-nilpotent minimal right ideal of N contained in $(U:V)$. It follows that $eR = R$ where e is in R. As V is faithful there exists v in V^* such that $vR \neq \{0\}$. Thus $veR \neq \{0\}$ and since $e \in (U : V)$, ve is in U. As required it has been shown $s(R) \cap U \neq \emptyset$. $\diamondsuit$

A corollary of 8.3 and 8.4 is that a pointed 3-tame N-group (N with $DCCR$) is either primary or has a unique minimal N-subgroup and this is of order two. To verify this note that by 8.3 the result will follow if it is shown that when V has a nonring minimal submodule then it is unique (amongst minimal submodules). This is because it is easy enough to see such an N-group is primary while if V were to have more than one minimal submodule it would mean they are C_2's and a sum of two such minimal submodules cannot occur. But now 8.4 supplies us with the fact that when V has a minimal submodule which is nonring it is unique. Indeed suppose H_1 and H_2 were two minimal submodules with H_1 being nonring. Clearly they are not of the same N-isomorphism type and by 8.4 there is an h_1 in H_1 such that $h_1R \neq \{0\}$ for some R in $mr(N)$. This means R is N-isomorphic to H_1 and $(h_1 + h_2)R = h_1R$ ($h_2 \neq 0$ in H_2) as h_2R cannot be N-isomorphic to R. This contradicts the fact that $s(R)$ is a singleton (both h_1 and $h_1 + h_2$ would be in it). Our corollary is verified.

Corollary 8.5. *Suppose V is a 3-tame N-group where N has $DCCR$. If V is pointed then it is either primary or has a unique minimal N-subgroup and this is C_2.*

By 8.5 the proof of 8.1 is essentially reduced to showing V of 8.5 having a unique minimal submodule which is a C_2, is in fact this minimal submodule.

Lemma 8.6. *Suppose V is a 3-tame N-group where N has $DCCR$. If V is pointed and has a unique minimal submodule which is a C_2 then this submodule is V.*

Proof. For such a V all elements of $mr(N)$ have order two and are ring modules. Let M be a right N-subgroup of N minimal for the property that $M + soc(N) = N$. If it is shown $M = \{0\}$, then N is semisimple and V is completely reducible so that $V = soc(V)$ and V has order two. Thus in order to prove 8.6 it is sufficient to show $M = \{0\}$.

It is not that difficult to see that $M \cap soc(N) = soc(M)$ so that if $M \neq \{0\}$, $M \cap soc(N) \neq \{0\}$. However by 4.4, $M \cap soc(N)$ is a right ideal of N and M contains an element R of $mr(N)$. We note also M contains a left identity e. This follows because $1 = \alpha + \beta$ (α in M and β in $soc(N)$) so that $M \leq \alpha M + soc(N)$ and $\alpha M = M$. Thus by [6] and 5.7 of [8], M being self-monogenic necessarily contains e.

If it is shown $M = M_1 + R$ where M_1 is a right N-subgroup of N such that $M_1 \cap R = \{0\}$ then because, $N = M_1 + soc(N)$ the minimality of M will conclude our proof of 8.6. Clearly $e\rho = \rho$ for all ρ in R and there exists v in V such that $ve\rho \neq 0$ for some ρ in R. This means ve is the unique element of $s(R)$ and $(ve+ve\rho)R = \{0\}$. From this we see $(e+\rho)R$ cannot be $= R$ as otherwise $vR = \{0\}$. Thus $(e+\rho)R = \{0\}$. From [6] and [8] this clearly means $(e+\rho)M < M$ (otherwise $(0 : e+\rho) \cap M = \{0\}$). It follows that $M = (e+\rho)M + R$ (e is in $(e+\rho)e + R$). A contradiction to M being minimal has been obtained and the lemma is fully proved. $\Diamond$

This section concludes by showing how 8.2 to 8.6 establish 8.1.

Proof of 8.1. Suppose the V of 8.1 is pointed. 8.2 implies it is faithful. 8.5 and 8.6 imply that either V is primary or a C_2.

