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ON THE CONJECTURE OF GENERALIZED TRIGONOMETRIC AND HYPERBOLIC FUNCTIONS

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Abstract: In this paper we prove the conjecture posed by Klén et al. in [13], and give optimal inequalities for generalized trigonometric and hyperbolic functions.

1. Introduction

In 1995, P. Lindqvist [15] studied the generalized trigonometric and hyperbolic functions with parameter $p > 1$. Thereafter several authors became interested to work on the equalities and inequalities of these generalized functions, e.g., see [4, 5, 6, 3, 7, 10, 11, 18] and the references therein. Recently, Klén et al. [13] were motivated by many results on

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these generalized trigonometric and hyperbolic functions, and they generalized some classical inequalities in terms of generalized trigonometric and hyperbolic functions, such as Mitrinović–Adamović inequality, Huygens' inequality, and Wilker's inequality. In this paper we prove the conjecture posed by Klén et al. in [13], and in Th. 1.4 we generalize the inequality

$$\frac{1}{\cosh(x)^a} < \frac{\sin(x)}{x} < \frac{1}{\cosh(x)^b},$$

where $a = \log(\pi/2)/\log(\cosh(\pi/2)) \approx 0.4909$ and $b = 1/3$, due to Neuman and Sándor [17, Th. 2.1].

For the formulation of our main results we give the definitions of the generalized trigonometric and hyperbolic functions as below.

The increasing homeomorphism function $F_p : [0,1] \to [0, \pi_p/2]$ is defined by

$$F_p(x) = \mathrm{arcsin}_p(x) = \int_0^x (1-t^p)^{-1/p}\, dt,$$

and its inverse $\sin_{p,q}$ is called generalized sine function, which is defined on the interval $[0, \pi_p/2]$, where

$$\mathrm{arcsin}_p(1) = \pi_p/2.$$

The function $\sin_p$ is strictly increasing and concave on $[0, \pi_p/2]$, and it is also called the eigenfunction of the Dirichlet eigenvalue problem for the one-dimensional p-Laplacian [9]. In the same way, we can define the generalized cosine function, the generalized tangent, and also the corresponding hyperbolic functions.

The generalized cosine function is defined by

$$\frac{d}{dx}\sin_p(x) = \cos_p(x), \quad x \in [0, \pi_p/2]\,.$$

It follows from the definition that

$$\cos_p(x) = (1 - (\sin_p(x))^p)^{1/p}\,,$$

and

$$|\cos_p(x)|^p + |\sin_p(x)|^p = 1, \quad x \in \mathbb{R}. \tag{1.1}$$

Clearly we get

$$\frac{d}{dx}\cos_p(x) = -\cos_p(x)^{2-p}\sin_p(x)^{p-1}.$$

The generalized tangent function $\tan_p$ is defined by

$$\tan_p(x) = \frac{\sin_p(x)}{\cos_p(x)}.$$

For $x \in (0, \infty)$, the inverse of generalized hyperbolic sine function $\sinh_p(x)$ is defined by

$$\operatorname{arsinh}_p(x) = \int_0^x (1 + t^p)^{-1/p} dt,$$

and generalized hyperbolic cosine and tangent functions are defined by

$$\cosh_p(x) = \frac{d}{dx}\sinh_p(x), \quad \tanh_p(x) = \frac{\sinh_p(x)}{\cosh_p(x)},$$

respectively. It follows from the definitions that

$$|\cosh_p(x)|^p - |\sinh_p(x)|^p = 1. \tag{1.2}$$

From above definition and (1.2) we get the following derivative formulas,

$$\frac{d}{dx}\cosh_p(x) = \cos_p(x)^{2-p}\sin_p(x)^{p-1}, \quad \frac{d}{dx}\tanh_p(x) = 1 - |\tanh_p(x)|^p.$$

Note that these generalized trigonometric and hyperbolic functions coincide with usual functions for $p = 2$.

