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STRICTLY ORDER-PRESERVING MAPS INTO $\mathbb{Z}$, I

A PROBLEM OF DAYKIN FROM THE 1984 BANFF CONFERENCE ON GRAPHS AND ORDER

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Abstract: Let S be a subset of a poset P, and let $g : S \to \mathbb{Z}$ be a strictly order-preserving map into the linearly ordered set of integers. Necessary and sufficient conditions are found for there to be a strictly order-preserving map $\Psi : P \to \mathbb{Z}$ extending g. This solves a problem of Daykin from the 1984 Banff Conference on Graphs and Order. In 1985, Daykin and Daykin asked for a solution to the extension problem both for the case where g and Ψ are strictly order-preserving maps – which is settled in this note – and for the case where g and Ψ are one-to-one order-preserving maps – which remains unsettled, but regarding which the following is shown in the present work:
Let P be a poset and S a subset such that P is the convex hull of S. Let $g : S \to \mathbb{Z}$ be an injective order-preserving map. Necessary and sufficient conditions are found for when $g : S \to \mathbb{Z}$ has an injective order-preserving extension $\Psi : P \to \mathbb{Z}$ to all of P. The task of trying to prove this theorem was set by Skilton in 1985.

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1. Motivation

Let P and Q be posets. A function $f : P \to Q$ is *strictly order-preserving* if whenever $p, p' \in P$ and $p < p'$, then $f(p) < f(p')$.

Suppose P is a poset, S a subset and $g : S \to \mathbb{Z}$ a strictly order-preserving map from S into the linearly ordered set of integers. At the 1984 Banff Conference on Graphs and Order, David Daykin, of the celebrated Ahlswede–Daykin "Four Functions" Theorem [1], asked for necessary and sufficient conditions (assuming P is countable and locally finite) for there to exist a strictly order-preserving map $\Psi: P \to \mathbb{Z}$ extending g.

We solve this problem without any cardinality assumptions (Th. 4.12).

If g is in addition one-to-one, Daykin asked for necessary and sufficient conditions guaranteeing that g has an extension to P that is also one-to-one. (See [3], Problem 8.1 and [8], pp. 532–533. While in [8], p. 532, Daykin says, "There are really two problems here. . . ," note that in [3], the two questions were posed as a single problem and also posed "for P countably infinite and for P noncountably infinite." Also note that Daykin's terminology is different than ours.[1])

If $\ell_P[p, q]$ is the length of the interval $[p, q]$ in the poset P (the cardinality of the largest chain in the interval, minus 1), then an obvious necessary condition is that every interval must have finite length, and that, morever, for all $s, t \in S$ such that $s < t$

$$\ell_P[s, t] \leq |g(t) - g(s)|.$$

Daykin and Daykin proved that this condition is sufficient for finite P for the first problem.

If we assume that g is injective, and we wish to extend it to an injective order-preserving map, we might first ask: When is it the case that there exists *some* injective order-preserving map from P to $\mathbb{Z}$? Skilton [9], Th. 1 has shown that such a map exists if and only if P is

[1]Daykin [8], p. 532 refers to "strict order-preserving maps" and "arbitrary order-preserving maps," whereas Daykin and Daykin [3] use the terms "order preserving injection" and "order preserving map," specifying that, by the latter expression, they mean a "strict order preserving" map, adding, "[W]e omit the word strict." The term "locally finite" is not defined in [8], p. 532.

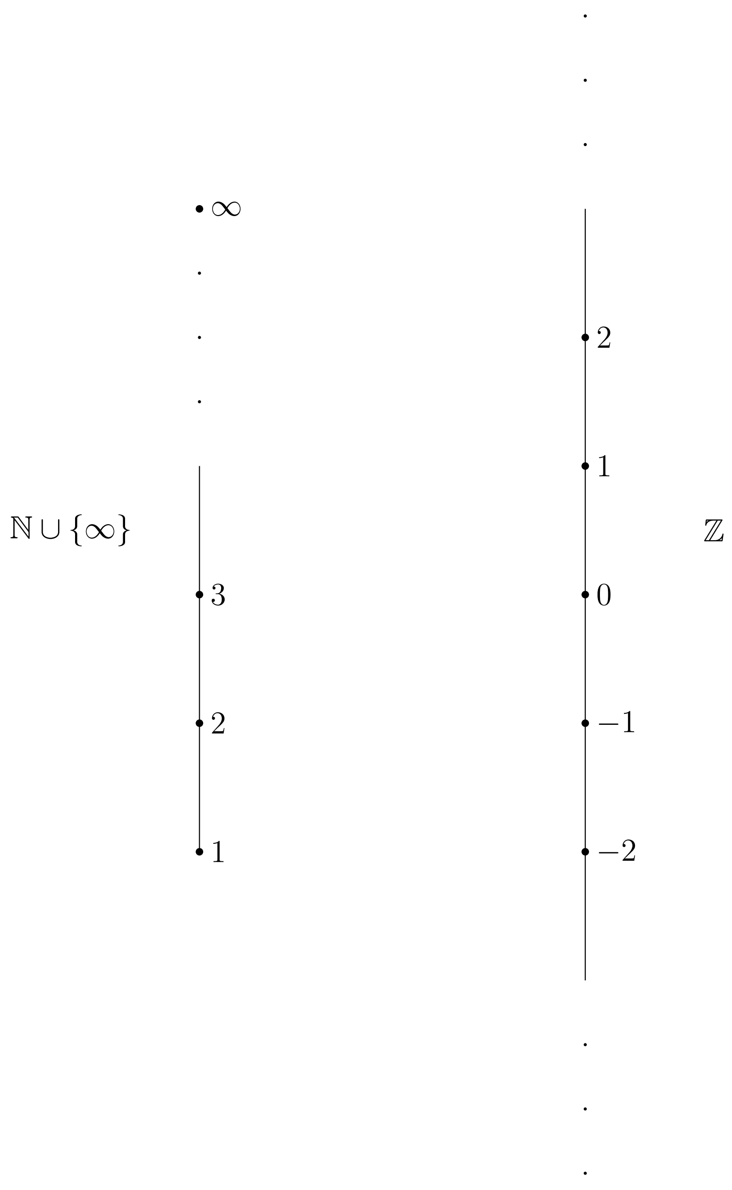

Figure 1.1. A poset that admits no injective order-preserving map into $\mathbb{Z}$

countable and every interval is finite. (For example, there is no injective order-preserving map from the chain $\mathbb{N} \cup \{\infty\}$ to $\mathbb{Z}$; see Fig. 1.1.)

There are other necessary conditions: Suppose s, $s' \in S$ and $s \leq s'$. Then if an extension $\Psi : P \to \mathbb{Z}$ of $g : S \to \mathbb{Z}$ exists, every element of the interval $[s, s']$ in P must go to a different element of the interval $[g(s), g(s')]$ of $\mathbb{Z}$. So we must have $\big|[s, s']\big| \leq \big|[g(s), g(s')]\big|$.

In general, we must have

$$\Bigg| \bigcup_{\substack{v,v' \in V \\ v \leq v'}} [v, v'] \Bigg| \leq \Bigg| [\min_{v \in V} g(v), \max_{v \in V} g(v)] \Bigg|$$

for all finite $V \subseteq S$.

Daykin and Daykin proved that, if P is finite, this condition is also sufficient [3], Th. 8.1. Skilton proved that the condition is still sufficient [9], Th. 3 even if P is infinite, provided that S is finite. (We are assuming, of course, that P *can* be mapped injectively into $\mathbb{Z}$ in an order-preserving fashion.)

Example 1.1. Let P be the poset $\{a, b, c, x, y, u, v\}$ where $a, u < x < v, b$; $u < c < v, y$; $y < b$; and the only other comparabilities are the necessary ones (Fig. 1.2). Let $S = \{a, b, c\}$.