Now suppose V is faithful and either primary or a C_2. In the primary case 7.1 yields the fact that the relationship Ψ is a one-one correspondence. Clearly this implies V is pointed. If V is a C_2 the only possibility for a faithful N is that it is the field of order two and the $a \neq 0$ in V is the unique element of V with $aN \neq \{0\}$. 8.1 has been completely proved. $\Diamond$

9. When perfect implies conformed

It is of real enough value to know that up to N-bijectivity faithful primary 3-tame N-groups are unique (see 7.3). An obvious question here is whether something like this holds for faithful semiprimary 3-tame N-groups? Indications are that this is not the case. However by imposing a further condition on such N-groups uniqueness is obtained.

This and the next section is about N-groups dual to semiprimary ones. It is by considering perfect N-groups that the result indicated above is obtained. The general result on perfect 3-tame N-groups is noteworthy. In itself it answers a question stemming from G. Peterson's work in [1]. There it was proved that, with a finiteness condition, two faithful d.g. compatible perfect N-groups have N-isomorphic Fitting factors. It will be shown here that in this statement, d.g. can be removed and compatibility replaced by 3-tame.

The dual notion to semiprimary is that of being perfect. Perfect N-groups have already received considerable attention. Quite a bit of [10] centers on such N-groups. The well known definition here is that an

N-group V is perfect if there is no proper submodule H with V/H a ring module. However this section is also about conformed N-groups. An N-group V is conformed if there do not exist distinct proper submodules H_i, $i = 1, 2$, of V with V/H_1, N-isomorphic to V/H_2.

Although non-conformed 2-tame N-groups are easy to obtain there is an important situation where 2-tame N-groups are necessarily conformed. As is seen in this section this will be the case when N has $DCCR$ and V is perfect. Thus the theorem to be proved is as follows:

Theorem 9.1. *If V is a perfect 2-tame N-group (N with $DCCR$) then it is conformed.*

Two preliminary results will make the proof of 9.1 relatively easy.

If V is an N-group of the form $V_1 \oplus V_2$ (V_2 a copy of V_1 – under the N-isomorphism μ say), then $diag(V)$ will denote all N-subgroups (of V) with elements of the form $v_1 + v_1\lambda$, v_1 in V_1 and λ any given N-isomorphism of V_1 onto V_2. Such subsets are certainly N-subgroups of V. Moreover, there always exists such N-subgroups (the set of all $v_1 + v_1\mu$ is an example). In certain interesting situations $diag(V)$ contains only one element.

Proposition 9.2. *If V is an N-group of the form $V_1 \oplus V_2$ (V_2 a copy of V_1) then, when V_1 is perfect and 2-tame, $diag(V)$ has only one element.*

Proof. Suppose there were to exist two distinct N-isomorphisms μ_i, $i = 1, 2$, of V_1 onto V_2. In this case $\mu_1\mu_2^{-1}$ is a non-trivial N-automorhism of V_1. Now $1-\mu_1\mu_2^{-1}$ is non-zero and an N-endomorphism of V_1. Because V_1 is a central sum of $V_1\mu_1\mu_2^{-1}$ and $V_1(1 - \mu_1\mu_2^{-1})$ with $V_1\mu_1\mu_2^{-1}$ $(= V_1)$ containing $V_1(1-\mu_1\mu_2^{-1})$ we see $V_1(1-\mu_1\mu_2^{-1})$ is a non-zero ring module. This means $V_1/ker(1 - \mu_1\mu_2^{-1})$ is a non-zero ring module contrary to V_1 being perfect. 9.2 is proved. $\Diamond$

The result just proved was very straightforward. What is really required is information as to when a subdirect sum of the V of 9.2 is 2-tame. Here conformity is enough to imply the only such N-subgroup is the unique element of $diag(V)$.

Lemma 9.3. *Suppose V is an N-group of the form $V_1 \oplus V_2$ where V_1 is both perfect and conformed and V_2 a copy of V_1. A perfect 2-tame subdirect sum W of V (an N-subgroup with projections into the V_i being V_i) must be D (D the unique element of $diag(V)$ – see 9.2).*

Proof. Let π_i, $i = 1, 2$, be the projections of V onto V_i and μ the unique (see 9.2) N-isomorphism of V_1 onto V_2. We have $W\pi_i = V_i$, $ker\pi_1 = V_2$

and $ker\pi_2 = V_1$. Now $W/(W \cap ker\pi_1)$ is N-isomorphic to $W\pi_1$ ($= V_1$ which is also N-isomorphic to V_2). This of course means the V_i are 2-tame and D exists. Set K as the direct sum $(W \cap ker\pi_1) \oplus (W \cap ker\pi_2)$. It will be shown $W \leq D + K$.

