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FERRERO CATEGORIES

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Abstract: We use categorical methods to clarify the fundamental relations between planar nearrings and block designs by introducing new concepts of Ferrero categories and Clay functors.

1. Introduction

We start with a quote from [11]: "Nearrings are generalized rings: commutativity of addition is not assumed and only one distributive law is required. The most famous example is the collection of all maps from an additive group into itself. ... The theory of nearrings is a sophisticated theory which has found numerous applications...". The definition of nearrings will be given in Sec. 4. As for general references for nearrings

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we take [3] and [10].

Some part of the theory of nearrings has a heavy combinatorial favour. In particular, this is the case in planar nearrings and balanced incomplete block designs. Due to this, much effort was spent on sets and equations. Here, we propose a new point of view by emphasizing the the role played by the yet neglected morphisms.

First, we expand the work of Ferrero [4] by introducing suitable concepts of morphisms and show that we can define a category of Ferrero pairs which is complete and cocomplete. We indicate why this cannot be an additive category. Then we construct the category of Ferrero triples from which we construct a functor to the category of nearrings. The category point of view makes relations between objects involved more precise. For example in order to reverse the direction of the functor from Ferrero triples to nearrings we show that it is an equivalence of categories, and yet this is not the same as an isomorphism.

The final category we introduce is that of balanced incomplete block designs (BIBD) ([13]). This is usually not treated as a category. But then our interest is not pure combinatorics. We are more interested in the structures involved. Finally we impose conditions on the morphisms allowed to refine the categories that will appear in our construction of the Clay functor.

It is fair to say that the construction on the objects are known classically. Our contribution is to provide the missing morphisms to give a categorial structure to the theory of nearrings. The functors we construct here have never been introduced in the standard theory of nearrings. We go further than earlier attempts at a categorical study made by Clay [3, Chapter 3] and Pilz [12]. We give a categorical analysis of the celebrated papers of Ferrero [4] and Clay [2] by introducing Ferrero categories and the Clay functor. We hope that our new perspective not only clarifies the fundamental relations between nearrings and block designs but also opens the way to further developments in theory of nearrings.

2. Ferrero pairs

Given a set Φ we can consider the category of all groups N equipped with an action by Φ as endomorphisms, with morphisms the group homomorphisms $\phi : N \to N'$ commuting with all elements ϕ of Φ, i.e.

$T\phi = \phi T$. This category has most of the properties of the category of modules over a given ring (see [5]). For the purpose of studying nearrings we extend this concept to groups with operators.

Given a group N with group operation written additively (though this may not be commutative) and a group Φ of automorphisms of N such that every $\phi \in \Phi \setminus \{\mathrm{id}_N\}$ is fixed point free ($\phi(x) = x$ only if $x = 0$) in N. Here id_N denotes the identity map from N to N itself. We call $F = (N, \Phi)$ a *Ferrero pair*. Note that this is not the same in general as the Ferrero pairs given by Clay, which would be planar Ferrero pairs given in Sec. 4.

By a morphism of Ferrero pairs F and F' we mean a pair $t : N \to N'$ and $T : \Phi \to \Phi'$ of group homomorphisms such that for all $\phi \in \Phi$ the following diagram is commutative:

$$\begin{array}{ccc} N & \xrightarrow{t} & N' \\ {\scriptstyle\phi}\downarrow & & \downarrow{\scriptstyle T\phi} \\ N & \xrightarrow{t} & N' \end{array} .$$

Given two morphisms of of Ferrero pairs $(t, T) : F \to F'$ and $(t', T') : F' \to F''$, the following diagram shows that we can define composition as $(t', T') \circ (t, T) = (t' \circ t, T' \circ T)$:

$$\begin{array}{ccccc} N & \xrightarrow{t} & N' & \xrightarrow{t'} & N'' \\ {\scriptstyle\phi}\downarrow & & \downarrow{\scriptstyle T\phi} & & \downarrow{\scriptstyle T'(T\phi)} \\ N & \xrightarrow{t} & N' & \xrightarrow{t'} & N'' \end{array} .$$

The following proposition is clear.

Proposition 2.1. *Composition is associative and identity morphisms are* $(\mathrm{id}_N, \mathrm{id}_\Phi)$.

