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DISTRIBUTIVE LATTICES OF SMALL WIDTH, I

A question of Rosenberg from the 1981 Banff Conference on Ordered Sets

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Abstract: The finite posets with the same width as their lattices of order ideals are characterized, answering a question of Ivo Rosenberg from the 1981 Banff Conference on Ordered Sets.

At the 1981 Banff Conference on Ordered Sets, Ivo Rosenberg asked to describe those finite posets that had the same width as their lattices of order ideals [4, p. 805]. We answer this question in Cor. 28.

For terminology, notation, and basic facts about lattices, please see [1], which calls order ideals "down-sets." We invoke Trotter's Axiom: All posets are finite. Let $\mathrm{w}(Q)$ be the width of the poset Q; let $\mathcal{O}(Q)$ be the lattice of down-sets of Q. For a distributive lattice L, let $\mathcal{J}(L)$ denote its poset of join-irreducible elements. A poset Q is *ranked or graded* if all maximal chains have the same length; the rank of an element x in Q is one less than the size of the largest chain whose top element is x; the rank of Q is the rank of a maximal element. If a and b are elements of a poset, $a \prec b$ means a is a lower cover of b. Unless otherwise stated, P is a poset of width w and $L \cong \mathcal{O}(P)$, so that by Birkhoff's Theorem

$P \cong \mathcal{J}(L)$. By *interval* we mean a set $\{x \in P : a \leq x \leq b\}$ for some $a, b \in P$ such that $a \leq b$; we denote it "$[a, b]$."

The *ordinal sum* of two posets P and Q with disjoint underlying sets is the poset $P \oplus Q$ where $p < q$ for all p in P and q in Q, and the restriction of the partial ordering to P or Q gives you the original ordering on P or Q, respectively; if P is a poset with a top element 1 and Q has a bottom element 0, the *coalesced ordinal sum* of P and Q is the poset $P \boxplus Q$ obtained by identifying 1 and 0. We can define both types of ordinal sum for more than two posets. As one finds in [1], for any posets P and Q, $\mathcal{O}(P \oplus Q) \cong \mathcal{O}(P) \boxplus \mathcal{O}(Q)$. One easily sees that the width of a non-empty ordinal sum or coalesced ordinal sum equals the width of one of the summands.

Following an observation of Edelman [5, pp. 156, 177–178], if $w = w(L)$ then $w \leq 3$, for if P has an antichain of size $k > 3$, then L has an antichain of size $\binom{k}{\lfloor \frac{k}{2} \rfloor} \geq \binom{k}{2} > k$. So if $w = w(L)$, then L can have no more than 3 elements of each rank. Thus L will be a coalesced ordinal sum of copies of the two-element chain $\mathbf{2}$ and the following sorts of lattices, whose structure it behooves us to determine:

Definition 1. A $2,3$-*lattice* L is a distributive lattice of rank $n \geq 2$ such that for $0 < r < n$ there are exactly 2 or 3 elements of rank r. A *segment* of L is the set of all elements of L that have a given rank $r < n$ and their upper covers, considered as a subposet of L. We will refer to any poset that can occur as a segment of a $2,3$-lattice as a "segment."

Note: A segment simply consists of the elements of two consecutive ranks.

Our argument follows closely that for "3-lattices" in [2].

Definition 2. Let S be a segment of a $2,3$-lattice L. If s is a minimal element of S and has rank r in L, then we say that S has *level* r in L and we write level $(S) = r$. If T is a segment of L with level $(T) = \text{level}(S)+1$, then we say that S *precedes* T and T *follows* S in L.

Lemma 3. *If L is a $2,3$-lattice of rank n and S is a segment of L with $0 < \text{level}(S) < n-1$, then S has the following properties:*

(1) *S has 2 or 3 minimal elements and 2 or 3 maximal elements.*
(2) *Every element of S is either minimal or maximal but not both.*
(3) *For every s in S, there exists t in S such that $s \neq t$ and s is comparable to t.*