First note that π_1 induces an N-isomorphism of W/K onto $V_1/(W \cap \cap ker\pi_2)$ (here $w+K$ in W/K with w in W is mapped to $w\pi_1+(W\cap ker\pi_2)$) and π_2 induces a similar map of W/K onto $V_2/(W\cap ker\pi_1)$ (here $w+K$ is mapped to $w\pi_2 + (W \cap ker\pi_1)$).

Now applying the N-isomorphism induced by μ to $V_1/(W \cap ker\pi_2)$ we map $w\pi_1+(W\cap ker\pi_2)$ to $w\pi_1\mu+(W\cap ker\pi_2)\mu$ and see $V_2/(W\cap ker\pi_1)$ is N-isomorphic to $V_2/(W \cap ker\pi_2)\mu$ (remember $V_1/(W \cap ker\pi_2)$ was N-isomorphic to $V_2/(W \cap ker\pi_1)$ – both N-isomorphic to W/K). Now V_2 is conformed (follows readily enough from the fact that V_1 is) and $W \cap ker\pi_1$ therefore coincides with $(W \cap ker\pi_2)\mu$ so that a map δ can be given that takes $w\pi_2 + (W \cap ker\pi_1)$ (w in W) of $V_2/(W \cap ker\pi_1)$ to $w\pi_1\mu+(W\cap ker\pi_2)\mu$. The well defined nature of this map follows because, if w in W is such that $w\pi_2$ is in $W \cap ker\pi_1$, then $w - w\pi_2$ is in $W \cap ker\pi_1$ and $w\pi_1\mu$ is in $(W\cap ker\pi_2)\mu$. There is no real difficulty checking that δ is an N-isomorphism. In fact it is an N-automorphism on $V_2/(W \cap ker\pi_1)$ (remember $w\pi_1\mu$ is in V_2 and $(W\cap ker\pi_2)\mu$ is $W\cap ker\pi_1$). However from 9.2, δ is the identity. This must therefore mean that $w\pi_1\mu+(W\cap ker\pi_2)\mu$ is (for all w in W) equal to $w\pi_2 + (W \cap ker\pi_1)$. It now follows that

$$w\pi_1 + w\pi_2 + K = w\pi_1 + (W \cap ker\pi_2) + w\pi_1\mu + (W \cap ker\pi_2)\mu$$

which in turn equals

$$w\pi_1 + (W \cap ker\pi_2) + w\pi_2 + (W \cap ker\pi_1)$$

or simply $w + K$. But because $w\pi_1 + w\pi_1\mu$ is a typical element of D it follows that $D + K = W + K$ and $W \leq D + K$.

The proof of 9.3 is now easily finished. First observe that K (contained in W) is the direct sum of the submodules $W \cap ker\pi_2$ and $(W\cap ker\pi_2)\mu$ (this being just $W\cap ker\pi_1$) and being N-isomorphic under μ we conclude K is a ring module. From what is proved above (remember $K \leq W$) it follows that $W = (D \cap W) + K$. Now factoring out $D \cap W$ (a submodule of W) would give a ring factor unless $K \leq D \cap W$ and $W = D\cap W$. It is now easily seen that because the N-subgroup W of the diagonal N-subgroup D has projections to V_i being V_i it is not proper. The lemma is proved. $\Diamond$

We are now ready to prove 9.1.