Thus we have defined the category of Ferrero pairs which we shall denote by $\mathcal{C}_{\text{F.P.}}$. A Ferrero pair $F = (N, \Phi)$ is said to be *finite* if both N and Φ are finite. The full subcategory of finite Ferrero pairs will be denote by $\mathcal{C}_{\text{f.F.P.}}$.

We shall say that the morphism $(t, T) : F \to F'$ of Ferrero pairs is an *isomorphism* if both t and T are isomorphisms. The following proposition is clear.

Proposition 2.2. *The morphism* $(t, T) : F \to F'$ *of Ferrero pairs is an isomorphism if and only if* $T\phi = t\phi t^{-1}$.

For groups in additive notation a morphism $0 : N \to N'$ is a zero morphism if it factors through the zero group:

$$\begin{tikzcd} N \arrow[rr, "0"] \arrow[dr] & & N' . \\ & 0 \arrow[ur] & \end{tikzcd}$$

For groups in multiplicative notation a zero morphism is one which factor through the group 1 and in this case we write 1 for the "zero" morphism.

The zero morphism in the Ferrero category $\mathcal{C}_{\text{F.P.}}$ is $(0, 1)$.

Given a morphism $(t, T) : (N, \Phi) \to (N', \Phi')$ of Ferrero pairs we set $K = \operatorname{Ker} t$, $\Psi = \operatorname{Ker} T$, and let $i : K \to N$, $I : \Psi \to \Phi$ be the inclusion homomorphisms. For $k \in K$ and $\phi \in \Phi$ we have $t\phi k = T\phi i k = 0$ and so $\phi K \subset K$. It follows that $(i, I) : (K, \Psi) \to (N, \Phi)$ is a morphism in the category $\mathcal{C}_{\text{F.P.}}$ of Ferrero pairs.

Proposition 2.3. $(i, I) : (K, \Psi) \to (N, \Phi)$ *is the kernel of* $(t, T) : (N, \Phi) \to (N', \Phi')$ *in* $\mathcal{C}_{\text{F.P.}}$.

Proof. The proposition says that (i, I) is the equalizer of (T, A) and $(0, 1)$. So we need to show that if $(s, S) : (Y, \Xi) \to (N, \Phi)$ is a morphism such that $(t, T) \circ (s, S) = (0, 1) \circ (s, S)$, then there is a unique morphism $(r, R) : (Y, \Xi) \to (K, \Psi)$ such that the following diagram commutes

$$\begin{tikzcd} (K, \Psi) \arrow[r, "{(i,I)}"] & (N, \Phi) \arrow[r, shift left, "{(t,T)}"] \arrow[r, shift right, "{(0,1)}"'] & (N', \Phi') . \\ (Y, \Xi) \arrow[u, dashed, "{(r,R)}"] \arrow[ur, "{(s,S)}"'] & & \end{tikzcd}$$

But by construction we have in the category of groups unique homomorphisms $R : X \to K$ and $C : \Xi \to \Psi$ such that

$$\begin{tikzcd} K \arrow[r, "i"] & N \arrow[r, bend left, "t"] \arrow[r, bend right, "0"'] & N' \\ Y \arrow[u, dashed, "r"] \arrow[ur, "s"'] & & \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \Psi \arrow[r, "I"] & \Phi \arrow[r, bend left, "T"] \arrow[r, bend right, "1"'] & \Phi' . \\ \Xi \arrow[u, dashed, "R"] \arrow[ur, "S"'] & & \end{tikzcd}$$

Moreover the image of r is in K, the image of R is in Ψ and so in fact $r = s$ and $S = R$. By assumption (s, S) is a morphism in $\mathcal{C}_{\text{F.P.}}$ and so is (r, R). $\Diamond$

In the same way one can establish that co-equalizers exist in the category of Ferrero pairs and we can prove the isomorphism theorems as in the case of groups with operators. See also [12].

Next we show that the category of Ferrero pairs has products.