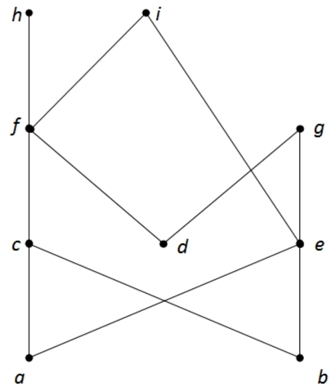

Figure 1

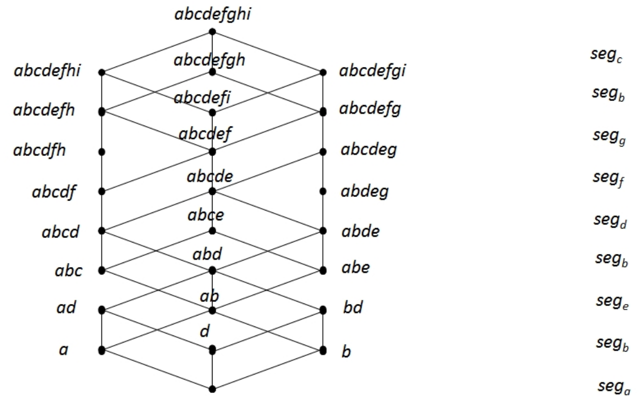

Figure 2

(4) *For any distinct maximal a, b in S, there is at most one c in S such that $c \prec a, b$ and for any distinct minimal a, b in S, there is at most one c in S such that $a, b \prec c$.*

Proposition 4. *Let L be a 2, 3-lattice of rank n and let S be a segment of L such that $0 <$ level $S < n-1$. For any a in S such that a is minimal in S, there exists b in S such that b is minimal in S and $a, b \prec a \vee b$ in L.*

Proof. See the proof of [2, Lemma 3.2]. ♢

Example 5. Fig. 1 is [2, Fig. A.1(ii)]. Its lattice of down-sets is Fig. 2 [2, Fig. 3].

Example 6. The poset of Fig. 3 has the lattice of down-sets of Fig. 4.

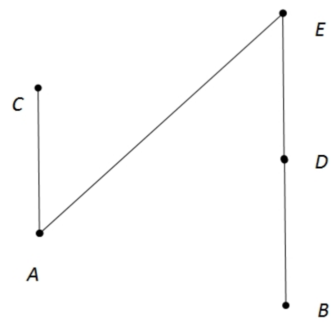

Figure 3

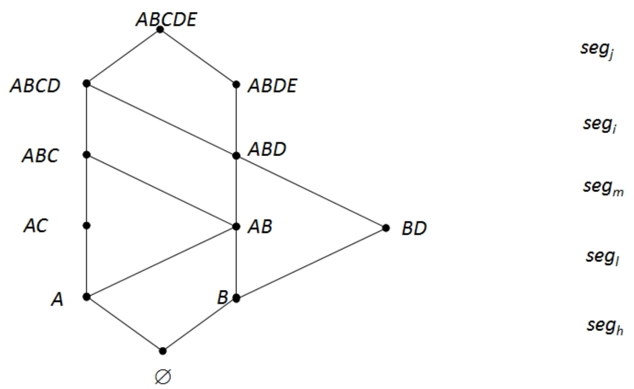

Figure 4

Example 7. The poset of Fig. 5 has the lattice of down-sets of Fig. 6.

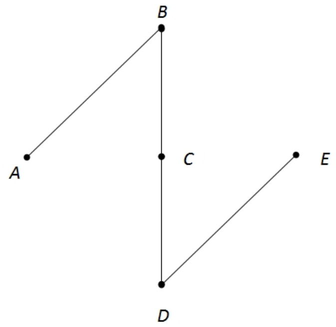

Figure 5

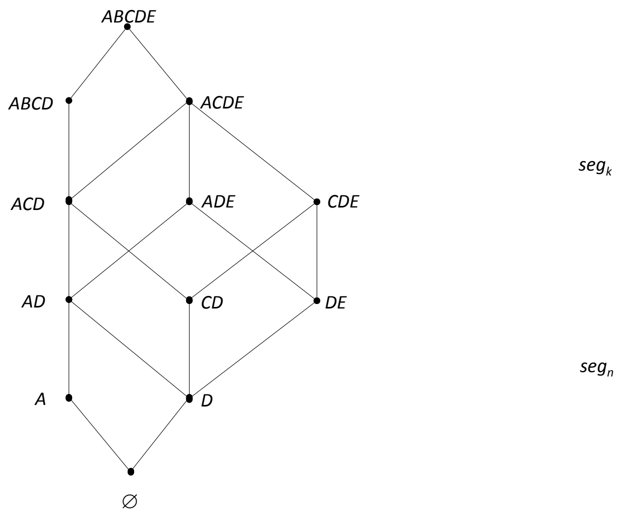

Figure 6

Corollary 8. *The posets in Fig. 7 are all the segments.*

Proof. Examples 5 through 7 show that the posets seg_a through seg_n are all segments.