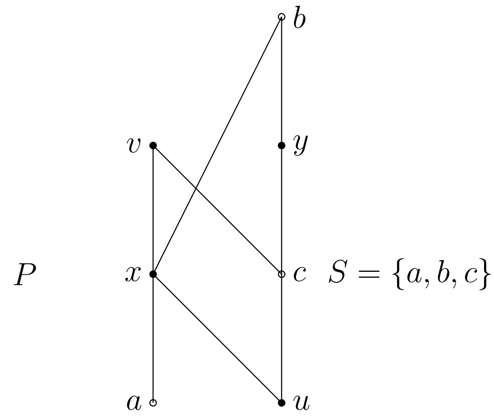

Figure 1.2. The poset P and the subset S

Suppose $g : S \to \mathbb{Z}$ is given by $g(a) = -1$, $g(b) = 3$, and $g(c) = 0$ (Fig. 1.3). Then $g : S \to \mathbb{Z}$ *does* have an injective order-preserving extension; for example, $\Psi(x)=1$, $\Psi(y)=2$, $\Psi(u)=-666$, and $\Psi(v)=42$ (Fig. 1.4).

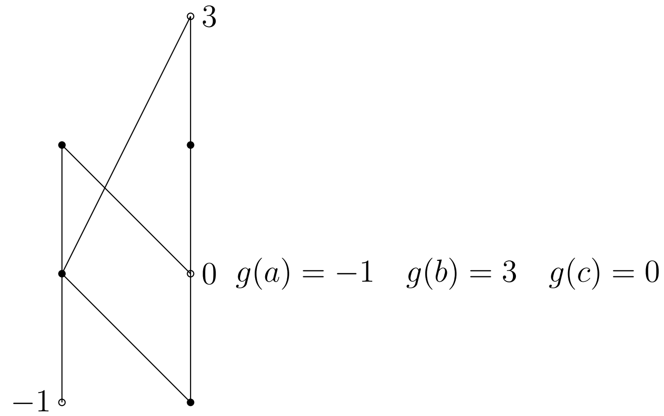

Figure 1.3. A partial injective map from P to $\mathbb{Z}$

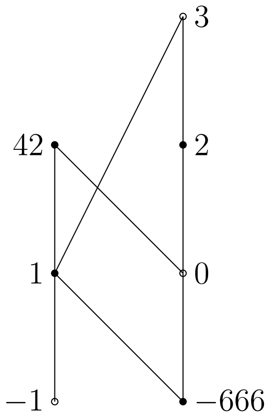

Figure 1.4. An injective order-preserving extension

$\Psi : P \to \mathbb{Z}$ of the map of Figure 1.3

On the other hand, if $g : S \to \mathbb{Z}$ is given by $g(a) = -1$, $g(b) = \mathbf{2}$, and $g(c) = 0$ (Fig. 1.5), then $g : S \to \mathbb{Z}$ has *no* injective order-preserving extension $\Psi : P \to \mathbb{Z}$: It is easy to see that such a map $\Psi : P \to \mathbb{Z}$ must send both x and y to 1.

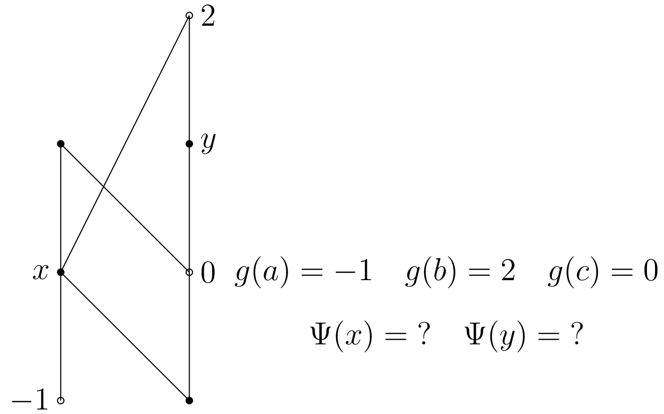

Figure 1.5. Another partial injective map from P to $\mathbb{Z}$

We can also use the Daykin–Daykin criterion: Letting $V = S$, we see that

$$\bigcup_{\substack{v,v' \in V \\ v \leq v'}} [v, v']$$

has 5 elements ($\{a, b, c, x, y\}$), but

$$[\min_{v \in V} g(v), \max_{v \in V} g(v)] = [-1, 2]$$

has only 4.

As an initial step towards the solution of the general extension problem, Skilton proposed tackling the case where the entire poset is the convex hull of S, that is,

$$P = \bigcup_{\substack{s,s' \in S \\ s \leq s'}} [s, s'].$$

We prove that, in this case, the same conditions used above are both necessary and sufficient (Th. 5.3).

2. Definitions and notation

See [2] for definitions, notation, and basic results.

Let $\mathbb{N}_0 := \{0, 1, 2, \dots\}$ and let $2\mathbb{Z}$ denote the set of even integers (and $\mathbb{Z} \setminus 2\mathbb{Z}$ the set of odd integers). If S is a set, let $|S|$ denote the cardinality of S. Given sets T and U and a function $f : T \to U$, let $f[T] = \{ f(t) \mid t \in T \}$.

Let P be a poset. Given $p \in P$, let $\downarrow p = \{ q \in P \mid q \leq p \}$ and let $\uparrow p = \{ q \in P \mid p \leq q \}$. Given $Q \subseteq P$, let

$$\downarrow Q = \bigcup_{q \in Q} \downarrow q$$

and let

$$\uparrow Q = \bigcup_{q \in Q} \uparrow q;$$

if $p \in P$, let $\downarrow_Q p = Q \cap \downarrow p$ and let $\uparrow_Q p = Q \cap \uparrow p$; we also define $\downarrow_Q R$ and $\uparrow_Q R$ for a subset $R \subseteq P$. For $p, q \in P$ with $p \leq q$, the *interval* $[p, q]$ is the set $\uparrow p \cap \downarrow q$. A poset is *locally chain-bounded* if every interval has finite length. (If a poset has a strictly order-preserving map into $\mathbb{Z}$, it must be locally chain-bounded.) A poset is *locally finite* if every interval is finite [10], p. 98. Given $Q \subseteq P$, the *convex hull* of Q is the set

$$\overline{Q} = \uparrow Q \cap \downarrow Q = \bigcup_{\substack{q,q' \in Q \\ q \leq q'}} [q, q'].$$

(We will only use this notation for the convex hull in §5.) A subset S of P is *convex* if $[p, q] \subseteq S$ for all $p, q \in S$ such that $p \leq q$. Note that Q is a convex subset of P if and only if $Q = \overline{Q}$. Skilton calls a subset $Q \subseteq P$ *dense* if $P = \overline{Q}$. (This is different from some other uses of the word "dense" in the literature.)

A maximal antichain $A \subseteq P$ is *separating* if, for all $p \in \downarrow A$ and $p' \in \uparrow A$, there exists $a \in [p, p'] \cap A$ whenever $p \leq p'$.

For $p, q \in P$, we write $p \lessdot q$ if $p < q$ and there is no element of P strictly between p and q.

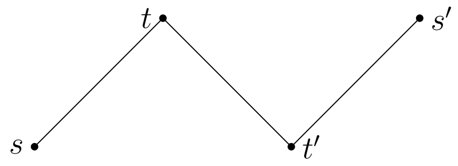

Figure 2.1. The poset Q

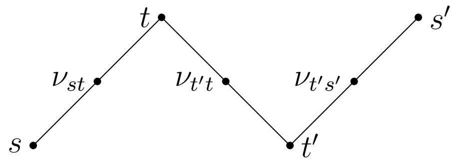

Figure 2.2. The poset $\nu(Q)$

Given a locally chain-bounded poset Q, let $\nu(Q)$ be a new poset consisting of Q and the elements

$$\{\, \nu_{qr} \mid q, r \in Q \text{ and } q \lessdot r \,\},$$

where, for all $q, r \in Q$ such that $q \lessdot r$, we have $q < \nu_{qr} < r$, and no other comparabilities hold but the necessary ones. (See Figures 2.1 and 2.2.)

