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SERIES IDENTITIES AND ASSOCIATED HYPERGEOMETRIC TRANSFORMATION AND REDUCTION FORMULAS

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Abstract: Transformation and reduction formulas for various families of hypergeometric functions in two and more variables are potentially useful in many diverse areas. The main object of this paper is to show that several substantially more general results on this subject than those proven recently by Pogány and Rathie [18] can be derived rather systematically by applying a family of series identities. Relevant connections of the results presented in this paper with those in the earlier works are also pointed out.

1. Introduction, definitions and preliminaries

Throughout our present investigation, we use the following standard notations:

$$\mathbb{N} := \{1, 2, 3, \cdots\}, \quad \mathbb{N}_0 := \{0, 1, 2, 3, \cdots\} = \mathbb{N} \cup \{0\}$$

and

$$\mathbb{Z}^- := \{-1, -2, -3, \cdots\} = \mathbb{Z}_0^- \setminus \{0\}.$$

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Also, as usual, $\mathbb{Z}$ denotes the set of integers, $\mathbb{R}$ denotes the set of *real* numbers, $\mathbb{R}^+$ denotes the set of *positive* real numbers and $\mathbb{C}$ denotes the set of *complex* numbers.

In the vast and widely-scattered literature on the theory and applications of hypergeometric functions, one can find a considerable large number of summation, transformation and reduction formulas for the Gauss, Kummer and Clausen (and other higher-order) hypergeometric functions (see, for details, [3, 5, 7, 8, 12, 13, 14, 19, 20, 29, 30]). Quite a few of these known summation, transformation and reduction formulas have been found to be useful in deriving the corresponding results for Kampé de Fériet and Lauricella multivariable hypergeometric functions and also for their generalizations and extensions due to Srivastava and Daoust [21, 21] (see, for example, [4, 16, 24]; see also [26, Chapters 1 and 9] *and* the references to the earlier works cited therein).

Recently, by applying some known product formulas for the Kummer confluent hypergeometric function ${}_1F_1$ due to Bailey [2], Ramanujan (see, for example, [10]) and Preece [17], Pogány and Rathie [18] derived several interesting transformation and reduction formulas involving the Kampé de Fériet and Srivastava–Daoust hypergeometric functions in two variables. In this present sequel to the work of Pogány and Rathie [18], we show that several substantially more general results on this subject than those proven by Pogány and Rathie [18] can be derived rather systematically by applying a family of series identities which we derive here.

We begin by recalling the following definition of the aforementioned Srivastava–Daoust hypergeometric function in r (real or complex) variables $z_1, \cdots, z_r$ (see [22, p. 454]; see also [21] for the case when $r = 2$):

(1.1)

$$F^{A:B';\cdots;B^{(r)}}_{C:D';\cdots;D^{(r)}}\left(\begin{matrix}\left[(a):\theta',\cdots,\theta^{(r)}\right]:\left[(b'):\phi'\right];\cdots;\left[b^{(r)}:\phi^{(r)}\right]; \\ \left[(c):\psi',\cdots,\psi^{(r)}\right]:\left[(d'):\delta'\right];\cdots;\left[d^{(r)}:\delta^{(r)}\right];\end{matrix}\; z_1,\cdots,z_r\right)$$

$$:= \sum_{m_1,\cdots,m_r=0}^{\infty} \frac{\prod\limits_{j=1}^{A}(a_j)_{m_1\theta'_j+\cdots+m_r\theta^{(r)}_j}\prod\limits_{j=1}^{B'}(b'_j)_{m_1\phi'_j}\cdots\prod\limits_{j=1}^{B^{(r)}}\left(b^{(r)}_j\right)_{m_r\phi^{(r)}_j}}{\prod\limits_{j=1}^{C}(c_j)_{m_1\psi'_j+\cdots+m_r\psi^{(r)}_j}\prod\limits_{j=1}^{D'}(d'_j)_{m_1\delta'_j}\cdots\prod\limits_{j=1}^{D^{(r)}}\left(d^{(r)}_j\right)_{m_r\delta^{(r)}_j}}\,\frac{z_1^{m_1}}{m_1!}\cdots\frac{z_r^{m_r}}{m_r!},$$

where (*and* in what follows) an empty product is interpreted to be 1 and

$(\lambda)_\nu$ $(\lambda, \nu \in \mathbb{C})$ denotes the generalized Pochhammer symbol (or the *shifted* factorial) defined, in terms of the familiar Gamma function, by

(1.2)
$$(\lambda)_\nu := \frac{\Gamma(\lambda+\nu)}{\Gamma(\lambda)} = \begin{cases} 1 & (\nu = 0;\ \lambda \in \mathbb{C} \setminus \{0\}), \\ \lambda(\lambda+1)\cdots(\lambda+n-1) & (\nu = n \in \mathbb{N};\ \lambda \in \mathbb{C}), \end{cases}$$

it being understood *conventionally* that $(0)_0 := 1$ and assumed *tacitly* that the Γ-quotient exists (see, for details, [28, p. 21 *et seq.*]).