First suppose a segment has exactly 1 minimal element. Since the rank of a $2,3$-lattice is at least 2, the segment must be seg_a or seg_h.

Now consider a segment with exactly 2 minimal elements. If it has exactly 1 maximal element, it is seg_j. If it has exactly 2 maximal elements, it is seg_i by Lemma 3(3), Lemma 3(4), and Prop. 4. Now assume it has 3 maximal elements. If both minimal elements have exactly 2 upper covers, by Lemma 3(4) we get seg_l. If one has only 1 upper cover, the other must have 3 by Lemma 3, so we get seg_n. If one has 2 or 3 upper covers, the other cannot have 3 by Lemma 3(4).

Since our list seg_a–seg_n is self-dual, by duality we need only now consider a segment with 3 minimal and 3 maximal elements. If one minimal element has 3 upper covers, then by Lemma 3(3) and Lemma 3(4), the other minimals have exactly 1 upper cover, and we get seg_e or seg_g. So assume no element has 3 upper covers and no element has 3 lower covers. If all three minimals have exactly 2 upper covers, we must get seg_b by Lemma 3(4). If exactly two minimals have exactly 2 upper covers, we get seg_f. If exactly one minimal has exactly 2 upper covers, then by Prop. 4 we would get seg_m (and not 3 maximal elements). By

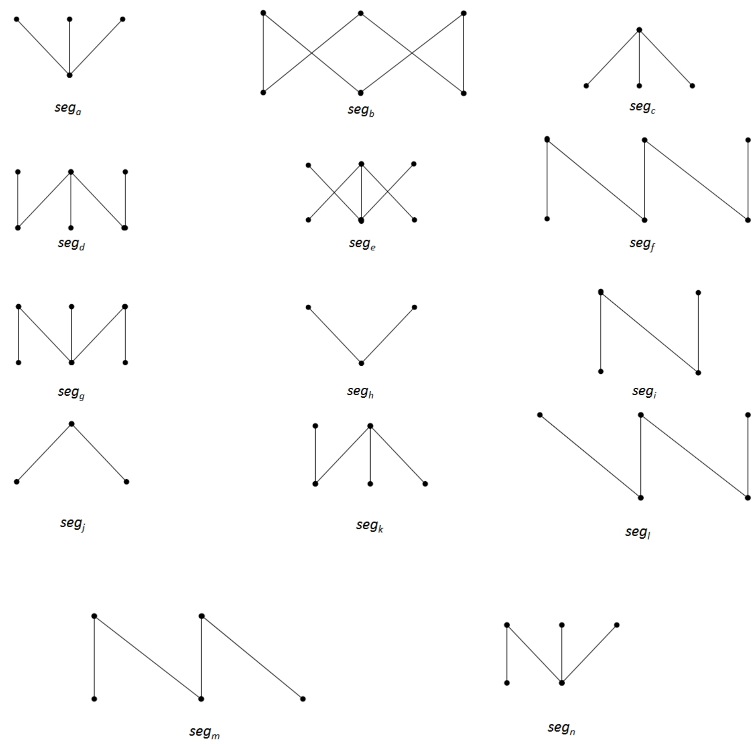

Figure 7

Prop. 4, all three minimals cannot have exactly 1 upper cover. ◊

Proposition 9. *Let S and T be segments of a $2,3$-lattice L. If S has a minimal element with 3 upper covers, and T follows S in L, then T is isomorphic to* seg_b.

Proof. See the proof of [2, Lemma 3.3]. ◊

Corollary 10. *Let L be a $2,3$-lattice and let S and T be segments of L with* $\mathrm{level}(T) = \mathrm{level}(S) + 1$. *If S is isomorphic to* seg_a, seg_e, seg_g *or* seg_n, *then T is isomorphic to* seg_b. *Dually, if T is isomorphic to* seg_c, seg_d, seg_e, *or* seg_k, *then S is isomorphic to* seg_b.