Remark. The diagram for $\nu(Q)$ is just a subdivision of the diagram for Q. Compare this with the construction in the proof of [6], Th. 7.

Let P be a poset. Let $S \subseteq P$. Let $p \in P$. Let $g : S \to \mathbb{Z}$ be a function. Define

$$\begin{aligned} {}_g p_P &= \sup\{\, g(s) + \ell_P[s,p] \mid s \in{\downarrow_S}\, p \,\} \in \mathbb{Z} \cup \{-\infty, \infty\}, \\ p_P^g &= \inf\{\, g(s) - \ell_P[p,s] \mid s \in{\uparrow_S}\, p \,\} \in \mathbb{Z} \cup \{-\infty, \infty\}. \end{aligned}$$

If the poset in which ${}_g p_P$ and p_P^g are being calculated is understood, we will write ${}_g p$ and p^g, respectively.

Let $\mathrm{Min}P$ and $\mathrm{Max}P$ denote the sets of minimal and maximal elements of a poset P, respectively. A poset in which every element is minimal or maximal is called *bipartite*. A *braid* is obtained from a bipartite poset P such that $\mathrm{Min}P \cap \mathrm{Max}P = \emptyset$ by replacing every edge with a finite chain of positive length.

Example 2.1. Let $P := \{a, b, c, d\}$ be the bipartite poset such that $a, b < c, d$ with no other non-trivial comparabilities (Fig. 2.3).

Let $P' := \{a, b, c, d, x, y, z\}$ be the braid in which $a < x < c$; $a < d$; $b < c$; and $b < y < z < d$, with no other non-trivial comparabilities (Fig. 2.4).

Let P be a braid. For all $p \in P$, let $\ell_P^p : \mathrm{Min}P \to \mathbb{N}_0$ be the partial function defined for all $m \in \mathrm{Min}P$ by

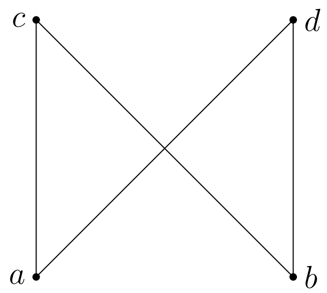

Figure 2.3. The bipartite poset P

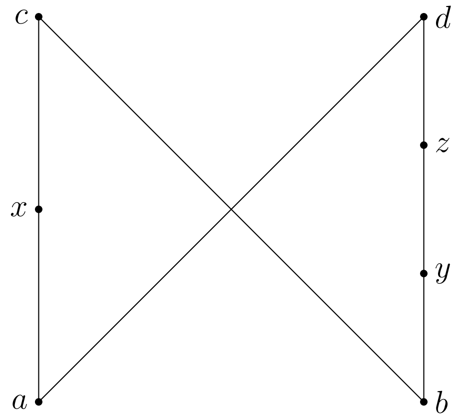

Figure 2.4. The braid poset P'

$$\ell_P^p(m) := \begin{cases} \ell_P[m,p] & \text{if } m \leq p, \\ \text{undefined} & \text{otherwise.} \end{cases}$$

Let C be a set. Let $f : C \to \mathbb{N}_0$ be a partial function with domain $\mathrm{Dom} f$ and let $g : C \to \mathbb{N}_0$ be a function. We say f *dominates* g if

$$\sup\{f(c) - g(c) | c \in \mathrm{Dom} f\} = \infty.$$

Let $\mathcal{B}$ be the class of braid posets P such that, for all $g : \mathrm{Min} P \to \mathbb{N}_0$, there exists $p \in P$ for which ℓ_P^p dominates g.

Example 2.2 (B. S. W. Schröder, personal communication)**.** Let B be a braid poset with minimal elements $\{x_n | n \in \mathbb{N}_0\}$ and maximal elements

$$\{y_f | f : \mathbb{N}_0 \to \mathbb{N}_0\}.$$

For $n \in \mathbb{N}_0$ and $f : \mathbb{N}_0 \to \mathbb{N}_0$, suppose $x_n < y_f$ and let $[x_n, y_f]$ be a chain of length $f(n)+1$. Let no other non-trivial comparabilities hold. Clearly $B \in \mathcal{B}$, for every $g : \mathrm{Min} B \to \mathbb{N}_0$ may be interpreted as a function $g' : \mathbb{N}_0 \to \mathbb{N}_0$ which is dominated by $f(n) := g'(n) + n$. Clearly $\ell_P^{y_f}$ dominates g.

3. Posets that admit a strictly order-preserving map into $\mathbb{Z}$

In our solution to the extension problem for strictly order-preserving maps, we require the assumption that the poset P have *some* strictly order-preserving map into $\mathbb{Z}$. We do not have a nice characterization of such posets, but we show that there is a class $\mathcal{B}$ of simple posets that do not admit such maps, and any poset that does not admit such a map must be related in a clear way to a poset in $\mathcal{B}$. Ways in which our results ought to be improved will be suggested at the end of the section.

Lemma 3.1. *Let Q be a locally chain-bounded poset. Then $P := \nu(Q)$ is locally chain-bounded and every maximal antichain of P is separating.*

Proof. Clearly P is locally chain-bounded. Let A be a maximal antichain of P. Assume that $p \in\downarrow A$; $p' \in\uparrow A$; and $p < p'$. Suppose for a contradiction that $A \cap [p, p'] = \emptyset$.

Choose $r \in P$ maximal in $\downarrow A \cap [p, p']$. Choose $s \in P$ minimal in $\uparrow A \cap [r, p']$. Then $r < s$.

If it is false that $r \lessdot s$, then there exists $t \in P$ such that $r < t \lessdot s$. Hence $t \notin\uparrow A$, so that $t \in\downarrow A$ by the maximality of A, contradicting the maximality of r. Therefore $r \lessdot s$.

Either $r \in P \setminus Q$ or $s \in P \setminus Q$, so that either $r \in A$ or $s \in A$, and $A \cap [p, p'] \neq \emptyset$ after all. $\Diamond$

Lemma 3.2. *Let C be a set and $g : C \to \mathbb{N}_0$ a function such that*

$$\sup\{g(c) | c \in C\} = \infty.$$

Define $f : C \to \mathbb{N}_0$ by $f(c) := 2^{g(c)}$ for all $c \in C$.

Then f dominates g.

Here is another characterization of the braids in the class $\mathcal{B}$.

Lemma 3.3. *Let P be a braid poset. The following are equivalent:*

(1) *$P \in \mathcal{B}$;*

(2) *For all $g : \mathrm{Min}P \to \mathbb{N}_0$, there exist $p \in \mathrm{Max}P$ and a countably infinite set $C \subseteq \mathrm{Dom}\ell_P^p$ such that $g(c) \leq \ell_P^p(c)$ for all $c \in C$ and*

$$\sup_{c \in C} \ell_P^p(c) = \infty.$$

Proof. Assume (2). Given $g : \mathrm{Min}P \to \mathbb{N}_0$, let $p \in P$ be such that there exists a countably infinite $C \subseteq \mathrm{Dom}\ell_P^p$ where $2^{g(c)} \leq \ell_P^p(c)$ for all $c \in C$ and $\sup_{c \in C} \ell_P^p = \infty$. Then $\sup_{c \in C}\{\ell_P^p(c) - g(c)\} = \infty$. $\Diamond$

Lemma 3.4 (cf. B. S. W. Schröder)**.** *Let Q be a poset such that $Q = \uparrow \mathrm{Min}Q$. The following are equivalent:*

(1) *There exists a strictly order-preserving map $\Phi : Q \to \mathbb{Z}$.*

(2) *There exists a map $g \colon \mathrm{Min}Q \to \mathbb{Z}$ such that, for all $q \in Q$, ${}_g q < \infty$.*

In case (2)*, we may define $\Phi : Q \to \mathbb{Z}$ for all $q \in Q$ by $\Phi(q) := {}_g q$; in case* (1)*, we may define $g : \mathrm{Min}Q \to \mathbb{Z}$ by $g := \Phi \restriction \mathrm{Min}Q$.*

Lemma 3.5. *Let P be a braid. If $\Phi : P \to \mathbb{Z}$ is a strictly order-preserving map, then there exists a strictly order-preserving map $\Psi : P \to \mathbb{Z}$ such that $\Psi[\mathrm{Min}P] \subseteq \{-1, -2, -3, \dots\}$ and $\Psi[\mathrm{Max}P] \subseteq \{1, 2, 3, \dots\}$.*

Corollary 3.6. *Let $P \in \mathcal{B}$. Then there is no strictly order-preserving map from P to $\mathbb{Z}$.*

Proof. The corollary follows from Lemma 3.4(1) and Lemma 3.5. $\Diamond$

The following lemma is essentially due to B. S. W. Schröder.