Proof. In order to obtain a contradiction suppose there exist perfect 2-tame N-groups (N with $DCCR$), having factors of the form V/H which are N-isomorphic but not the same. Out of all such N-groups suppose V is chosen with a composition series of minimal length (a perfect 2-tame N-group is cyclic and has a composition series because it is N-isomorphic to a factor of N – see 10.5 of [10]). Let V/H_i, $i = 1, 2$, be two distinct N-isomorphic factors of V. Clearly in $V/(H_1 \cap H_2)$ the N-isomorphic factors $[V/(H_1 \cap H_2)]/[H_1/(H_1 \cap H_2)]$ and $[V/(H_1 \cap H_2)]/[H_2/(H_1 \cap H_2)]$ are not the same since the V/H_i are distinct. The minimality of V forces $H_1 \cap H_2$ to be $\{0\}$. Now V is embedded in a natural way subdirectly into the direct sum $V/H_1 \oplus V/H_2$ ($= X$ say). Let the image of this embedding be W. Also V/H_1 and V/H_2 are N-isomorphic and V/H_1 is, by the minimality of V, conformed. Indeed both H_i cannot be zero so one of the V/H_i is conformed and the other, being N-isomorphic to it must also be (it is not hard to establish an N-group N-isomorphic to a conformed one is conformed). By 9.3 we have that $W = D$ where D is the unique element of $diag(X)$. It is not difficult to see D is N-isomorphic to the conformed N-group V/H_1 and V being N-isomorphic to it is also conformed contrary to our choice of V. The theorem of this section has been proved. $\diamondsuit$

10. A substantial result

In this section we prove a theorem outlined in the previous. It was shown in [1] that, in the d.g. case, Fitting factors of faithful perfect compatible N-groups are N-isomorphic (a finiteness condition was assumed). The theorem proved here not only removes the d.g. requirement but replaces compatibility with something which, superficially at least, looks to be much weaker. This is the 3-tame assumption. As outlined previously this has consequences for semiprimary N-groups.

The Fitting factor result is substantial. Its corollary adds to semiprimary understanding. The perfect nature of such N-groups appears necessary for the result. However these matters are not all this section is about. Of considerable interest is how perfect 2-tame N-groups embed subdirectly into direct sums. In this direction a result rather satisfying in its own right, facilitates the proof about Fitting factors. A lemma providing the main step of the proof is now given.

Lemma 10.1. *There is at most one perfect* 2*-tame* N*-subgroup (*N *with DCCR) of a direct sum of* N*-groups* V_i*,* $i = 1, 2$*, with projections into each* V_i *being* V_i*.*

Proof. Suppose W_j, $j = 1, 2$ are perfect 2-tame N-subgroups of $V_1 \oplus V_2$ with projections into each V_i, $i = 1, 2$, being V_i. As the V_i, $i = 1, 2$, are N-homomorphic images of W_1 they are perfect and 2-tame. Considering the faithful action of N on the V_i we have by 10.5 of [10] that V_i and $V_1 \oplus V_2$ have composition series. The proof that $W_1 = W_2$ will be by induction on the composition length of $V_1 \oplus V_2$.

Clearly if V_1 or V_2 is $\{0\}$ then there is nothing to prove. Taking a minimal submodule H of V_1 we have $(W_j + H)/H$, $j = 1, 2$, satisfy the conditions of the lemma with $V_1 \oplus V_2$ replaced by $(V_1/H) \oplus V_2$. This means $(W_1+H)/H = (W_2+H)/H$. If H were $\leq W_j$ for $j = 1$ and 2 then $W_1 = W_2$. If H were $\leq W_1$ but $H \cap W_2 = \{0\}$ we have $W_1 = W_2 \oplus H$. However $(0 : V_1 \oplus V_2) = (0 : W_1) = (0 : W_2)$ so that W_1 and W_2 are faithful tame $N/(0 : W_1)$-groups and H is N-isomorphic to a factor of W_2. This readily implies H is a ring module and W_1/W_2 a non-zero ring module N-homomorphic image of W_1, contrary to W_1 being perfect. By symmetry it can be assumed that no minimal submodule of either V_1 or V_2 is contained in either W_1 or W_2. Thus the V_i, $i = 1, 2$, intersect the W_j, $j = 1, 2$ in $\{0\}$ and it is not really difficult to see both W_j are in $diag(V_1 \oplus V_2)$ (V_i, $i = 1, 2$, are N-isomorphic to W_1 and W_2). By 9.2, $W_1 = W_2$ and the lemma holds. ◊

The proof of the theorem that was discussed above follows easily.