Proposition 11. *Let S and T be segments of a $2,3$-lattice L. If T is isomorphic to* seg_b, *and T follows S in L, then there is a minimal element of S that has 3 upper covers.*

Proof. See the proof of [2, Lemma 3.5]. ◊

Corollary 12. *Let L be a $2,3$-lattice and let S and T be segments of*

L with $\mathrm{level}(T) = \mathrm{level}(S) + 1$. *If S is isomorphic to* seg_b*, then T is isomorphic to* seg_c, seg_d, seg_e*, or* seg_k*. Dually, if T is isomorphic to* seg_b*, then S is isomorphic to* seg_a, seg_e, seg_g*, or* seg_n*.*

Proposition 13. *Let S and T be segments of a $2,3$-lattice L and let T follow S in L. If S has maximal elements a, b such that $a \wedge b \notin S$, then T is isomorphic to* seg_f, seg_g*, or* seg_m*.*

Proof. See the proof of [2, Lemma 3.7]. $\Diamond$

Corollary 14. *Let L be a $2,3$-lattice and let S and T be segments of L with* $\mathrm{level}(T) = \mathrm{level}(S) + 1$. *If S is isomorphic to* seg_d, seg_f*, or* seg_l*, then T is isomorphic to* seg_f, seg_g*, or* seg_m*. Dually, if T is isomorphic to* seg_f, seg_g*, or* seg_m*, then S is isomorphic to* seg_d, seg_f*, or* seg_l*.*

Proposition 15. *Let S and T be segments of a $2,3$-lattice L and suppose T follows S in L. If S has exactly 2 maximal elements, then T is isomorphic to* seg_i, seg_j, seg_l*, or* seg_n*.*

Corollary 16. *Let L be a $2,3$-lattice and let S and T be segments of L with* $\mathrm{level}(T) = \mathrm{level}(S) + 1$. *If S is isomorphic to* seg_h, seg_i, seg_k*, or* seg_m*, then T is isomorphic to* seg_i, seg_j, seg_l*, or* seg_n*. Dually, if T is isomorphic to* seg_i, seg_j, seg_l*, or* seg_n*, then S is isomorphic to* seg_h, seg_i, seg_k*, or* seg_m*.*

Recall the definition of concatenation function [2, pp. 1101–1102].

Lemma 17. *Suppose S and T are segments of a $2,3$-lattice L such that*

(i) *there exist maximal elements s_1 and s_2 of S that do not have a meet in S;*

(ii) *there exist minimal elements t_1 and t_2 of T that do not have a join in T;*

(iii) *T follows S in L.*

Let $\phi : S_{\max} \to T_{\min}$ be a concatenation function. If $S \&_\phi T$ is isomorphic to $S \cup T$ as a subposet of L, then $\phi[\{s_1, s_2\}] = [\{t_1, t_2\}]$.

Proof. See the proof of [2, Lemma 3.9]. $\Diamond$

Theorem 18. *Every $2,3$-lattice can be constructed via the finite-state diagram of Fig. 8, where the concatenation functions for successive segments are given by matching left- and right-most elements.*

N.B. To make the picture less cluttered, seg_j appears twice in the diagram.

Proof. For the state transitions, use Corollaries 10, 12, 14, and 16.

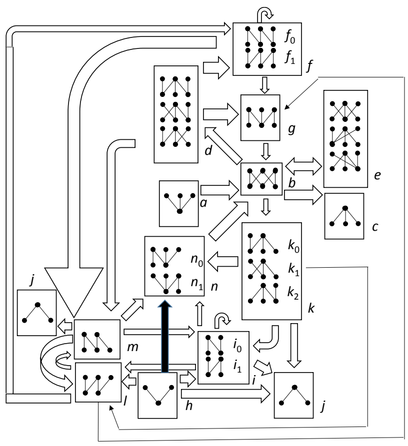

Figure 8

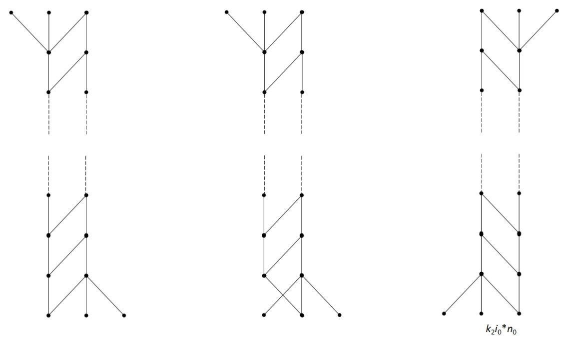

Figure 9

Every segment can be put in the orientations shown by rearranging the maximal elements and using Lemma 17. $\Diamond$