Theorem 2.4. *The category $\mathcal{C}_{\text{F.P.}}$ of Ferrero pairs is complete and co-complete.*

Proof. We start with a family of Ferrero pairs (N_i, Φ_i) indexed by $i \in I$. We take products in the category of groups to get products with projections $p_i : \prod_i N_i \to N_i$, $P_i : \prod_i \Phi_i \to \Phi_i$. We let $\prod_i \Phi_i$ acts on $\prod_i N_i$ componentwise. This means that the diagram

$$\begin{array}{ccc} \prod_i N_i & \xrightarrow{p_i} & N_i \\ {\scriptstyle \phi}\big\downarrow & & \big\downarrow{\scriptstyle P_i\phi} \\ \prod_i N_i & \xrightarrow{p_i} & N_i \end{array}$$

commutes for any $\phi \in \prod_i \Phi_i$. This says that

$$(p_i, P_i) : \left(\prod_i N_i, \prod_i \Phi_i\right) \to (N_i, \Phi_i)$$

is a morphism in $\mathcal{C}_{\text{F.P.}}$.

We shall show that this map defines a product in $\mathcal{C}_{\text{F.P.}}$. For this we take a family of $\mathcal{C}_{\text{F.P.}}$-morphisms $(t_i, T_i) : (X, \Xi) \to (N_i, \Phi_i)$, and we need to find a unique morphism $(t, T) : (X, \Xi) \to (\prod_i N_i, \prod_i \Phi_i)$ such that $(p_i, P_i) \circ (t, T) = (t_i, T_i)$ in $\mathcal{C}_{\text{F.P.}}$.

From the construction of $\prod_i N_i$ and $\prod_i \Phi_i$ we obtain unique homomorphisms $t : X \to \prod_i N_i$ and $T : \Xi \to \prod_i \Phi_i$ such that

$$p_i t = t_i \quad \text{and} \quad P_i T = T_i.$$

It remains to check that (t, T) is a morphism in $\mathcal{C}_{\text{F.P.}}$.

As (t_i, T_i) is a $\mathcal{C}_{\text{F.P.}}$-morphisms, for any $\xi \in \Xi$ we get

$$t_i \xi = T_i \xi t_i = P_i T \xi t_i = P_i T \xi p_i t$$

with the second equality coming from the following diagram

$$\begin{tikzcd} \Xi \arrow[d, dashed, "T"'] \arrow[dr, "T_i"] & \\ \prod_i \Phi_i \arrow[r, "P_i"'] & \Phi_i \end{tikzcd}$$

for the product $\prod_i \Phi_i$, and the third equality from $p_i t = t_i$ given above. From the fact that (p_i, P_i) is a morphism in $\mathcal{C}_{\text{F.P.}}$ and $T\Xi \in \prod_i \Phi_i$ we get $p_i t\xi p_i = p_i t\Xi$ and so

$$p_i t\xi = t_i \xi = p_i T\xi t.$$

Call this map $s_i : X \to N_i$. Then we have two maps $t\xi$ and $T\xi t$ from X to $\prod_i N_i$ making the following diagram commutative

$$\begin{tikzcd} X \arrow[d, dashed] \arrow[dr, "s_i"] & \\ \prod_i N_i \arrow[r, "p_i"'] & N_i \end{tikzcd} .$$

By the uniqueness we get $t\xi = T\xi t$, i.e.

$$\begin{array}{ccc} X & \xrightarrow{t} & \prod_i N_i \\ \xi \big\downarrow & & \big\downarrow T\xi \\ X & \xrightarrow{t} & \prod_i N_i \end{array}$$

commutes. This completes the proof that (t, T) is a morphism in $\mathcal{C}_{\text{F.P.}}$.

The case of coproducts follow the same lines. $\diamondsuit$

In general it is not possible to add two morphisms of Ferrero pairs using the usual addition of morphisms of additive group. This explains why it is difficult to give the structure of an additive category to the category of Ferrero pairs and thus blocks the usual techniques of homological algebra. However we have the following result.

A Ferrero pair $F = (N, \Phi)$ is said to be *abelian* if N is abelian. We note that the additive group of a planar nearring is automatically almost abelian.

Proposition 2.5. *Let $F = (N, \Phi)$ and $F' = (N', \Phi')$ be abelian Ferrero pairs, and fix a homomorphism $T : \Phi \to \Phi'$. The set $\mathrm{Hom}_T(F, F')$ of all morphisms from F to F' of the form (t, T) is an abelian group with addition given by $(t_1, T) + (t_2, T) = (t_1 + t_2, T)$ where $t_1 + t_2$ is defined as*

$$(t_1 + t_2)(a) = t_1(a) + t_2(a) \quad \text{for } a \in N.$$

The map $t_1 + t_2$ can also be described as the composition of the push-out of t_1, t_2 and the addition in N':

$$\begin{tikzcd} N \arrow[r, "t_2"] \arrow[d, "t_1"'] & N' \arrow[d] & \\ N' \arrow[r] & N' \oplus N' \arrow[r, "\alpha"] & N' \end{tikzcd}$$

with $\alpha(b_1, b_2) = b_1 + b_2$.