Lemma 3.7. *Let Q be a poset. Let $A \subseteq Q$ be a separating antichain. The following are equivalent:*

(1) *There exists a strictly order-preserving map $\Phi : Q \to \mathbb{Z}$.*

(2) *There exists a map $g : A \to \mathbb{Z}$ such that ${}_g q < \infty$ if $q \in \uparrow A$, and $q^g > -\infty$ if $q \in \downarrow A$.*

In case (2)*, for all $q \in Q$, let*

$$\Phi(q) = \begin{cases} {}_g q & \text{if } q \in \uparrow A, \\ q^g & \text{if } q \in \downarrow A. \end{cases}$$

In case (1)*, let $g := \Phi \restriction A$.*

Lemma 3.8. *Let Q be a poset. Let $A \subseteq Q$ be a separating antichain. Define a new poset P as follows. Let P have $\uparrow A$ as a subposet; order $D := Q \setminus \uparrow A$ as an antichain. For all $a \in A$, $d \in D$ such that $d \leq a$, add a chain from d to a of length $\ell_Q[d, a]$ whose elements (except for the endpoints) are disjoint from the rest of the poset; if $\ell_Q[d, a] = \infty$, let this chain (minus the endpoints) be order-isomorphic to the poset of negative integers.*

The following are equivalent:

(1) *There exists a strictly order-preserving map $\Phi : Q \to \mathbb{Z}$.*

(2) *There exists a strictly order-preserving map $\Psi : P \to \mathbb{Z}$.*

Lemma 3.9. *Let Q be a poset. Let $p, q \in Q$ be such that $\ell[p, q] = \infty$. Then there exist $B \in \mathcal{B}$ and a strictly order-preserving map $\Psi : B \to Q$. Moreover, there is no strictly order-preserving map from Q to $\mathbb{Z}$.*

Proof. Take any $B \in \mathcal{B}$ (for instance, the braid of Ex. 2.2) and send the minimal elements to p, the maximal elements to q, and the internal chains to chains in $[p, q]$ that are long enough. $\Diamond$

Proposition 3.10. *Let Q be a poset with a separating antichain. The following are equivalent:*

(1) *There is no strictly order-preserving map from Q to $\mathbb{Z}$.*

(2) *For some $B \in \mathcal{B}$, there exists a strictly order-preserving map $\Psi : B \to Q$.*

Proof. Cor. 3.6 shows that (2) implies (1). Now assume (1). If $\ell[p,q] = \infty$ for some $p, q \in Q$ with $p \leq q$, use Lemma 3.9. Otherwise, form the poset P of Lemma 3.8. Now, considering $\mathrm{Min}P = D \cup (A \cap \mathrm{Min}Q)$ as the separating antichain in Lemma 3.7, we see that for any function $g : \mathrm{Min}P \to \mathbb{Z}$, there exists $p \in P$ such that ${}_g p_P = \infty$; it is clear that $p \in (\uparrow_Q A) \setminus \mathrm{Min}Q$. Also, for all $q \in (\uparrow_Q A) \setminus \mathrm{Min}Q$, ${}_g q_P \leq {}_g q_Q$. Now form the braid B such that

$$\mathrm{Min}B = \mathrm{Min}P \quad \text{and} \quad \mathrm{Max}B = Q \setminus \mathrm{Min}P,$$

and for $m \in \mathrm{Min}B$, $n \in \mathrm{Max}B$ such that $m \leq_Q n$, there is a chain of length $\ell_Q[m, n]$. If $f : \mathrm{Min}B \to \mathbb{N}_0$, let $g := -f$. Then there exists $p \in \mathrm{Max}B$ and a countably infinite set $C \subseteq \mathrm{Dom}\ell_B^p$ such that

$$\sup_{c \in C} \{g(c) + \ell_Q[c, p]\} = \infty,$$

so without loss of generality $\ell_B[c, p] - f(c) \geq 0$ and $\sup_{c \in C} \ell_B[c, p] = \infty$. By Lemma 3.3, $B \in \mathcal{B}$. Clearly there is a strictly order-preserving map from B to Q. $\Diamond$

Lemma 3.11. *Let Q be a locally chain-bounded poset. The following are equivalent:*

(1) *For some $C \in \mathcal{B}$, there exists a strictly order-preserving map $\Phi : C \to Q$.*

(2) *For some $B \in \mathcal{B}$, there exists a strictly order-preserving map $\Psi : B \to$ $\to \nu(Q)$.*

Proof. Assume (2). Without loss of generality $\Psi[\mathrm{Max}B], \Psi[\mathrm{Min}B] \subseteq Q$. Construct the braid C by letting $\mathrm{Min}C := \mathrm{Min}B$ and $\mathrm{Max}C := \mathrm{Max}B$, only remove every other node of the "internal" chains. Each chain of C is therefore at least half the size of the corresponding chain of B. The map Φ can then be constructed using Ψ so that $\Phi[C] \subseteq Q$. Lemma 3.3 tells us that $C \in \mathcal{B}$. $\Diamond$

Lemma 3.12. *Let Q be a locally chain-bounded poset. Let $P := \nu(Q)$. The following are equivalent:*

(1) *There exists a strictly order-preserving map $\Phi : Q \to \mathbb{Z}$.*

(2) *There exists a strictly order-preserving map $\Psi : P \to \mathbb{Z}$.*

Moreover, we can assume $\Psi[Q] \subseteq 2\mathbb{Z}$ and $\Psi[P \setminus Q] \subseteq \mathbb{Z} \setminus 2\mathbb{Z}$.

Proof. Assume (1). We may assume $\Phi[Q] \subseteq 2\mathbb{Z}$. Then, for all $p, q \in Q$ such that $p \lessdot q$ in Q, we may let $\Psi(p) = \Phi(p)$ and $\Psi(\nu_{pq}) = \Phi(p) + 1$. $\Diamond$

Theorem 3.13. *Let P be a poset. The following are equivalent:*

(1) *There is no strictly order-preserving map from P to $\mathbb{Z}$.*

(2) *For some $B \in \mathcal{B}$ there exists a strictly order-preserving map $\Psi : B \to P$.*

Proof. If P is not locally chain-bounded, use Lemma 3.9. Otherwise, use Lemma 3.1, Prop. 3.10, Lemma 3.11, and Lemma 3.12. $\Diamond$

Th. 3.13 is admittedly unsatisfactory. It would be better (if possible) to have a forbidden subposet characterization, or to find a class $\mathcal{C}$ that could serve in place of $\mathcal{B}$ in that theorem, but which consists of a nicer class of posets, perhaps even a single poset.

4. The extension problem for strictly order-preserving maps into $\mathbb{Z}$

Lemma 4.1. *Let Q be a poset; let $\widehat{\Phi} : Q \to 2\mathbb{Z}$ be a strictly order-preserving map. Let $P := \nu(Q)$. Let $S \subseteq Q$ and let $g : S \to 2\mathbb{Z}$ be a strictly order-preserving map.*

The following are equivalent:

(1) *There exists a strictly order-preserving map $\Phi : Q \to 2\mathbb{Z}$ extending g.*

(2) *There exists a strictly order-preserving map* $\Psi : P \to \mathbb{Z}$ *extending* g *such that* $\Psi[Q] \subseteq 2\mathbb{Z}$.