Theorem 10.2. *There is at most one perfect* 2*-tame* N*-subgroup (*N *with DCCR) of a direct sum of* N*-groups* V_i*,* $i = 1, \ldots, k$*, (*$k \geq 1$ *an integer) with projections into each* V_i *being* V_i*.*

Proof. The theorem will be proved by induction on k. Clearly if $k = 1$ the result holds. In fact there are either no N-subgroups of the type specified or V_1 is perfect and 2-tame and V_1 is the only such N-subgroup. Suppose $k \geq 2$. Let $V = V_1 \oplus \cdots \oplus V_k$ and W_1 and W_2 be perfect 2-tame N-subgroups of V with projections into V_i being V_i. Let $U = = V_1 \oplus \cdots \oplus V_{k-1}$, and δ the projection of V onto U. We have the restriction of the projections π_i of U onto V_i to the W_j, $j = 1, 2$, $i = 1, \ldots, k-1$, are such that $W_j\delta\pi_i = V_i$ and the 2-tame perfect nature of the $W_j\delta$, $j = 1, 2$, (N-homomorphic images of W_j) supplies the fact that they are, by induction, equal. Thus W_j, $j = 1, 2$, are both contained in

$W_j\delta \oplus V_k$ (here the first components are the same) and are contained there in such a way that their projections (δ and π_k) onto the first and second components are respectively $W_j\delta$ (i.e. $W_1\delta$) and V_k. By 10.1, $W_1 = W_2$. 10.2 is proved. $\Diamond$

This paper covers a number of quite meaningful results. Two might be regarded as the more significant. This is 11.2 of the next section and the result that follows.

Theorem 10.3. *If N is a nearring with $DCCR$ then all Fitting factors of faithful perfect 3-tame N-groups (if such faithfuls exist) are N-isomorphic.*

Proof. This will follow like part of the proof of 39.2 of [10].

Let V_j, $j = 1, 2$, be two faithful perfect 3-tame N-groups and $\Delta_1, \ldots, \Delta_n$, be the set of all N-isomorphism types of minimal N-groups ($n \geq 1$ is finite by 8.2 of [10]). Take U_i/W_i, $i = 1, \ldots, n$, as minimal factors of V_1 of type Δ_i (they exist by 8.10 of [10]). Also take H_i/K_i, $i = 1, \ldots, n$, as minimal factors of V_2 of type Δ_i. Now $F(V_1)$ and $F(V_2)$ are specified as in the proof of 39.2 of [10] (see the paragraph following on from Step 2). Let L_i and P_i, $i = 1, \ldots, n$, be the N-groups defined there and δ_i be the same N-isomorphisms (between L_i and P_i – see 32.3 of [10]). $V_1/F(V_1)$ and $V_2/F(V_2)$ are respectively embedded as subdirect sums into $L_1 \oplus \cdots \oplus L_n$ and $P_1 \oplus \cdots \oplus P_n$. Thus Q_j, $j = 1, 2$, (see 39.2 of [10]) are defined N-isomorphic to $V_j/F(V_j)$. δ provides an N-subgroup $Q_1\delta$ of $P_1 \oplus \cdots \oplus P_n$ with projections into each P_i being P_i (see 39.2's proof). Q_1 is N-isomorphic to $Q_1\delta$ (δ is an N-isomorphism) and $Q_1\delta = Q_2$ (this follows by 10.2 since Q_2 is perfect and 2-tame). But Q_j is, for $j = 1, 2$, a copy of $V_j/F(V_j)$. 10.3 is proved. $\Diamond$

As noted in the last section indications are that faithful 3-tame semiprimary N-groups (N with $DCCR$) may not necessarily be N-isomorphic. However, as their Fitting submodules are zero it follows that when perfect all is well.

Corollary 10.4. *Two faithful 3-tame semiprimary N-groups (N with $DCCR$) that are perfect must necessarily be N-isomorphic.*

This section finishes with a simple note. Although the faithful 3-tame perfect N-groups of 10.3 may not be finite it is certainly true that those of 10.4 will be (see the comment that follows 7.2).