Our main result reads as follows:

Theorem 1.3 ([13, Conj. 3.12])**.** *For $p \in [2, \infty)$, the function*

$$f(x) = \frac{\log(x/\sin_p(x))}{\log(\sinh_p(x)/x)}$$

is strictly increasing from $(0, \pi_p/2)$ onto $(1, p)$. In particular,

$$\left(\frac{x}{\sinh_p(x)}\right)^p < \frac{\sin_p(x)}{x} < \frac{x}{\sinh_p(x)}.$$

Theorem 1.4. *For $p \in [2,\infty)$, the function*
$$g(x) = \frac{\log(x/\sin_p(x))}{\log(\cosh_p(x))}$$
is strictly increasing in $x \in (0, \pi_p/2)$. In particular, we have
$$\frac{1}{\cosh_p(x)^\beta} < \frac{\sin_p(x)}{x} < \frac{1}{\cosh_p(x)^\alpha},$$
where $\alpha = 1/(1+p)$ and $\beta = \log(\pi_p/2)/\log(\cosh_p(\pi_p/2))$ are the best possible constants.

2. Preliminaries and proofs

The following lemmas will be used in the proof of the main result.

Lemma 2.1 ([2, Th. 2])**.** *For $-\infty < a < b < \infty$, let $f, g : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$, and be differentiable on (a,b). Let $g'(x) \neq 0$ on (a,b). If $f'(x)/g'(x)$ is increasing (decreasing) on (a,b), then so are*
$$\frac{f(x)-f(a)}{g(x)-g(a)} \quad \text{and} \quad \frac{f(x)-f(b)}{g(x)-g(b)}.$$
If $f'(x)/g'(x)$ is strictly monotone, then the monotonicity in the conclusion is also strict.

Lemma 2.2. *For $p \in [2,\infty)$, the function*
$$f(x) = \frac{p\sin_p(x)\log\left(x/\sin_p(x)\right)}{\sin_p(x) - x\cos_p(x)}$$
is strictly decreasing from $(0, \pi_p/2)$ onto $(1, p\log(\pi_p/2))$. In particular,
$$\exp\left(\frac{1}{p}\left(\frac{x}{\tan_p(x)} - 1\right)\right) < \frac{\sin_p(x)}{x} < \exp\left(\left(\log\frac{\pi_p}{2}\right)\left(\frac{x}{\tan_p(x)} - 1\right)\right).$$

Proof. Write
$$f_1(x) = p\sin_p(x)\log\left(x/\sin_p(x)\right), \quad f_2(x) = \sin_p(x) - x\cos_p(x),$$
and clearly $f_1(0) = f_2(0) = 0$. Differentiation with respect to x gives
$$\frac{f_1'(x)}{f_2'(x)} = \frac{(\sin_p(x))/x + \cos_p(x)(\log(x/\sin_p(x)) - 1)}{x\cos_p(x)^{2-p}\sin_p(x)^{p-1}} =$$
$$= \frac{1}{x\tan_p(x)^{p-1}}\left(\frac{\tan_p(x)}{x} + \log\left(\frac{x}{\sin_p(x)}\right) - 1\right),$$

which is the product of two decreasing functions, this implies that f_1'/f_2' is decreasing. Hence the function f is decreasing by Lemma 2.1. The limiting values follows from the l'Hôspital rule. $\diamondsuit$

Lemma 2.3. *For $p \in [2, \infty)$ the function*
$$g(x) = \frac{p \sinh_p(x) \log\left(\sinh_p(x)/x\right)}{x \cosh_p(x) - \sinh_p(x)}$$
is strictly increasing from $(0, \infty)$ onto $(1, p)$. In particular, we have
$$\exp\left(\frac{1}{p}\left(\frac{x}{\tanh_p(x)} - 1\right)\right) < \frac{\sinh_p(x)}{x} < \exp\left(\left(\frac{x}{\tanh_p(x)} - 1\right)\right).$$