3. Ferrero triples

Given a Ferrero pair $F = (N, \Phi)$ we denote the set of orbits of the the action $\Phi \times N \to N$ by $\Phi \backslash N$ and the natural projection by

$$\rho : N \to \Phi \backslash N : a \mapsto \Phi a.$$

By a *section* of F we shall mean a map $\sigma : \Phi \backslash N \to N$ such that $\rho\sigma = \mathrm{id}_N$. The set of all sections of F will be denoted by Σ_F. It is easy to check that if $\sigma \in \Sigma_F$ then $\phi\sigma \in \Sigma_F$ for $\phi \in \Phi$. Thus Φ also acts on Σ_F.

Let $(t, T) : F \to F'$ be a morphism of Ferrero pairs. We can write the condition on morphism as $t\phi = T\phi t$ and so for $a \in N$ we have

$$t\Phi a \subset T\Phi ta \subset \Phi' ta.$$

Thus it makes sense to say that the morphism (t, T) induces a map on the orbit sets

$$\mathcal{T} : \Phi \backslash N \to \Phi' \backslash N' : \Phi a \mapsto \Phi' ta.$$

A *Ferrero triple* is the data (N, Φ, σ) where $F = (N, \Phi)$ is a Ferrero pair and $\sigma \in \Sigma_F$ is a section of F. A morphism $(t, T) : F \to F'$ of Ferrero pairs is a morphism of Ferrero triples $(N, \Phi, \sigma) \to (N', \Phi', \sigma')$ if the following diagram commutes

$$\begin{tikzcd} N \arrow[d, "t"'] & \Phi \backslash N \arrow[l, "\sigma"'] \arrow[d, "\mathcal{T}"] \\ N & \Phi' \backslash N' \arrow[l, "\sigma'"] \end{tikzcd} .$$

The Ferrero triples with morphism defined as above form a category which we denote by $\mathcal{C}_{\text{F.T.}}$. The category of finite Ferrero triples is denoted by $\mathcal{C}_{\text{f.F.T.}}$. See [8, §2.3].

It is easy to check the following proposition.

Proposition 3.1. *An isomorphism $(t, T) : F \to F'$ of Ferrero pairs is an isomorphism of Ferrero triples $(N, \Phi, \sigma) \to (N', \Phi', \sigma')$ if and only if $t(\operatorname{Im}\sigma) = \operatorname{Im}\sigma'$.*

4. Nearrings

A *nearring* is a triple $(N, +, *)$ so that (1) $(N, +)$ is a group; (2) $(N, *)$ is a semigroup; and (3) $a*(b+c)=a*b+a*c$ for all $a, b, c \in N$. A morphism of nearrings $t : (N, +, *) \to (N', +', *')$ is a group homomorphism $t : (N, +) \to (N', +')$ such that $t(a * b) = ta *' tb$. We shall write in the future $*$, $+$ for $*'$, $+'$ respectively. The category of nearrings will be denoted by $\mathcal{C}_{\text{N.R.}}$.

For $a, b \in N$ we write $a \equiv_m b$ whenever $a * x = b * x$ for all $x \in N$. The relation $\equiv_m$ is an equivalence relation. Say $(N, +, *)$ is a *planar nearring* if the set $N/{\equiv_m}$ of equivalence classes has at least 3 elements and for each triple $a, b, c \in N$ with $a \not\equiv_m b$, the equation $a * x = b * x + c$ has a unique solution $x \in N$. In another words this is saying that two lines of different "slops" define a unique point. We say that a planar nearring N is *integral* if for $a \in N$, $a \equiv_m 0$ if and only if $a = 0$.