Proof. Clearly (2) implies (1).

Assume (1). For every $p, q \in Q$ such that $p \lessdot q$ in Q, we have $\Phi(p) < \Phi(p) + 1 < \Phi(q)$ since $\Phi(p), \Phi(q) \in 2\mathbb{Z}$, so let

$$\Psi(\nu_{pq}) = \Phi(p) + 1 \in \mathbb{Z} \setminus 2\mathbb{Z}.$$

For every $p \in Q$, let $\Psi(p) = \Phi(p)$.

Since $S \subseteq Q$, for all $s \in S$, $\Psi(s) = \Phi(s) = g(s)$. Hence Ψ extends g. Since $\Phi[Q] \subseteq 2\mathbb{Z}$, then $\Psi[Q] \subseteq 2\mathbb{Z}$. Let $p, p' \in P$ be such that $p < p'$. If $p, p' \in Q$, then $\Psi(p) = \Phi(p) < \Phi(p') = \Psi(p')$. If $p \in Q$, $p' \in P \setminus Q$, then assume $p' = \nu_{tu}$ where $t \lessdot u$ in Q; then $p \leq t$, so $\Psi(p) = \Phi(p) \leq \Phi(t) < \Psi(\nu_{tu}) = \Psi(p')$. If $p \in P \setminus Q$, $p' \in Q$, then assume $p = \nu_{tu}$ where $t \lessdot u$ in Q; then $u \leq p'$, so $\Psi(p) < \Phi(u) \leq \Phi(p') = \Psi(p')$. If $p, p' \in P \setminus Q$, then assume $p = \nu_{tu}$ and $p' = \nu_{vw}$ where $t \lessdot u$ in Q and $v \lessdot w$ in Q; then $u \leq v$, so $\Psi(p) = \Psi(\nu_{tu}) < \Phi(u) \leq \Phi(v) < \Psi(\nu_{vw}) = \Psi(p')$. Hence Ψ is strictly order-preserving. $\Diamond$

Lemma 4.2. *Let all be as in Lemma 4.1. Assume* S *is a convex subset of* Q. *Let* $R := S \cup \{\nu_{st} | s \lessdot t \text{ in } Q \text{ and } s, t \in S\}$. *Let* $f : R \to \mathbb{Z}$ *be a strictly order-preserving extension of* g.

Then R *is a convex subset of* P, *and the following are equivalent:*

(1) *There exists a strictly order-preserving map* $\Phi : Q \to 2\mathbb{Z}$ *extending* g.

(2) *There exists a strictly order-preserving map* $\Psi : P \to \mathbb{Z}$ *extending* f *such that* $\Psi[Q] \subseteq 2\mathbb{Z}$.

Proof. Let $r, r' \in R$ and let $p \in P$ be such that $r < p < r'$. There exist $s, s' \in S$ such that $s \leq r$ and $r' \leq s'$. If $p \in Q$, then, since S is convex in Q, $p \in S$; hence $p \in R$. So assume $p \notin Q$. Let $p = \nu_{tu}$ where $t, u \in Q$ and $t \lessdot u$ in Q. Hence $s \leq t \leq u \leq s'$ and $t, u \in S$ by convexity, so $p \in R$. Thus R is a convex subset of P.

Assume (2). Let $\Phi : Q \to \mathbb{Z}$ be the restriction of Ψ to Q. By hypothesis, Φ maps into $2\mathbb{Z}$. It is of course strictly order-preserving. Let $s \in S$. Then $s \in R$, so $\Phi(s) = \Psi(s) = f(s) = g(s)$. Hence (1) holds.

Now assume (1). Let $p, q \in Q$ be such that $p \lessdot q$ in Q. We have $\Phi(p) < \Phi(p) + 1 < \Phi(q)$ since $\Phi(p), \Phi(q) \in 2\mathbb{Z}$. If $p, q \in S$, then $\Phi(p) = = g(p) = f(p) < f(\nu_{pq}) < f(q) = g(q) = \Phi(q)$. Thus, let

$$\Psi(\nu_{pq}) := \begin{cases} f(\nu_{pq}) & \text{if } p, q \in S, \\ \Phi(p) + 1 & \text{otherwise.} \end{cases}$$

For every $p \in Q$, let $\Psi(p) = \Phi(p)$. We note that for all $p, q \in Q$ such that $p \lessdot q$ in Q, $\Phi(p) = \Psi(p) < \Psi(\nu_{pq}) < \Psi(q) = \Phi(q)$.

The previous proof shows that $\Psi : P \to \mathbb{Z}$ is strictly order-preserving, $\Psi[Q] \subseteq 2\mathbb{Z}$, and Ψ extends g.

Let $r \in R$. If $r \in S$, then $\Psi(r) = \Phi(r) = g(r) = f(r)$. If $r \notin S$, then let $r = \nu_{st}$ where $s, t \in S$ and $s \lessdot t$ in Q. Then $\Psi(r) = f(\nu_{st}) = f(r)$. Hence Ψ extends f. $\Diamond$

Lemma 4.3. *Let all be as in Lemma 4.1. Let $R \subseteq P$ be a convex subset. Assume $\widehat{\Psi} : P \to \mathbb{Z}$ is a strictly order-preserving map. Let $f : R \to \mathbb{Z}$ be a strictly order-preserving map. Assume*

$$\widehat{\Psi}[Q], f[Q \cap R] \subseteq 2\mathbb{Z},$$
$$\widehat{\Psi}[P \setminus Q], f[R \setminus Q] \subseteq \mathbb{Z} \setminus 2\mathbb{Z}.$$

Let A be a maximal antichain of $P \setminus (\uparrow R \cup \downarrow R)$ and let $\widetilde{R} := A \cup R$. Define $\tilde{f} : \widetilde{R} \to \mathbb{Z}$ for all $\tilde{r} \in \widetilde{R}$ by

$$\tilde{f}(\tilde{r}) := \begin{cases} f(\tilde{r}) & \text{if } \tilde{r} \in R, \\ \widehat{\Psi}(\tilde{r}) & \text{if } \tilde{r} \in A. \end{cases}$$

The following are equivalent:

(1) *There exists a strictly order-preserving map $\Psi : P \to \mathbb{Z}$ extending f.*

(2) *There exists a strictly order-preserving map $\widetilde{\Psi} : P \to \mathbb{Z}$ extending $\tilde{f}$.*

(3) *For all $p \in P$, ${}_{\tilde{f}}p_P < \infty$ if $p \in \uparrow \widetilde{R}$, and $p_P^{\tilde{f}} > -\infty$ if $p \notin \uparrow \widetilde{R}$.*

Moreover, we may choose $\widetilde{\Psi}$ so that $\widetilde{\Psi}[Q] \subseteq 2\mathbb{Z}$.

Proof. Assume (3). Define $\widetilde{\Psi} : P \to \mathbb{Z}$ for all $p \in P$ by

$$\widetilde{\Psi}(p) := \begin{cases} {}_{\tilde{f}}p_P & \text{if } p \in \uparrow \widetilde{R}, \\ p_P^{\tilde{f}} & \text{if } p \notin \uparrow \widetilde{R}. \end{cases}$$

We show that $\widetilde{\Psi}$ is strictly order-preserving. Let $p_1, p_2 \in P$ be such that $p_1 \lessdot p_2$. Without loss of generality, $p_1 \notin \uparrow \widetilde{R}$ and $p_2 \in \uparrow \widetilde{R}$; also $|Q \cap \{p_1, p_2\}| = 1$; now use the convexity of $\widetilde{R}$. Hence (2) holds. $\Diamond$

Corollary 4.4. *Let all be as in Lemma 4.3. The following are equivalent:*

(1) *There exists a strictly order-preserving map* $\Psi : P \to \mathbb{Z}$ *extending* f.