For convenience and brevity in the definition (1.1), we use (a) to abbreviate the array of A parameters $a_1, \cdots, a_A$ and $\left(b^{(\ell)}\right)$ abbreviates the array of $B^{(\ell)}$ parameters

$$b_1^{(\ell)}, \cdots, b_{B^{(\ell)}}^{(\ell)} \qquad (\ell = 1, \cdots, r),$$

with similar interpretations for (c) and $\left(d^{(\ell)}\right)$ $(\ell = 1, \cdots, r)$. Furthermore, assuming that the parameters θ, ϕ, ψ and δ are all real and nonnegative and that there are no zeros in the denominator on the right-hand side in (1.1), the multiple series in (1.1) is known to converge absolutely under the following conditions (see, for details, [23, pp. 157–158]):

(i) For all $z_1, \cdots, z_r \in \mathbb{C}$, when

(1.3)
$$\Delta_\ell := 1 + \sum_{j=1}^{C} \psi_j^{(\ell)} + \sum_{j=1}^{D^{(\ell)}} \delta_j^{(\ell)} - \sum_{j=1}^{A} \theta_j^{(\ell)} - \sum_{j=1}^{B^{(\ell)}} \phi_j^{(\ell)} > 0 \qquad (\ell = 1, \cdots, r);$$

(ii) For $|z_\ell| < R_\ell$ $(\ell = 1, \cdots, r)$, when $\Delta_\ell = 0$ $(\ell = 1, \cdots, r)$, where

(1.4)
$$R_\ell := \min_{\mu_1, \cdots, \mu_r > 0} \left\{ \mu_\ell^{\,1+\sum_{j=1}^{D^{(\ell)}} \delta_j^{(\ell)} - \sum_{j=1}^{B^{(\ell)}} \phi_j^{(\ell)}} \frac{\prod_{j=1}^{C} \left(\sum_{\ell=1}^{r} \mu_\ell \psi_j^{(\ell)} \right)^{\psi_j^{(\ell)}} \prod_{j=1}^{D^{(\ell)}} \left(\delta^{(\ell)}\right)^{\delta^{(\ell)}}}{\prod_{j=1}^{A} \left(\sum_{\ell=1}^{r} \mu_\ell \theta_j^{(\ell)} \right)^{\theta_j^{(\ell)}} \prod_{j=1}^{B^{(\ell)}} (\phi^{(\ell)})^{\phi^{(\ell)}}} \right\}$$
$$(\ell = 1, \cdots, r).$$

Indeed, when all $\Delta_\ell < 0$ $(\ell = 1, \cdots, r)$, the series in (1.1) would diverge except for $z_1 = \cdots = z_r = 0$ in which case it reduces obviously to the first term 1 (see also [9]).

The case $r = 2$ of the Srivastava–Daoust multivariable hypergeometric function

$$F^{A:B';\cdots;B^{(r)}}_{C:D';\cdots;D^{(r)}}$$

was introduced and investigated by Srivastava and Daoust [21]. This case $r = 2$ would provide a generalization of the following relatively more familiar Kampé de Fériet hypergeometric function in two variables (see, for details, [1]; see also [26, p. 27 *et seq.*]):

(1.5)

$$F^{A:B;B'}_{C:D;D'}\left[\begin{matrix}(a):(b);(b');\\(c):(d);(d');\end{matrix}x,y\right]:=\sum_{m,n=0}^{\infty}\frac{\prod\limits_{j=1}^{A}(a_j)_{m+n}\prod\limits_{j=1}^{B}(b_j)_m\prod\limits_{j=1}^{B'}(b'_j)_n}{\prod\limits_{j=1}^{C}(c_j)_{m+n}\prod\limits_{j=1}^{D}(d_j)_m\prod\limits_{j=1}^{D'}(d'_j)_n}\frac{x^m}{m!}\frac{y^n}{n!},$$

provided that the double series in (1.5) converges absolutely. Moreover, the case $r = 1$ of the Srivastava–Daoust multivariable hypergeometric function