Let (N, Φ, σ) be a Ferrero triple. The image C of σ is a complete set of orbit representatives of Φ in N. Then $N = \cup_c \Phi(c)$. For $a, b \in N$ define

$$a * b = \begin{cases} \phi(b) & \text{if } a = \phi(c), c \in C \setminus \{0\}, \phi \in \Phi, \\ 0 & \text{otherwise.} \end{cases}$$

Then $(N, +, *)$ is an integral nearring. We shall also write

$$\eta((N, \Phi, \sigma)) = (N, +, *).$$

We note that the definition actually says that $\phi(b) = \phi(c) * b$ for any nonzero $c \in C$!

Suppose that $(t, T) : (N, \Phi, \sigma) \to (N', \Phi', \sigma')$ is a morphism of Ferrero triples. By definition, $t : (N, +) \to (N', +)$ is a homomorphism. Now from $\phi(c) * b = \phi(b)$ we get

$$t(\phi(c) * b) = t(\phi(b)) = (T\phi)tb = (T\phi)(tc) * tb = t(\phi c) * tb\,.$$

This shows that t is a nearring homomorphism. We denote this by $\eta(t, T)$, i.e. $\eta(t, T) = t$.

We say a Ferrero pair F is *planar* if $|\Phi| \geq 2$ and $-\mathrm{id}_N + \phi$ is surjective for every $\phi \in \Phi \setminus \mathrm{id}_N$. The condition $|\Phi| \geq 2$ is to ensure that there are at least two different "slopes" in the system. The planarity, i.e. the second condition, says that if a and b are two non-equivalent elements (so they correspond to different elements in Φ), then the equation $ax = bx + c$ has a unique solution. When F is finite with $|\Phi| \geq 2$, it is automatically planar. See [3, §4].

We introduce the category $\mathcal{C}_{\text{f.p.F.T.}}$ whose objects are finite planar Ferrero triples and morphisms are those morphisms (t, T) of Ferrero triples such that t is an isomorphism. We also define the category $\mathcal{C}_{\text{f.i.p.N.R.}}$ with objects which are finite integral planar nearring and we stipulate that the morphisms between objects in this category are isomorphisms of nearring.

Theorem 4.1. (1) *Let F, F' be two finite planar Ferrero triples. The map*

$$\eta : \mathrm{Hom}_{\mathcal{C}_{\text{f.p.F.T.}}}(F, F') \longrightarrow \mathrm{Hom}_{\mathcal{C}_{\text{f.i.p.N.R.}}}(\eta F, \eta F')$$

is bijective.

(2) *For any integral planar nearring $(N, +, *)$ there exists a Ferrero triple (N, Φ, σ) such that*

$$\eta((N, \Phi, \sigma)) = (N, +, *).$$

(3) *We have an equivalence of categories given by the functor*

$$\eta : \mathcal{C}_{\text{f.p.F.T.}} \longrightarrow \mathcal{C}_{\text{f.i.p.N.R.}}.$$

Proof. (1) follows from 2.2 which characterizes when t is an isomorphism, and T is uniquely determined as $\phi \mapsto t\phi t^{-1}$.

(2) can be found in [3, p. 47].

(3) It is clear that $\eta(\mathrm{id}_N, \mathrm{id}_\Phi) = \mathrm{id}_N$ says that

$$\eta(\mathrm{id}_{(N,\Phi,\sigma)}) = \mathrm{id}_{\eta((N,\Phi,\sigma))}.$$

Also $\eta(s, S) \circ \eta(t, T) = \eta(s \circ t, S \circ T)$. This shows that $\eta : \mathcal{C}_{\text{F.T.}} \to \mathcal{C}_{\text{N.R.}}$ is a functor. Now parts (1) and (2) says that η is an equivalence of categories. $\Diamond$

Remark 4.2. The above theorem shows that the situation is quite "rigid". However it does open the way to further investigation as to what will happen if we relax the conditions on the choice of morphisms for the categories involved.

Another point is this. Given two finite planar Ferrero triples F and F'. Then the category $\mathrm{Hom}_{\mathcal{C}_{\text{f.p.F.T.}}}(F, F')$ is nonempty if and only if $\mathrm{Hom}_{\mathcal{C}_{\text{f.p.F.T.}}}(\eta F, \eta F')$ is nonempty. This is the same as saying ηF and $\eta F'$ are isomorphic if and only if there exists an isomorphism $t : N \to N'$ such that $\Phi' = t\Phi t^{-1}$. See [6].