(2) *For all* $p \in P$, ${}_f p_P < \infty$ *and* $p_P^f > -\infty$.

Moreover, we can assume $\Psi[Q] \subseteq 2\mathbb{Z}$.

Corollary 4.5. *Let all be as in Lemma 4.2. The following are equivalent:*

(1) *There exists a strictly order-preserving map* $\Psi : Q \to 2\mathbb{Z}$ *extending* g.

(2) *For all* $q \in Q$, ${}_g q_P < \infty$ *and* $q_P^g > -\infty$.

Proof. Clearly (1) implies (2). Now assume (2). By Lemma 3.12, there exists a strictly order-preserving map $\widehat{\Xi} : P \to \mathbb{Z}$ such that $\widehat{\Xi}[Q] \subseteq 2\mathbb{Z}$ and $\widehat{\Xi}[P \setminus Q] \subseteq \mathbb{Z} \setminus 2\mathbb{Z}$. Let us assume that $f : R \to \mathbb{Z}$ is a strictly order-preserving extension of g such that $f[R \setminus S] \subseteq \mathbb{Z} \setminus 2\mathbb{Z}$. We have (1) by Cor. 4.4. $\Diamond$

Corollary 4.6. *Let* Q *be a poset. Let* $S \subseteq Q$ *be a convex subset of* Q. *Let* $\widehat{\Phi} : Q \to \mathbb{Z}$ *be a strictly order-preserving map. Let* $g : S \to \mathbb{Z}$ *be a strictly order-preserving map.*

The following are equivalent:

(1) *There exists a strictly order-preserving map* $\Psi : Q \to \mathbb{Z}$ *extending* g.

(2) *For all* $q \in Q$, ${}_g q_Q < \infty$ *and* $q_Q^g > -\infty$.

Proof. Assume (2). Define $\widehat{\Omega} : Q \to 2\mathbb{Z}$ by $\widehat{\Omega} := 2\widehat{\Phi}$. Define $\overline{g} : S \to 2\mathbb{Z}$ by $\overline{g} := 2g$. Define $P := \nu(Q)$. For all $q \in Q$ and all $s \in \downarrow_S q$, we have $\overline{g}(s) + \ell_P[s, q] = 2g(s) + 2\ell_Q[s, q]$. Hence for all $q \in Q$, ${}_{\overline{g}} q_P < \infty$ and $q_P^{\overline{g}} > -\infty$. By Cor. 4.5, there exists a strictly order-preserving map $\overline{\Psi} : Q \to 2\mathbb{Z}$ extending $\overline{g}$. Let $\Psi := \frac{1}{2}\overline{\Psi}$. $\Diamond$

Lemma 4.7. *Let* $n \in \mathbb{N}_0$. *Let* Q *be a poset. Let* $S \subseteq Q$. *Let* $T := \uparrow S \cap \downarrow S$. *Let* $\widehat{\Phi} : Q \to \mathbb{Z}$ *be a strictly order-preserving map. Let* $g : S \to \mathbb{Z}$ *be a strictly order-preserving map. Let* $h : T \to \mathbb{Z}$ *be a strictly order-preserving map extending* g. *Let* $p \in (\uparrow S) \setminus T$ *be such that* ${}_g p < \infty$.

Define $k : T \to \mathbb{Z}$ *as follows: For all* $t \in T$, *let*

$$k(t) := \begin{cases} \max\{n, {}_g p\} - \ell[t, p] & \text{if } t \leq p \text{ and } h(t) + \ell[t, p] > \max\{n, {}_g p\}, \\ h(t) & \text{otherwise.} \end{cases}$$

Then $k : T \to \mathbb{Z}$ *is a strictly order-preserving map extending* g *and* $k(t) \leq h(t)$ *for all* $t \in T$. *Moreover, for all* $q \in (\uparrow S) \setminus T$, *if* ${}_h q < \infty$ *then*

${}_kq < \infty$; and for all $q \in (\downarrow S) \setminus T$, if $q^h > -\infty$, then $q^k > -\infty$. Finally, ${}_kp < \infty$.

Lemma 4.8. *Let α be an ordinal. Let Q be a poset. Let $S \subseteq Q$. Let*

$$T := \uparrow S \cap \downarrow S.$$

Let $(p_\beta)_{\beta<\alpha}$ be a sequence in $(\uparrow S) \setminus T$. Let $\widehat{\Phi} : Q \to \mathbb{Z}$ be a strictly order-preserving map. Let $g : S \to \mathbb{Z}$ be a strictly order-preserving map. Let $h_0 : T \to \mathbb{Z}$ be a strictly order-preserving map extending g. Assume that for all $q \in (\uparrow S) \setminus T$ we have ${}_gq < \infty$ and that for all $q \in (\downarrow S) \setminus T$ we have $q^g > -\infty$.

If $(n_\beta)_{\beta<\alpha}$ is a sequence in $\mathbb{N}_0$, define a sequence of functions

$$(h_\beta : T \to \mathbb{Z})_{\beta\leq\alpha}$$

as follows:

Assume $\beta < \alpha$ and $h_\beta : T \to \mathbb{Z}$ is defined.

Case (a) *${}_{h_\beta}p_\beta < \infty$. Let $h_{\beta+1} := h_\beta$.*

Case (b) *${}_{h_\beta}p_\beta = \infty$. Define $h_{\beta+1} : T \to \mathbb{Z}$ for all $t \in T$ by*

$$h_{\beta+1}(t) := \begin{cases} \max\{n_\beta, {}_gp_\beta\} - \ell[t, p_\beta] & \text{if } t \leq p_\beta \text{ and} \\ & h_\beta(t) + \ell[t, p_\beta] > \max\{n_\beta, {}_gp_\beta\}, \\ h_\beta(t) & \text{otherwise.} \end{cases}$$

Assume $\beta \leq \alpha$ is a limit ordinal and $h_\gamma : T \to \mathbb{Z}$ is defined for all $\gamma < \beta$. Define $h_\beta : T \to \mathbb{Z}$ for all $t \in T$ by $h_\beta(t) := \lim_{\gamma\to\beta} h_\gamma(t)$.

Then for all $\beta \leq \alpha$:

(1_β) *$h_\beta : T \to \mathbb{Z}$ is a strictly order-preserving map extending g. If β is a limit ordinal, then for all $t \in T$ there exists $\gamma < \beta$ such that $h_\beta(t) = h_\delta(t)$ whenever $\gamma \leq \delta < \beta$.*

(2_β) *For all $\gamma \leq \beta$, the following holds: if $q \in (\uparrow S) \setminus T$ and ${}_{h_\gamma}q < \infty$, then ${}_{h_\beta}q < \infty$.*

(3_β) *If $\beta < \alpha$ then ${}_{h_{\beta+1}}p_\beta < \infty$.*

Proof. By hypothesis (1_0) and (2_0) hold, and (3_0) holds by Lemma 4.7. Now assume $\beta < \alpha$ and (1_γ)–(3_γ) hold for all $\gamma \leq \beta$.

Case (a) ${}_{h_\beta}p_\beta < \infty$. Since $h_{\beta+1} = h_\beta$, then ($1_{\beta+1}$) and ($2_{\beta+1}$) hold. By Lemma 4.7, ($3_{\beta+1}$) holds.

Case (b) ${}_{h_\beta}p_\beta = \infty$. By Lemma 4.7, $(1_{\beta+1})$ and $(2_{\beta+1})$ hold. If $\beta + 1 < \alpha$ but ${}_{h_{\beta+2}}p_{\beta+1} = \infty$, then ${}_{h_{\beta+1}}p_{\beta+1} = \infty$ so by Lemma 4.7, ${}_{h_{\beta+2}}p_{\beta+1} < \infty$, a contradiction. Thus $(3_{\beta+1})$ holds.