$$F^{A:B';\cdots;B^{(r)}}_{C:D';\cdots;D^{(r)}}$$

corresponds to the Fox–Wright function

$$_p\Psi_q \quad (p,q\in\mathbb{N}_0) \qquad \text{or} \qquad {}_p\Psi_q^* \quad (p,q\in\mathbb{N}_0),$$

which is a *further* generalization of the widely- and extensively-investigated generalized hypergeometric function ${}_pF_q$ $(p,q\in\mathbb{N}_0)$, with p numerator parameters $a_1, \cdots, a_p$ and q denominator parameters $b_1, \cdots, b_q$ such that

(1.6) $a_j\in\mathbb{C} \quad (j=1,\cdots,p)$ and $b_j\in\mathbb{C}\setminus\mathbb{Z}_0^- \quad (j=1,\cdots,q)$,

defined by (see, for details, [7, p. 183] and [26, p. 21]; see also [11, p. 56], [15, p. 30], [25, p. 19], [27], [28, p. 50])

(1.7)
$$ {}_p\Psi_q^*\left[\begin{matrix}(a_1,A_1),\cdots,(a_p,A_p);\\(b_1,B_1),\cdots,(b_q,B_q);\end{matrix}z\right]:=\sum_{n=0}^{\infty}\frac{(a_1)_{A_1n}\cdots(a_p)_{A_pn}}{(b_1)_{B_1n}\cdots(b_q)_{B_qn}}\frac{z^n}{n!}=$$

$$=\frac{\Gamma(b_1)\cdots\Gamma(b_q)}{\Gamma(a_1)\cdots\Gamma(a_p)}\,{}_p\Psi_q\left[\begin{matrix}(a_1,A_1),\cdots,(a_p,A_p);\\(b_1,B_1),\cdots,(b_q,B_q);\end{matrix}z\right]$$

$$\left(A_j>0\ (j=1,\cdots,p);\ B_j>0\ (j=1,\cdots,q);\ 1+\sum_{j=1}^{q}B_j-\sum_{j=1}^{p}A_j\geqq 0\right),$$

where the equality in the convergence condition holds true for suitably bounded values of $|z|$ given by

$$(1.8) \qquad |z| < \nabla := \left(\prod_{j=1}^{p} A_j^{-A_j} \right) \cdot \left(\prod_{j=1}^{q} B_j^{B_j} \right).$$

In the particular case when

$$A_j = B_k = 1 \qquad (j = 1, \cdots, p; \ \ k = 1, \cdots, q),$$

we have the following relationship (see, for details, [26, p. 21]):

$$(1.9) \qquad {}_p\Psi_q^* \left[\begin{array}{c} (a_1, 1), \ \cdots, (a_p, 1); \\ (b_1, 1), \ \cdots, (b_q, 1); \end{array} z \right] = {}_pF_q \left[\begin{array}{c} a_1, \ \cdots, a_p; \\ b_1, \ \cdots, b_q; \end{array} z \right] =$$

$$= \frac{\Gamma(b_1) \ \cdots \Gamma(b_q)}{\Gamma(a_1) \ \cdots \Gamma(a_p)} \ {}_p\Psi_q \left[\begin{array}{c} (a_1, 1), \ \cdots, (a_p, 1); \\ (b_1, 1), \ \cdots, (b_q, 1); \end{array} z \right],$$

in terms of the generalized hypergeometric function ${}_pF_q$ $(p, q \in \mathbb{N}_0)$.

2. Series identities and their applications

We first state and prove the series identity (2.1) asserted by Th. 1 below.

Theorem 1. *Let* $\{\Omega_n\}_{n \in \mathbb{N}_0}$ *be a bounded sequence of complex numbers. Then*

$$(2.1) \quad \sum_{n=0}^{\infty} \Omega_n \, \frac{\Gamma(\lambda n+1)}{n!} \left(\frac{x}{(1-y)^{\lambda}} \right)^n = \Omega_0 + \lambda x \sum_{m,n=0}^{\infty} \Omega_{m+1} \, \Gamma\big(\lambda(m+1)+n\big) \, \frac{x^m}{m!} \frac{y^n}{n!}$$

$$(\lambda \in \mathbb{C}),$$

provided that each member of (2.1) *exists.*

Proof. We denote, for convenience, the first member of the assertion (2.1) of Th. 1 by $\mathfrak{S}_\lambda(x, y)$. Then, in view of (1.2) and the binomial expansion:

$$(1 - z)^{-\lambda} = \sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} \, z^n = \frac{1}{\Gamma(\lambda)} \sum_{n=0}^{\infty} \Gamma(\lambda + n) \, \frac{z^n}{n!} \qquad (|z| < 1; \ \lambda \in \mathbb{C}),$$

we find that

$$\mathfrak{S}_\lambda(x,y) := \sum_{m=0}^{\infty} \Omega_m \, \frac{\Gamma(\lambda m+1)}{m!} \left(\frac{x}{(1-y)^\lambda}\right)^m =$$

$$= \Omega_0 + \lambda \sum_{m=1}^{\infty} \frac{\Gamma(\lambda m)}{(m-1)!} \, x^m \, (1-y)^{-\lambda m} =$$

$$= \Omega_0 + \lambda x \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} \Omega_m \, \Gamma(\lambda m + n) \, \frac{x^{m-1}}{(m-1)!} \frac{y^n}{n!},$$

which, upon setting $m \mapsto m+1$, yields the second member of the series identity (2.1) asserted by Th. 1. The final result (2.1) holds true, by the principle of analytic continuation, whenever each member of (2.1) exists. $\Diamond$

Upon setting

$$\Omega_n = \frac{\Gamma\left(a_1 + A_1 n\right) \, \cdots \Gamma\left(a_p + A_p n\right)}{\Gamma\left(b_1 + B_1 n\right) \, \cdots \Gamma\left(b_q + B_q n\right)} \qquad (n \in \mathbb{N}_0), \tag{2.2}$$

in the series identity (2.1), if we make use of the definitions (1.5) and (1.7), Th. 1 would reduce to Cor. 1 below.

Corollary 1. *The following hypergeometric transformation formula holds true:*

(2.3)

$${}_{p+1}\Psi_q \left[\begin{array}{r} (1,\lambda), (a_1,A_1)\,,\ \cdots\,, (a_p,A_p)\,; \\ (b_1,B_1)\,,\ \cdots\,, (b_q,B_q)\,; \end{array} \ \frac{x}{(1-y)^\lambda} \right] = \frac{\prod\limits_{j=1}^{p} \Gamma\left(a_j\right)}{\prod\limits_{j=1}^{q} \Gamma\left(b_j\right)} \cdot$$

$$\cdot \left[1 + \lambda x F^{1:p;0}_{0:q;0} \left(\begin{array}{r} [\lambda:\lambda,1] : [a_1 + A_1 : A_1], \cdots, [a_p + A_p : A_p]; \text{———}; \\ \text{———} : [b_1 + B_1 : B_1], \cdots, [b_q + B_q : B_q]; \ \text{———}; \end{array} \ x, y \right) \right],$$

provided that each member of (2.3) *exists.*

Remark 1. If, in Cor. 1, we set $p = q = 0$ or (alternatively)

$$q = p, \quad a_j = b_j \qquad \text{and} \qquad A_j = B_j \quad (j = 1, \cdots, p),$$

the hypergeometric transformation formula (2.3) would correspond to the first of the four main results of Pogány and Rathie [18, Th. 2.1].

Next, for a bounded sequence $\{\Omega_n\}_{n\in\mathbb{N}_0}$ of complex numbers, we consider the following double sum:

$$\mathfrak{S}_{\lambda,\mu}^{\rho,\sigma}(x,y) := \sum_{m,n=0}^{\infty} \Omega_{m+n} \frac{(\lambda)_m(\mu)_n}{(\rho)_m(\sigma)_n} \frac{x^m}{m!}\frac{y^n}{n!}, \tag{2.4}$$

which is assumed to be absolutely convergent. By using series rearrangement (with $m+n = N$) along with identities for the Pochhammer symbol defined by (1.2), it is easily seen from (2.4) that

$$\mathfrak{S}_{\lambda,\mu}^{\rho,\sigma}(x,y) = \sum_{N=0}^{\infty} \Omega_N \frac{(\lambda)_N}{(\rho)_N} \frac{x^N}{N!} \, {}_3F_2 \left[\begin{array}{r} -N, 1-\rho-N, \mu; \\ \sigma, 1-\lambda-n; \end{array} \; -\frac{y}{x} \right], \tag{2.5}$$