5. BIBD's

A *tactical configuration* is a pair $(X, \mathcal{X})$ such that the following properties are satisfied:

(1) X is a finite set of elements called points or treatments,

(2) $\mathcal{X}$ is a collection (i.e., multiset) of nonempty subsets of X called blocks, and

(3) each block in $\mathcal{X}$ has a fixed size k.

Let v, k, and λ be positive integers such that $v > k \geq 2$. A (v, k, λ)-*balanced incomplete block design* (which is abbreviated to (v, k, λ)-BIBD) is a tactical configuration $(X, \mathcal{X})$ such that the following properties are satisfied:

(1) $|X| = v$,

(2) each block contains exactly k points, and

(3) every pair of distinct points is contained in exactly λ blocks.

The number λ is called the *concurrence parameter.* A BIBD is also called a 2-design because of the concurrent condition. When the blocks of a 2-design are all distinct, the design is said to be *simple.* In the following, a design is understood to be a 2-design.

For a map $f : X \to Y$ and $A \in \mathcal{X}$ we write $f(A) = \{f(x) : x \in A\}$. We define a morphism from a design $(X, \mathcal{X})$ to a design $(Y, \mathcal{Y})$ to be a pair of maps $f : X \to Y$, $F : \mathcal{X} \to \mathcal{Y}$ such that for every $A \in \mathcal{X}$ we have $f(A) = F(A)$. Clearly composition of morphisms of designs is a morphism of design and composition is associative. Therefore we can talk about the category $\mathcal{C}_{\text{BIBD}}$ of BIBD's.

Remark 5.1. We could have said that a design morphism is a map $f : X \to Y$ such that $f(A) \in \mathcal{Y}$ for all $A \in \mathcal{X}$. But we shall see how the second map F makes certain categorical argument easy.

A design morphism (m, M) is said to be *monic* if $(m, M) \circ (f, F) = = (m, M) \circ (g, G)$ implies $(f, F) = (g, G)$. A design morphism (s, S) is said to be *epic* if $(f, F) \circ (s, S) = (g, G) \circ (s, S)$ implies $(f, F) = (g, G)$.

Proposition 5.2. *If $(f, F) : (X, \mathcal{X}) \to (Y, \mathcal{Y})$ is both monic and epic then $f : X \to Y$ is bijective and*

$$\{f(A) : A \in \mathcal{X}\} = \mathcal{Y}.$$

Proof. The fact that f is monic and epic on sets means that f is bijective. The same is true for F. In fact from (f, F) is monic follows that both f and F are injective, and so if $f(A_1) = f(A_2)$ then $F(A_1) = F(A_2)$ and hence $A_1 = A_2$. Assuming (f, F) is epic, if $B \in \mathcal{Y}$, then there exists $A \in \mathcal{X}$ such that $F(A) = B$. But $f(A) = F(A)$, so $f(A) = B$. $\Diamond$

The conclusion in the proposition is the usual definition of equivalence for designs.

We say a design morphism $(f, F) : (X, \mathcal{X}) \to (Y, \mathcal{Y})$ is an isomorphism if there is a design morphism $(g, G) : (Y, \mathcal{Y}) \to (X, \mathcal{X})$ such that $(g, G) \circ (f, F) = \mathrm{id}_{(X,\mathcal{X})}$ and $(f, F) \circ (g, G) = \mathrm{id}_{(Y,\mathcal{Y})}$.

6. Block Ferro pairs

We define the category $\mathcal{C}_{\text{b.F.P.}}$ of *block Ferrero pairs* to have finite planar Ferrero pairs as objects and require a morphism $(t, T) : (N, \Phi) \to (N', \Phi')$ in this category satisfies the conditions that t is injective and T is surjective. We remark that the properties of being injective or surjective is stable under composition.

Given a finite Ferrero pair (N, Φ), we set $\mathcal{N}$ to be the set of subsets of N of the form $\Phi a + b$ with $a, b \in N$ and $a \neq 0$. It is known that $\Phi a + b = \Phi c + d$ if and only if $b = d$ and $\Phi a = \Phi c$.

Theorem 6.1 ([2, Th. 2])**.** *Let (N, Φ) be a finite planar Ferrero pair. Then $(N, \mathcal{N})$ is a BIBD with parameters $(|N|, |\Phi|, |\Phi| - 1)$.*

Clay called this construction one of the milestones of this theory [2, p. 96], even though he never introduced this as a functor.