11. Semiprimary nearrings

In a natural way the nearring N is an N-group. It makes complete sense therefore, when this N-group is semiprimary, to call N semiprimary. This is the definition that is adopted. There is however a notion that might at first give the impression of being an alternative. It is that the nearring N has a faithful semiprimary N-group. However this possibility does not really make much sense. Here N may be very far removed from having no ring right ideals. It may even be a ring. One just has to take V as a finite simple non-abelian group and N (acting on V in the natural way) as the ring of integers *mod* n where n is the exponent of V.

In the above discussion the definition of N being semiprimary was given. Also a very doubtful alternative definition was presented. It was seen how as a possibility this does not stack up. This however is not the case for tame nearrings with $DCCR$. There N being semiprimary is equivalent to N having a faithful tame semiprimary N-group.

Proposition 11.1. *Suppose N is a tame nearring with $DCCR$. N is semiprimary if and only if it has a faithful tame semiprimary N-group.*

Proof. Suppose N has a faithful tame semiprimary N-group V but N is not semiprimary. In this case there is a non-zero ring right ideal R of N. As a v in V with $vR \neq \{0\}$ supplies a non-zero ring submodule (i.e. vR) of V we have a contradiction. Thus the existence of V implies N is semiprimary.

Now suppose N is semiprimary. Being tame it has a faithful tame N-group and having $DCCR$ there is such an N-group W with a composition series. Take H as a submodule of W maximal such that $(H:W) = \{0\}$. Clearly W/H is a faithful tame N-group. It will be shown it is semiprimary. Suppose $H_1 > H$ is a submodule of W such that H_1/H is a ring module. It is clear $(H_1/H : W/H)$ $(= (H_1 : W)/(H : W))$ is a non-zero ideal of N (i.e. of $N/\{0\}$). Since this ideal of N must be a ring submodule of N (follows readily from H_1/H being a ring submodule of the faithful N-group W/H) we obtain a contradiction completing the proof. $\Diamond$

Prop. 11.1 was elementary. It gave us an 'if and only if' condition that N is semiprimary. It turns out there is also quite a deep result about tame N along these lines. It again gives 'if and only if' conditions that certain N are semiprimary. However for this to hold it appears N (with

$DCCR$) must not only be 3-tame but $soc(N)$ must be homogeneous. The surprising thing about this result is that it tells us that many 3-tame N (N with $DCCR$ and $soc(N)$ homogeneous) have non-zero centre.

Theorem 11.2. *If N is a 3-tame nearring with $DCCR$ and $soc(N)$ homogeneous then the following are equivalent*

(a) *the centre $Z(N)$ of N is $\{0\}$,*

(b) *N is semiprimary and*

(c) *N has a faithful 3-tame primary N-group.*

Theorem 11.2 is easy to prove once a substantial lemma is in place. The lemma follows:

Lemma 11.3. *Suppose N with $DCCR$ is 3-tame on V and $soc(V)$ is a sum of copies of a minimal submodule U. If U is a ring module then $Z(N) \neq \{0\}$.*

Proof. The proof of this will be accomplished in four steps.

Step 1. Here it is shown that there exists a self-monogenic right N-subgroup M of N with a unique maximal submodule H and that M can be found so that M/H is N-isomorphic to U.

As $N/J(N)$ is a direct sum of minimal right ideals of $N/J(N)$ N-isomorphic to minimal factors of V (each N-isomorphism type occurs) a right ideal $R_1 > J(N)$ of N can be found with $R_1/J(N)$, N-isomorphic to U. Take $M \leq R_1$ as a right N-subgroup of N minimal for not being $\leq J(N)$ (see 5.7 of [8]). M is minimal for being non-nilpotent ($J(N)$ is nilpotent but $(M + J(N))/J(N) = R_1/J(N)$ is not) and $M \cap J(N)$ contains all right N-subgroups of N that are $< M$. Also $(M + J(N))/J(N)$ (N-isomorphic to $M/(M \cap J(N))$) is N-isomorphic to U. Because M is self-monogenic (a non-nil element α of M is such that $\alpha M = M$) Step 1 is complete (here $H = M \cap J(N)$).

Step 2. Here it is shown that if R is the right ideal of N generated by M, then $R \cap soc(N)$ is $\leq Z(R)$.