Proof. Write
$$g_1(x) = \sinh_p(x) \log\left(\frac{\sinh_p(x)}{x}\right), \quad g_2(x) = x \cosh_p(x) - \sinh_p(x),$$
clearly $g_1(0) = g_2(0) = 0$. Differentiation with respect to x gives
$$\frac{g_1'(x)}{g_2'(x)} = \frac{\cosh_p(x)(1 + \log(\sinh_p(x)/x) - \sinh_p(x)/x}{x \cosh_p(x)^{2-p} \sinh_p(x)^{p-1}} =$$
$$= \frac{\sinh_p(x)}{x} \frac{\cosh_p(x)(1 + \log(1 + \sinh_p(x)/x) - \sinh_p(x)/x}{\cosh_p(x) \tanh_p(x)^p},$$
which is increasing, this implies that g is increasing. The limiting values follows from the l'Hôspital rule. $\diamondsuit$

Lemma 2.4. *For all $x > 0$ and $p > 1$, we have*
$$\log(\cosh_p(x)) > \frac{x}{p} \tanh_p(x)^{p-1} x.$$

Proof. Let
$$f(x) = \log(\cosh_p(x)) - \frac{x}{p} \tanh_p(x)^{p-1}.$$

A simple computation yields
$$f'(x) = \tanh_p(x)^{p-1} - \left(\frac{\tanh_p(x)^{p-1}}{p} + \frac{(p-1)}{p} \frac{x \tanh_p(x)^{p-2}}{\cosh_p(x)^p}\right) =$$
$$= \frac{p-1}{p} \tanh_p(x)^{p-2} \left(\tanh_p(x) - \frac{x}{\cosh_p(x)^p}\right),$$

which is positive because $\sinh_p(x) > x$ and $\cosh_p(x) > 1$ for all $x > 0$. Thus $f(x)$ is strictly increasing and $f(x) > f(0) = 0$, this implies the proof. $\diamondsuit$

Proof of Theorem 1.3. Write $f(x) = f_1(x)/f_2(x)$ for $x \in (0, \pi_p/2)$, where

$$f_1(x) = \log\left(\frac{x}{\sin_p(x)}\right), f_2(x) = \log\left(\frac{\sinh_p(x)}{x}\right).$$

For the proof of the monotonicity of the function f, it is enough to prove that

$$f'(x) = \frac{f_1'(x)f_2(x) - f_1(x)f_2'(x)}{f_2(x)^2}$$

is positive. After simple computation, this is equivalent to write

$$x\left(f_2(x)\right)^2 f'(x) = \frac{\sin_p(x) - x\cos_p(x)}{\sin_p(x)} f_2(x) - \frac{x\cosh_p(x) - \sinh_p(x)}{\sinh_p(x)} f_1(x),$$

which is positive by Lemmas 2.2 and 2.3. Hence, f is strictly increasing, and limiting values follows by applying the l'Hôpital rule. This completes the proof. $\diamondsuit$

Proof of Theorem 1.4. Write $g(x) = g_1(x)/g_2(x)$ for $x \in (0, \pi_p/2)$, where $g_1(x) = \log(x/\sin_p(x))$, $g_2(x) = \log(\cosh_p(x))$. Here we give the same argument as in the proof of Th. 1.3, and compute similarly

$$\begin{aligned}
&\left(\log\left(\cosh_p(x)\right)\right)^2 g'(x) = \\
&= \frac{\sin_p(x) - x\cos_p(x)}{x\sin_p(x)} \log\cosh_p(x) - \tanh_p(x)^{p-1} \log\left(\frac{x}{\sin_p(x)}\right) > \\
&> \frac{\sin_p(x) - x\cos_p(x)}{x\sin_p(x)\tanh_p(x)^{1-p}} \frac{x}{p} - \frac{\sin_p(x) - x\cos_p(x)}{p\sin_p(x)\tanh_p(x)^{1-p}} \\
&= 0,
\end{aligned}$$

by Lemmas 2.2 and 2.4. The limiting values follow from the l'Hôspital rule easily, hence the proof is obvious. $\diamondsuit$

The following corollary follows from [13, Lemma 3.3] and Th. 1.4.

Corollary 2.5. *For $p \in [2, \infty)$ and $x \in (0, \pi_p/2)$, we have*

$$\cos_p(x)^\beta < \frac{1}{\cosh_p(x)^\beta} < \frac{\sin_p(x)}{x} < \frac{1}{\cosh_p(x)^\alpha} < 1,$$

where α and β are as in Th. 1.4.

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