which, in the special case when $\mu = \lambda$ and $\sigma = \rho$, yields

$$\mathfrak{S}_{\lambda,\lambda}^{\rho,\rho}(x,y)=\sum_{N=0}^{\infty}\Omega_N \frac{(\lambda)_N}{(\rho)_N} \frac{(x+y)^N}{N!} \, {}_3F_2\left[\begin{array}{r} -\frac{1}{2}N, -\frac{1}{2}N+\frac{1}{2}, \rho-\lambda; \\ \rho, 1-\lambda-n; \end{array} \; \frac{4xy}{(x+y)^2}\right], \tag{2.6}$$

where we have also used the familiar Whipple's transformation (see, for example, [7, p. 190, Eq. 4.5(1)]) for the Clausen hypergeometric function ${}_3F_2$. Upon expressing the second member of (2.6) as a double sum, we are thus lead to the following theorem due to Buschman and Srivastava [4, p. 437, Th. 2].

Theorem 2. *Let $\{\Omega_n\}_{n\in\mathbb{N}_0}$ be a bounded sequence of complex numbers. Then*

$$\sum_{m,n=0}^{\infty} \Omega_{m+n} \frac{(\lambda)_m(\lambda)_n}{(\rho)_m(\rho)_n} \frac{x^m}{m!}\frac{y^n}{n!} = \sum_{m,n=0}^{\infty} \Omega_{m+2n} \frac{(\lambda)_{m+n}(\rho-\lambda)_n}{(\rho)_{m+2n}(\rho)_n} \frac{(x+y)^m}{m!}\frac{(-xy)^n}{n!} \tag{2.7}$$
$$(\lambda \in \mathbb{C};\ \rho \in \mathbb{C}\setminus\mathbb{Z}_0^-),$$

provided that each member of the double-series identity (2.7) *exists.*

Remark 2. A special case of (2.7) when $y = -x$ and the sequence $\{\Omega_n\}_{n\in\mathbb{N}_0}$ is given by (2.2) *with*

$$A_j = B_k = 1 \qquad (j = 1, \cdots, p;\ k = 1, \cdots, q) \tag{2.8}$$

yields a reduction formula for the Kampé de Fériet function, which was recorded in the above-cited paper by Buschman and Srivastava [4, p. 439, Eq. (3.4)] (see also [26, p. 31, Eq. 1.3(46)]). Furthermore, in its special case when $y = x$ and the sequence $\{\Omega_n\}_{n\in\mathbb{N}_0}$ is given by (2.2) *without* the parametric restrictions in (2.8), Th. 2 can be applied to deduce Cor. 2 below.

Corollary 2. *The following hypergeometric transformation formula holds true:*
(2.9)

$$F^{p:1;1}_{q:1;1}\left(\begin{matrix}[a_1:A_1,A_1],\cdots,[a_p:A_p,A_p]:[\lambda:1];[\lambda:1];\\ [b_1:B_1,B_1],\cdots,[b_q:B_q,B_q]:[\rho:1];[\rho:1];\end{matrix}\; x,x\right)=$$

$$=F^{p+1:0;1}_{q+1:0;1}\left(\begin{matrix}[a_1:A_1,2A_1],\cdots,[a_p:A_p,2A_p],[\lambda:1,1]:\text{———};[\rho-\lambda:1];\\ [b_1:B_1,2B_1],\cdots,[b_q:B_q,2B_q],[\rho:1,2]:\text{———};\quad[\rho:1];\end{matrix}\; 2x,-x^2\right),$$

provided that each member of (2.9) *exists.*

Just as in the derivations of Th. 2 and Cor. 2, by applying some suitable hypergeometric summation, transformation and reduction formulas (see also [2], [10] and [17]), one can similarly establish each of the double-series identities and the associated hypergeometric transformations which are asserted by Th. 3 and Cor. 3 below. The details are being omitted here.

Theorem 3. *Let* $\{\Omega_n\}_{n\in\mathbb{N}_0}$ *be a bounded sequence of complex numbers. Then*

$$\sum_{m,n=0}^{\infty}\Omega_{m+n}\,\frac{(\lambda)_m(\mu)_n}{(2\lambda)_m(2\mu)_n}\,\frac{x^{m+n}}{m!\,n!}= \tag{2.10}$$

$$=\sum_{m,n=0}^{\infty}\Omega_{m+2n}\,\frac{\left(\dfrac{\lambda+\mu}{2}\right)_n\left(\dfrac{\lambda+\mu+1}{2}\right)_n}{\left(\lambda+\frac{1}{2}\right)_n\left(\mu+\frac{1}{2}\right)_n(\lambda+\mu)_n}\,\frac{x^m}{m!}\,\frac{\left(\frac{1}{4}x^2\right)^n}{n!}$$