Now assume $\beta \le \alpha$ is a limit and (1_γ)–(3_γ) hold for all $\gamma < \beta$. Then $h_\beta : T \to \mathbb{Z}$ is well defined by local chain-boundedness and the fact that, for all $\gamma < \beta$ and for all $t \in T$, there exists $s \in S$ such that $s \le t$ and $h_\gamma(t) \ge h_\gamma(s) = g(s) \in \mathbb{Z}$. Indeed, for all $t \in T$, there exists $\gamma < \beta$ such that $h_\beta(t) = h_\delta(t)$ whenever $\gamma \le \delta < \beta$. Clearly (1_β) holds. Also (2_β) holds since for all $\gamma < \beta$ and for all $t \in T$, $h_\beta(t) \le h_\gamma(t)$. If $\beta < \alpha$ and ${}_{h_{\beta+1}}p_\beta = \infty$, then ${}_{h_\beta}p_\beta = \infty$, so by Lemma 4.7, ${}_{h_{\beta+1}}p_\beta < \infty$, a contradiction. Hence (3_β) holds. $\Diamond$

Lemma 4.9. *Let all be as in Lemma 4.8. Let $q \in (\downarrow S) \setminus T$. Assume $q^{h_0} > -\infty$ but $q^{h_\beta} = -\infty$ for some $\beta \le \alpha$.*

Then there exist a strictly increasing sequence $(\beta_i)_{i<\omega}$ of ordinals less than β and a sequence $(t_i)_{i<\omega}$ of elements of T with $t_i \in [q, p_{\beta_i}]$ for $i < \omega$ such that

(1) $\lim_{i\to\omega} \ell[q, p_{\beta_i}] = \infty$ *and*

(2) *for all $i < \omega$, $\ell[q, p_{\beta_i}] > n_{\beta_i}$.*

Proof. Let β be the least ordinal $\gamma \le \alpha$ such that $q^{h_\gamma} = -\infty$. By Lemma 4.7, β is a limit. There exist $t_0, t_1, t_2, \ldots$ in $(\uparrow_T q) \setminus S$ such that
$$\inf\{h_\beta(t_i) - \ell[q, t_i] | i < \omega\} = -\infty.$$
Without loss of generality, $(h_\beta(t_i) - \ell[q, t_i])_{i<\omega}$ is a strictly decreasing sequence of negative integers less than q^{h_0}. As $q^{h_0} > -\infty$, for each $i < \omega$, there exists $\beta_i < \beta$ such that $h_{\beta_i}(t_i) \ne h_{\beta_i+1}(t_i) = h_\beta(t_i)$; this means $t_i \le p_{\beta_i}$. No ordinal can appear infinitely often in the sequence $(\beta_i)_{i<\omega}$ by the minimality of β. Thus without loss of generality $\beta_0 < \beta_1 < \beta_2 < \cdots$ [4], Lemma 6.8.

For $i < \omega$,
$$h_\beta(t_i) + \ell[t_i, p_{\beta_i}] \ge n_{\beta_i}$$
and
$$h_\beta(t_i) - \ell[q, t_i] < 0$$
so
$$\ell[q, t_i] + \ell[t_i, p_{\beta_i}] > n_{\beta_i}$$
and thus
$$\ell[q, p_{\beta_i}] > n_{\beta_i}.$$

If $M := \sup\{\ell[q, p_{\beta_i}] | i < \omega\} < \infty$, then $\inf_{i<\omega} h_\beta(t_i) = -\infty$; but then

$$\begin{aligned} h_\beta(t_i) &= \max\{n_{\beta_i}, {}_g p_{\beta_i}\} - \ell[t_i, p_{\beta_i}] \geq \\ &\geq n_{\beta_i} - M \geq -M, \end{aligned}$$

a contradiction. $\Diamond$

Corollary 4.10. *Let all be as in Lemma 4.8. Then there exists a sequence $(n_\beta)_{\beta<\alpha}$ in $\mathbb{N}_0$ such that, for all $q \in (\downarrow S) \setminus T$, if $q^{h_0} > -\infty$ then $q^{h_\alpha} > -\infty$.*

Proof. Suppose not. Consider the braid B formed in a certain way from the sets $Y = (\downarrow S) \setminus T$ and $Z = \{p_\beta | \beta < \alpha\}$, where Max$B$ as a set is $Y \cap \downarrow_Q Z$ and Min$B$ as a set is $Z \cap \uparrow_Q Y$, but both are of course ordered as antichains, and for $y \in \text{Max}B$ and $z \in \text{Min}B$ such that $y \leq z$ in Q, there is a chain in B of length $\ell_Q[y, z]$.

Every function from the set Min$B$ to $\mathbb{N}_0$ "extends" to a sequence $(n_\beta)_{\beta<\alpha}$ in $\mathbb{N}_0$. But for every sequence $(n_\beta)_{\beta<\alpha}$ in $\mathbb{N}_0$, there is a $q \in \in (\downarrow S) \setminus T$ such that $q^{h_0} > -\infty$ but $q^{h_\alpha} = -\infty$ (so $q \in \text{Max}B$ as well). By Lemma 4.9, there exists a countably infinite subset $\{\beta_i\}_{i<\omega}$ of ordinals less than α such that $\ell^q_B(p_{\beta_i}) > n_{\beta_i}$ for all $i < \omega$ and $\sup_{i<\omega} \ell^q_B(p_{\beta_i}) = \infty$.

By Lemma 3.3, $B \in \mathcal{B}$. Since there is obviously a strictly order-preserving map from B into the dual of Q, this contradicts Th. 3.13. $\Diamond$

Corollary 4.11. *Let Q be a poset. Let $S \subseteq Q$. Let $T := \uparrow S \cap \downarrow S$. Let $\widehat{\Phi} : Q \to \mathbb{Z}$ be a strictly order-preserving map. Let $g : S \to \mathbb{Z}$ be a strictly order-preserving map. Assume that for all $q \in (\uparrow S) \setminus T$ we have ${}_g q < \infty$ and for all $q \in (\downarrow S) \setminus T$ we have $q^g > -\infty$.*

Then the following are equivalent:

1. *There exists a strictly order-preserving map $h_0 : T \to \mathbb{Z}$ extending g.*
2. *There exists a strictly order-preserving map $h : T \to \mathbb{Z}$ extending g such that, for all $q \in (\uparrow S) \setminus T$ we have ${}_h q < \infty$ and for all $q \in (\downarrow S) \setminus T$ we have $q^h > -\infty$.*

Proof. Assume (1) holds. Let α equal

$$|\{q \in (\uparrow S) \setminus T : {}_{h_0} q = \infty\}| + \omega$$

(if there is a $q \in (\uparrow S) \setminus T$ such that ${}_{h_0} q = \infty$). By Cor. 4.10, there is a sequence $(n_\beta)_{\beta<\alpha}$ in $\mathbb{N}_0$ such that for all $q \in (\downarrow S) \setminus T$, if $q^{h_0} > -\infty$ then

$q^{h_\alpha} > -\infty$. By Lemma 4.8, $h_\alpha : T \to \mathbb{Z}$ is a strictly order-preserving map extending g and for all $q \in (\uparrow S) \setminus T$, we have ${}_{h_\alpha}q < \infty$.