Remark 6.2. Beside the above construction, there are yet many other ways to obtain BIBD's from planar nearrings. See [3, §7] and [14]. One may also find a unifying approach to various methods in [1].

Let us write $\beta((N,\Phi)) = (N,\mathcal{N})$.

Next we have to complete the picture by adding the effects of the morphisms. Given a morphism $(t,T) : (N,\Phi) \to (N',\Phi')$ in the category $\mathcal{C}_{\text{b.F.P.}}$ we have the homomorphism $t : N \to N''$ and we define

$$\beta T(\Phi a + b) = t(\Phi a + b) = \Phi' ta + tb.$$

Note that t is assumed to be injective so that $a \neq 0$ implies $ta \neq 0$ and T is surjective implies that $T\Phi = \Phi'$. We have also used the fact that $t(\phi a) = T\phi(ta)$ for $a \in N$ and $\phi \in \Phi$. In this way we attach to a morphism (t,T) a design morphism $(t,\beta T)$ which we will also denote by $\beta(t,T)$.

Now if we have morphisms

$$(t,T) : (N,\Phi) \to (N',\Phi'), \quad (s,S) : (N',\Phi') \to (N'',\Phi'')$$

in the category $\mathcal{C}_{\text{b.F.P.}}$, then from

$$\Phi a + b \xrightarrow{\beta(t,T)} \Phi' ta + tb \xrightarrow{\beta(s,S)} \Phi'' sta + stb = \beta(ST)(\Phi a + b)$$

we see that

$$\beta(st, ST) = \beta(s,S) \circ \beta(t,T).$$

Since $\mathrm{id}_{(N,\Phi)} = (\mathrm{id}_N, \mathrm{id}_\Phi)$ we see that $\beta(\mathrm{id}_{(N,\Phi)}) = \mathrm{id}_{(N,\mathcal{N})}$.

We have proved the following proposition.

Proposition 6.3. *$\beta : \mathcal{C}_{\text{b.F.P.}} \to \mathcal{C}_{\text{BIBD}}$ is a functor.*

Remark 6.4. Certainly there are many more BIBD's which cannot be constructed from Ferrero pairs. Those that can be constructed from Ferrero pairs have parameter $\lambda = k-1$. Even if among $\lambda = k-1$ BIBD's, there are many which cannot constructed from Ferrero pairs. There is still no characterization about Ferrero pairs derived BIBD's, not even conjectures.

Next we introduce a faithful functor $\theta : \mathcal{C}_{\text{f.p.F.T.}} \longrightarrow \mathcal{C}_{\text{b.F.P.}}$. On objects θ forgets the section, thus, $\theta(N,\Phi,\sigma) = (N,\Phi)$. On morphisms it is just the inclusion map taking (t,T) to (t,T).

7. A diagram

Since the functor $\eta : \mathcal{C}_{\text{f.p.F.T.}} \to \mathcal{C}_{\text{f.i.p.N.R.}}$ in Th. 4.1 is an equivalence of categories, there is a functor $\xi : \mathcal{C}_{\text{f.i.p.N.R.}} \to \mathcal{C}_{\text{f.p.F.T.}}$ such that $\xi\eta = \mathrm{id}_{\mathcal{C}_{\text{f.p.F.T.}}}$ and $\eta\xi = \mathrm{id}_{\mathcal{C}_{\text{f.i.p.N.R.}}}$ (see [9, Th. 1, p. 91]). We can now summarize our discussions by a commutative diagram

$$\begin{tikzcd} \mathcal{C}_{\text{f.p.F.T.}} \arrow[r, "\theta"] & \mathcal{C}_{\text{b.F.P.}} \arrow[d, "\beta"] \\ \mathcal{C}_{\text{f.i.p.N.R.}} \arrow[u, "\xi"] \arrow[r, "\kappa"'] & \mathcal{C}_{\text{BIBD}} \end{tikzcd} .$$

We call the functor κ defined by the diagram (so $\kappa = \beta\theta\xi$) the *Clay functor.* It was Clay [2] who first assigned the BIBD to a finite planar nearring, and one of our goals is to make this precise. With the Ferrero category and the Clay functor set up, we completed our initial categorical analysis. We shall follow up in a future paper.

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