$$(2\lambda,2\mu\in\mathbb{C}\setminus\mathbb{Z}_0^-),$$

$$\sum_{m,n=0}^{\infty}\Omega_{m+n}\,\frac{(\lambda)_m\,(\rho-\lambda)_n}{(\rho)_m\,(\rho)_n}\,\frac{x^{m+n}}{m!\,n!}= \tag{2.11}$$

$$= \sum_{m,n=0}^{\infty} \Omega_{m+2n} \frac{(\lambda)_n (\rho-\lambda)_n}{(\rho)_n \left(\frac{1}{2}\rho\right)_n \left(\frac{1}{2}\rho+\frac{1}{2}\right)_n} \frac{x^m}{m!} \frac{\left(\frac{1}{4}x^2\right)^n}{n!} \qquad (\rho \in \mathbb{C} \setminus \mathbb{Z}_0^-)$$

and

(2.12)

$$\sum_{m,n=0}^{\infty} \Omega_{m+n} \frac{(\lambda)_m (1-\lambda)_n}{(\rho)_m (2-\rho)_n} \frac{x^{m+n}}{m!\, n!} =$$

$$= \sum_{m,n=0}^{\infty} \Omega_{m+2n} \frac{\left(\lambda-\frac{1}{2}\rho+\frac{1}{2}\right)_n \left(\frac{1}{2}\rho-\lambda+\frac{1}{2}\right)_n}{\left(\frac{1}{2}\right)_n \left(\frac{1}{2}\rho+\frac{1}{2}\right)_n \left(\frac{3}{2}-\frac{1}{2}\rho\right)_n} \frac{x^m}{m!} \frac{\left(\frac{1}{4}x^2\right)^n}{n!} +$$

$$+ \frac{(2\lambda-\rho)(1-\rho)}{\rho(2-\rho)} \sum_{m,n=0}^{\infty} \Omega_{m+2n+1} \frac{\left(\lambda-\frac{1}{2}\rho+1\right)_n \left(\frac{1}{2}\rho-\lambda+1\right)_n}{\left(\frac{3}{2}\right)_n \left(\frac{1}{2}\rho+1\right)_n \left(2-\frac{1}{2}\rho\right)_n} \frac{x^{m+1}}{m!} \frac{\left(\frac{1}{4}x^2\right)^n}{n!}$$

$$(\rho, 2-\rho \in \mathbb{C} \setminus \mathbb{Z}_0^-),$$

provided that both members of each of the double-series identities (2.10), (2.11) *and* (2.12) *exist.*

Corollary 3. *Each of the following hypergeometric transformation formulas holds true:*

(2.13)

$$F^{p:1;1}_{q:1;1}\left(\begin{matrix}[a_1:A_1,A_1],\cdots,[a_p:A_p,A_p]: & [\lambda:1]; & [\mu:1]; \\ [b_1:B_1,B_1],\cdots,[b_q:B_q,B_q]: & [2\lambda:1]; & [2\mu:1];\end{matrix}\; x, x\right)$$

$$= F^{p:0;2}_{q:0;3}\left(\begin{matrix}[a_1:A_1,2A_1],\cdots,[a_p:A_p,2A_p]: \text{———} ; & \left[\frac{1}{2}(\lambda+\mu):1\right], \left[\frac{1}{2}(\lambda+\mu+1):1\right]; \\ [b_1:B_1,2B_1],\cdots,[b_q:B_q,2B_q]: \text{———} ; & \left[\lambda+\frac{1}{2}:1\right], \left[\mu+\frac{1}{2}:1\right], [\lambda+\mu:1];\end{matrix}\; x, \frac{1}{4}x^2\right),$$

(2.14)

$$F^{p:1;1}_{q:1;1}\left(\begin{matrix}[a_1:A_1,A_1],\cdots,[a_p:A_p,A_p]: & [\lambda:1]; & [\rho-\lambda:1]; \\ [b_1:B_1,B_1],\cdots,[b_q:B_q,B_q]: & [\rho:1]; & [\rho:1];\end{matrix}\; x, x\right) =$$

$$= F^{p:0;2}_{q:0;3}\left(\begin{matrix}[a_1:A_1,2A_1],\cdots,[a_p:A_p,2A_p]: \text{———} ; & [\lambda:1], [\rho-\lambda:1]; \\ [b_1:B_1,2B_1],\cdots,[b_q:B_q,2B_q]: \text{———} ; & [\rho:1], \left[\frac{1}{2}\rho:1\right], \left[\frac{1}{2}\rho+\frac{1}{2}:1\right];\end{matrix}\; x, \frac{1}{4}x^2\right)$$