Now do the same for $\{q \in (\downarrow S) \setminus T | q^{h_\alpha} = -\infty\}$. $\Diamond$

Theorem 4.12. *Let Q be a poset. Let $S \subseteq Q$. Let $g : S \to \mathbb{Z}$ be a strictly order-preserving map.*

Then (1) *and* (2) *are equivalent:*

(1) *There exists a strictly order-preserving map $\Psi : Q \to \mathbb{Z}$ extending g.*

(2)(a) *There exists a strictly order-preserving map from Q to $\mathbb{Z}$.*

(2)(b) *For all $s, t \in S$ such that $s < t$, $\ell_Q[s, t] \leq g(t) - g(s)$.*

(2)(c) *For all $q \in Q$, ${}_gq < \infty$ and $q^g > -\infty$.*

Proof. Assume (2). Let $T :=\uparrow S \cap \downarrow S$. Define $h_0 : T \to \mathbb{Z}$ for all $t \in T$ by $h_0(t) := {}_gt$. By (2)(a) and (b), h_0 is well defined, is strictly order-preserving, and extends g. By Cor. 4.11, there exists a strictly order-preserving map $h : T \to \mathbb{Z}$ extending g such that, for all $q \in (\uparrow S) \setminus T$ we have ${}_hq < \infty$ and for all $q \in (\downarrow S) \setminus T$ we have $q^h > -\infty$. By Cor. 4.6, there exists a strictly order-preserving map $\Psi : Q \to \mathbb{Z}$ extending h, hence g. $\Diamond$

As stated earlier, one defect of our theorem is (2)(a). Also, given the simplicity of the statement of the result – the "obvious" necessary conditions are also sufficient – it would not surprise us if there were a one-line proof of our theorem, which avoids altogether the use of the ancillary poset $\nu(Q)$ or transfinite induction.

5. Extending injective order-preserving maps into $\mathbb{Z}$ to convex hulls

We will use the following result.

Theorem 5.1 (Skilton [9], Th. 1)**.** *Let P be a poset. There exists an injective order-preserving map $\Psi : P \to \mathbb{Z}$ if and only if P is countable and locally finite.*

[*Mathematical Reviews* 86b:06002 erroneously states that *every* countable poset admits such an injection.]

Theorem 5.2 (Skilton [9], Th. 3)**.** *Let P be a countable locally finite poset. Let $S \subseteq P$ be a finite subset and suppose $g : S \to \mathbb{Z}$ is an injective order-preserving map. Then $g : S \to \mathbb{Z}$ has an injective order-preserving extension $\Psi : P \to \mathbb{Z}$ if and only if, for all $V \subseteq S$,*

$$\left|\overline{V}\right| \leq \left|\overline{g[V]}\right|.$$

Th. 5.2 extends a result of Daykin and Daykin for finite posets [3], Th. 8.1.

Skilton has suggested that "one might consider the problem of extending an [injective order-preserving map from] a *dense* subposet…" [9], §4.

In this section, we solve the problem suggested by Skilton.

Theorem 5.3. *Let P be a countable locally finite poset. Let $S \subseteq P$ be such that $P = \overline{S}$. Let $g : S \to \mathbb{Z}$ be a one-to-one order-preserving map. Then there exists a one-to-one order-preserving map $\Psi : P \to \mathbb{Z}$ extending g if and only if, for all finite subsets $V \subseteq S$, we have*

$$|\overline{V}| \leq |\overline{g[V]}|.$$

Proof. Necessity is clear. Now suppose the condition holds. For all $p \in P$, choose $s, t \in S$ such that $p \in [s, t]$ and let $A_p := [g(s), g(t)]$. For any finite $J \subseteq P$, by Th. 5.2 there exists an injective order-preserving map $\Theta_J : J \to \mathbb{Z}$ extending $g \restriction S \cap J$. By Rado's Selection Principle [7], Th. 4.1.1, there exists a one-to-one map $\Psi : P \to \mathbb{Z}$ such that, for all finite $J \subseteq P$, there exists a finite $K \subseteq P$ such that $J \subseteq K$ and $\Psi \restriction J = \Theta_K \restriction J$.

Let $p, q \in P$ be such that $p \leq q$. Let $J = \{p, q\}$. Then there exists a finite set $K \subseteq P$ such that $J \subseteq K$ and $\Psi(p) = \Theta_K(p)$ and $\Psi(q) = \Theta_K(q)$. But $\Theta_K(p) \leq \Theta_K(q)$. Thus Ψ is order-preserving.

Let $s \in S$. Let $J = \{s\}$. Then there exists a finite set $K \subseteq P$ such that $s \in K$ and $\Psi(s) = \Theta_K(s)$. But $\Theta_K(s) = g(s)$. Thus Ψ extends g. $\Diamond$

Something akin to the following result may be useful in solving the general problem.

Corollary 5.4. *Let P be a countable locally finite poset. Let $S \subseteq P$ and* *let* $g : S \to \mathbb{Z}$ *be a one-to-one order-preserving map. Let* $E \subseteq \mathbb{Z} \setminus g[S]$. *Let* $T := \overline{S}$.

There exists a one-to-one order-preserving map $h : T \to \mathbb{Z}$ *extending* g *such that* $E \cap h[T] = \emptyset$ *if and only if, for all finite subsets* $V \subseteq S$,

$$|\overline{V}| \leq |\overline{g[V]} \setminus E|.$$

Proof. Necessity is obvious. Now assume the condition. Let E' be a set $\{e' | e \in E\}$ of cardinality $|E|$ disjoint from P and ordered as an antichain. Let $S' := S \cup E'$ and let $g' : S' \to \mathbb{Z}$ be defined by $g'(s) = g(s)$ for all $s \in S$ and $g'(e') = e$ for all $e \in E$. Then $\overline{S'} = T \cup E'$ and for all finite $V' \subseteq S'$ – say $V := V' \cap S$ and $F' := V' \cap E'$ – we have $|\overline{V'}| = |\overline{V}| + |F'|$. Letting $F := \{f \in E \mid f' \in F'\}$ and $G := F \cap \overline{g[V]}$, we see that

$$\begin{aligned} & |\overline{V'}| \leq |\overline{g[V]} \setminus E| + |\overline{g[V]} \cap F| + |F \setminus \overline{g[V]}| \Rightarrow \\ \Rightarrow \quad & |\overline{V'}| \leq |\overline{g[V]}| + |F \setminus \overline{g[V]}| \Rightarrow \\ \Rightarrow \quad & |\overline{V'}| \leq |\overline{g[V]}| + |F \setminus G| \Rightarrow \\ \Rightarrow \quad & |\overline{V'}| \leq |\overline{g'[V']}|, \end{aligned}$$

so by Th. 5.3, there exists a one-to-one order-preserving map $h' : T \cup E' \to \to \mathbb{Z}$ extending g'. Thus $h := h' \restriction T$ extends g and $E \cap h[T] = \emptyset$. $\Diamond$

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References

[1] AHLSWEDE, R. and DAYKIN, D. E.: An inequality for the weights of two families of sets, their unions and intersections, *Zeitschrift für Wahrscheinlichkeitstheorie und verwandte Gebiete* **43** (1978), 183–185.

[2] DAVEY, B. A. and PRIESTLEY, H. A.: *Introduction to Lattices and Order*, second edition, Cambridge University Press, Cambridge, 2002.

[3] DAYKIN, D. E. and DAYKIN, J. W.: Order preserving maps and linear extensions of a finite poset, *SIAM Journal on Algebraic and Discrete Methods* **6** (1985), 738–748.

[4] FARLEY, J. D.: Perfect sequences of chain-complete posets, *Discrete Mathematics* **167/168** (1997), 271–296.

[5] FARLEY, J. D. and SCHRÖDER, B. S. W.: Strictly order-preserving maps into $\mathbb{Z}$, II: A 1979 problem of Erné, *Order* **18** (2001), 381–385.

[6] GRILLET, P. A.: Maximal chains and antichains, *Fundamenta Mathematicae* **65** (1969), 157–167.

[7] MIRSKY, L.: *Transversal Theory: An Account of Some Aspects of Combinatorial Mathematics*, Academic Press, New York, 1971.

[8] RIVAL, I. (ed.): *Graphs and Order: The Role of Graphs in the Theory of Ordered Sets and Its Applications*, D. Reidel, Dordrecht, Holland, 1985.

[9] SKILTON, D. K.: Imbedding posets in the integers, *Order* **1** (1985), 229–233.

[10] STANLEY, R. P.: *Enumerative Combinatorics: Volume I*, corrected reprint, Cambridge University Press, Cambridge, 1997.