To prove this we first show that for each v in V with $vM \neq \{0\}$ all minimal submodules of vM are central in vM. A minimal submodule U_1 of vM is N-isomorphic to U and, if $vM > vH$, N-isomorphic to the minimal factor vM/vH of vM. By 14.2 of [10] we see that if $U_1 \leq vH$, then $U_1 \leq Z(vM)$. If $vH \cap U_1 = \{0\}$ then as vM has the unique maximal submodule vH, $U_1 = vM$ and the result holds. However vM cannot be $= vH$ as this would imply $M = H + M \cap (0 : v)$ contrary to the uniqueness of H. It has been shown any minimal submodule of vM is central in vM or $vM = \{0\}$.

Now R (the right ideal of N generated by M) is such that it can be embedded into a finite direct sum D of N-groups of the form vR (v coming from V) in such a way that $soc(N) \cap R$ embeds into the socle of D. However each vR component is a vM ($vR = vM$) and from above has socle central in D. The fact that $R \cap soc(N)$ centralizes R follows from the embedding.

Step 3. In this step it is shown that with e a left identity of M (exists by Step 1, [6] and [8]), $(0 : e) + R = N$ (R as in Step 2) and for some minimal right idea X of N contained in R, $X \cap (0 : e) = \{0\}$.

As e is a left identity of M it is certainly true $N = M + (0 : e)$, where $M \cap (0 : e) = \{0\}$. Since $R \geq M$, $(0 : e) + R$ is necessarily $= N$. It is not that difficult to see $soc(M) = M \cap soc(N)$. This implies $R \cap soc(N)$ (it contains $soc(M)$ which is $\neq \{0\}$) is not contained in $(0 : e)$. As $R \cap soc(N)$ is a direct sum of minimal right ideals of N contained in R the right ideal X exists.

Step 4. In this step the proof is finally completed. It is shown the X of Step 3 must be $\leq Z(N)$.

As the sum $(0 : e) + X$ is direct it follows $C_V(vX) \geq v(0 : e)$ for all v in V. However by Step 2, $C_V(vX)$ is $\geq vR$ for all v in V. Thus for all v in V, $C_V(vX) \geq v(0 : e) + vR = vN$ (see Step 3). This readily implies $X \leq Z(N)$. Indeed, if α and β are in N and η in X, it is an elementary matter to show that $v[(\alpha + \eta)\beta - \eta\beta - \alpha\beta] = 0$ for all v in V so that $(\alpha + \eta)\beta - \eta\beta - \alpha\beta = 0$. The lemma is proved. $\diamondsuit$

Proof of 11.2. A primary N-group is certainly semiprimary and therefore, by 11.1, (c) implies (b). That (b) implies (a) follows straight from the definition of N being semiprimary.

Suppose (a) holds. N being 3-tame has a faithful 3-tame N-group V_1 with minimal submodules N-isomorphic to a minimal right ideal R_1 of N (there may be others). If the sum of all such minimal submodules is W_1 and W_2 is a submodule of V_1 maximal for avoiding W_1 (i.e. a maximal complement – see [2] or [3]), then V_1/W_2 is a faithful 3-tame N-group. This is because no minimal right ideal R_2 of N is $\leq (W_2 : V_1)$ as R_2 is necessarily N-isomorphic to R_1 while if v_1R_2 ($v_1 \in V_1$) were $\neq \{0\}$ it would be $\leq W_2$. As any minimal submodule U of V_1/W_2 ($= V$ say) is, from the maximality of W_2, contained in $(W_1 + W_2)/W_2$ it must follow that $soc(V/W_2) = (W_1 + W_2)/W_2$ and $soc(V/W_2)$ is homogeneous. If U is not a ring module then it is easy enough to see it is the unique minimal submodule of $soc(V)$ (thus of V) and V is primary. On the other hand if U is a ring module 11.3 supplies us with a contradiction. Thus (a) implies (c) and 11.2 is proved. ♢

Acknowledgement. In conclusion I would like to thank the referee for helpful comments. As things were originally the proof of 10.2 was flawed. Providing 10.1 overcame this. The referee's pointing out my error meant that what might possibly be regarded as the most substantial result of this paper (viz. 10.3) stood firm.

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