and

(2.15)

$$F^{p:1;1}_{q:1;1}\left(\begin{matrix}[a_1:A_1,A_1],\cdots,[a_p:A_p,A_p]: & [\lambda:1]; & [1-\lambda:1]; \\ [b_1:B_1,B_1],\cdots,[b_q:B_q,B_q]: & [\rho:1]; & [2-\rho:1];\end{matrix}\; x, x\right) =$$

$$= F^{p:0;2}_{q:0;3}\left(\begin{matrix}[a_1:A_1,2A_1],\cdots,[a_p:A_p,2A_p]: \text{———} ; & \left[\lambda-\frac{1}{2}\rho+\frac{1}{2}:1\right], \left[\frac{1}{2}\rho-\lambda+\frac{1}{2}:1\right]; \\ [b_1:B_1,2B_1],\cdots,[b_q:B_q,2B_q]: \text{———} ; & \left[\frac{1}{2}:1\right], \left[\frac{1}{2}\rho+\frac{1}{2}:1\right], \left[\frac{3}{2}-\frac{1}{2}\rho:1\right];\end{matrix}\; x, \frac{1}{4}x^2\right)+$$

$$
+\frac{(2\lambda-\rho)(1-\rho)\prod\limits_{j=1}^{p}(\alpha_j)_{A_j}}{\rho(2-\rho)\prod\limits_{j=1}^{q}(\beta_j)_{B_j}}\cdot
$$

$$
\cdot x F^{p:0;2}_{q:0;3}\left(\begin{array}{l} [a_1+A_1:A_1,2A_1],\cdots,[a_p+A_p:A_p,2A_p]:\text{———};\left[\lambda-\frac{1}{2}\rho+1:1\right],\left[\frac{1}{2}\rho-\lambda+1:1\right]; \\ [b_1+B_1:B_1,2B_1],\cdots,[b_q+B_q:B_q,2B_q]:\text{———};\left[\frac{3}{2}:1\right],\left[\frac{1}{2}\rho+1:1\right],\left[2-\frac{1}{2}\rho:1\right]; \end{array} x,\frac{1}{4}x^2\right),
$$

provided that both members of each of the assertions (2.13), (2.14) *and* (2.15) *exist.*

Remark 3. The special cases of the hypergeometric transformations (2.13) and (2.14), which are asserted by Cor. 3, when

$$
p=q=1 \qquad \text{and} \qquad A_1=B_1=1
$$

correspond to the second and the third main results in the aforecited paper by Pogány and Rathie [18, Theorems 2.2 and 2.3] (see also Rem. 1 in connection with their first main result [18, Th. 2.1]). The assertion (2.15) of Cor. 3 when

$$
\lambda=\alpha, \quad p=q=1 \qquad \text{and} \qquad A_1=B_1=1
$$

would provide the *duly-corrected* version of the fourth and final main result of Pogány and Rathie [18, Th. 2.4].

3. Concluding remarks and observations

Our presentation in this paper is motivated essentially by the usefulness of various families of reduction, transformation and summation formulas for hypergeometric functions in one, two and more variables in many diverse areas. Our main results (Theorems 1, 2 and 3) and their such applications and consequences as those asserted by Corollaries 1, 2 and 3 exhibit the fact that several substantially more general results on this subject than those proven recently (and markedly differently) by Pogány and Rathie [18] can be derived rather systematically by applying various known hypergeometric summation and transformation formulas. As a by-product of this sequel to [18], we have also provided the *duly-corrected* version of one of the four main results in this recent paper by Pogány and Rathie [18, Th. 2.4].

The methods and techniques, which we have applied in this paper, differ markedly from those used by Pogány and Rathie [18] for proving the special cases mentioned in Remarks 1, 2 and 3.

It is sincerely hoped that this presentation will encourage and aid the interested reader to develop further results of the types considered here for the Kampé de Fériet, Lauricella, and Srivastava–Daoust hypergeometric functions in two and more variables. For example, in a (presumably) hitherto unnoticed case of (2.5) when

$$y=-x,\quad \lambda=\beta-\alpha,\quad \mu=1-\alpha,\quad \rho=\beta\quad \text{and}\quad \sigma=2-\beta,$$

we find that

(4.1)

$$\mathfrak{S}^{\beta,2-\beta}_{\beta-\alpha,1-\alpha}(x,-x)=\sum_{N=0}^{\infty}\Omega_N\,\frac{(\beta-\alpha)_N}{(\beta)_N}\,\frac{x^N}{N!}\,{}_3F_2\left[\begin{matrix}1-\beta-N,1-\alpha,-N;\\ 1-\beta+\alpha-N,2-\beta;\end{matrix}\;1\right],$$

where $\mathfrak{S}^{\rho,\sigma}_{\lambda,\mu}(x,y)$ is defined by (2.4). Thus, upon evaluating the *well-poised* Clausenian series in (4.1) by means of Dixon's summation theorem [20, p. 243, Entry (III.8)], we are led to the following seemingly interesting companion of the series identity (2.12) asserted by Th. 3:

$$\begin{aligned}(4.2)\quad &\sum_{m,n=0}^{\infty}\Omega_{m+n}\,\frac{(\beta-\alpha)_m\,(1-\alpha)_n}{(\beta)_m\,(2-\beta)_n}\,\frac{x^m}{m!}\,\frac{(-x)^n}{n!}=\\ &=\sum_{n=0}^{\infty}\Omega_{2n}\,\frac{\left(\alpha-\frac{1}{2}\beta+\frac{1}{2}\right)_n\left(\frac{1}{2}\beta-\alpha+\frac{1}{2}\right)_n}{\left(\frac{1}{2}\beta+\frac{1}{2}\right)_n\left(\frac{3}{2}-\frac{1}{2}\beta\right)_n}\,\frac{x^{2n}}{(2n)!}+\\ &+\frac{(\beta-1)(\beta-2\alpha)}{\beta(\beta-2)}\sum_{n=0}^{\infty}\Omega_{2n+1}\,\frac{\left(\alpha-\frac{1}{2}\beta+1\right)_n\left(\frac{1}{2}\beta-\alpha+1\right)_n}{\left(\frac{1}{2}\beta+1\right)_n\left(2-\frac{1}{2}\beta\right)_n}\,\frac{x^{2n+1}}{(2n+1)!}\\ &\qquad\qquad(\beta,2-\beta\in\mathbb{C}\setminus\mathbb{Z}_0^-),\end{aligned}$$

provided that each member of the series identities (4.2) exists. In its special case when the sequence $\{\Omega_n\}_{n\in\mathbb{N}_0}$ is given by (2.2), the series identity (4.2) would readily yield the following hypergeometric reduction formula:

(4.3)

$$\begin{aligned}&F^{p:1;1}_{q:1;1}\left(\begin{matrix}[a_1:A_1,A_1],\cdots,[a_p:A_p,A_p]:[\beta-\alpha:1];[1-\alpha:1];\\ [b_1:B_1,B_1],\cdots,[b_q:B_q,B_q]:\quad[\beta:1];[2-\beta:1];\end{matrix}\;x,-x\right)=\\ &=\;{}_{p+2}F_{q+3}\left[\begin{matrix}[a_1:2A_1],\cdots,[a_p:2A_p],\left[\alpha-\frac{1}{2}\beta+\frac{1}{2}:1\right],\left[\frac{1}{2}\beta-\alpha+\frac{1}{2}:1\right];\\ [b_1:2B_1],\cdots,[b_q:2B_q],\left[\frac{1}{2}:1\right],\left[\frac{1}{2}\beta+\frac{1}{2}:1\right],\left[\frac{3}{2}-\frac{1}{2}\beta:1\right];\end{matrix}\;\frac{1}{4}x^2\right]+\\ &\quad+\frac{(\beta-1)(\beta-2\alpha)\,\prod_{j=1}^{p}(\alpha_j)_{A_j}}{\beta(\beta-2)\,\prod_{j=1}^{q}(\beta_j)_{B_j}}\cdot\end{aligned}$$

$$\cdot\, x \,{}_{p+2}F_{q+3}\left[\begin{matrix}[a_1+A_1:2A_1],\cdots,[a_p+A_p:2A_p],\left[\alpha-\frac{1}{2}\beta+1:1\right],\left[\frac{1}{2}\beta-\alpha+1:1\right]; & \dfrac{1}{4}x^2\\ [b_1+B_1:2B_1],\cdots,[b_q+B_q:2B_q],\left[\frac{3}{2}:1\right],\left[\frac{1}{2}\beta+1:1\right],\left[2-\frac{1}{2}\beta:1\right]; & \end{matrix}\right],$$

which holds true whenever both sides exist. This last result (4.3) may be compared with the hypergeometric transformation formula (2.15).

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