Definition 19. A $2,3$-*stack* is a finite poset constructed via a path in the state diagram of Fig. 8, using the obvious concatenation function (left nodes of successive segments are identified, as are right nodes). If it starts with seg_a or seg_h and ends with seg_c or seg_j, it is a *complete* $2,3$-stack.

Note: (1) Every $2,3$-stack is a ranked poset. (2) We can represent $2,3$-stacks as words, each letter representing a segment – or a letter with a subscript when there is more than one orientation of the segment. Fig. 4 shows $hlmi_0j$.

Corollary 20. *For all a, b, and c in a $2,3$-stack P, if $a,b \prec c$ and a and b are not minimal, then there is some $d \in P$ such that $d \prec a,b$, and dually.*

Proof. Use Fig. 8. $\Diamond$

Lemma 21. *Let P be a $2,3$-stack and let I be an interval in P. If $a,b \in I$ are distinct and have a common upper cover in I, then they have a common lower cover in I, and dually.*

Proof. See the proof of [2, Lemma 3.11], changing "$a',b' \not\prec a,b$" to "$a' \not\prec a$". $\Diamond$

Lemma 22. *Let P be a $2,3$-stack and let $I=[a,b]$ be an interval in P. If I has rank greater than 2, then $I \setminus \{a,b\}$ is connected.*

Proof. See the proof of [2, Lemma 3.13]. $\Diamond$

Lemma 23. *Let P be a $2,3$-stack. If $I = [a, b]$ is an interval in P of rank 3, then I is a distributive lattice.*

Proof. See the proof of [2, Lemma 3.14]. $\Diamond$

Theorem 24. *A poset is a $2,3$-lattice if and only if it is a complete $2,3$-stack.*

Proof. By [3, Th. 5.2], Lemmas 22 and 23, a complete $2,3$-stack of rank at least 3 is a distributive lattice. The only other complete $2,3$-stack is $\mathbf{2}^2$, which is distributive. The converse is Th. 18. $\Diamond$

While the $2,3$-stacks only have at most 3 elements of each rank, they might have width greater than 3. We characterize the ones that do next.

Proposition 25. *Let Q be a $2,3$-stack. The following are equivalent:*

(1) *The width of Q is at least 4.*

(2) *The $2,3$-stack uses seg_e or has a substring of consecutive letters of the form $k_0{i_1}^*n_1$, $k_1{i_1}^*n_1$, or $k_2{i_0}^*n_0$, where x^* means zero or more occurrences of the letter "x."*

Proof. The segment e has width 4. So does any $2,3$-stack of the form $k_0{i_1}^*n_1$, $k_1{i_1}^*n_1$, or $k_2{i_0}^*n_0$ (Fig. 9).

Let us now assume (2) is false. Note that if two posets can each be covered by 3 chains, and one has exactly 3 maximal elements and the other exactly 3 minimal elements, then their concatenation can also be covered by 3 chains.

Any non-empty $2,3$-stack consisting just of segments that have at most 2 minimal elements and at most 2 maximal elements can be covered with 2 chains (one consisting of the left elements and one consisting of the right elements). If we get a $2,3$-stack by adding to one of this type a segment with 3 maximal elements, or by preceding it with a segment having 3 minimal elements, we can cover the resulting $2,3$-stack with 3 chains. Every segment but seg_e can be covered with 3 chains.

Now let us return to a $2,3$-stack without the forbidden substrings. We are done if we can show that every such $2,3$-stack of rank at least 2 with 3 minimal elements and 3 maximal elements and having 2 elements of every other rank can be covered with 3 chains. This $2,3$-stack has the form $\{k, m\}i^*\{l, n\}$, i.e., it starts with seg_k or seg_m, ends with seg_l or seg_n, and has 0 or more seg_i's in between.

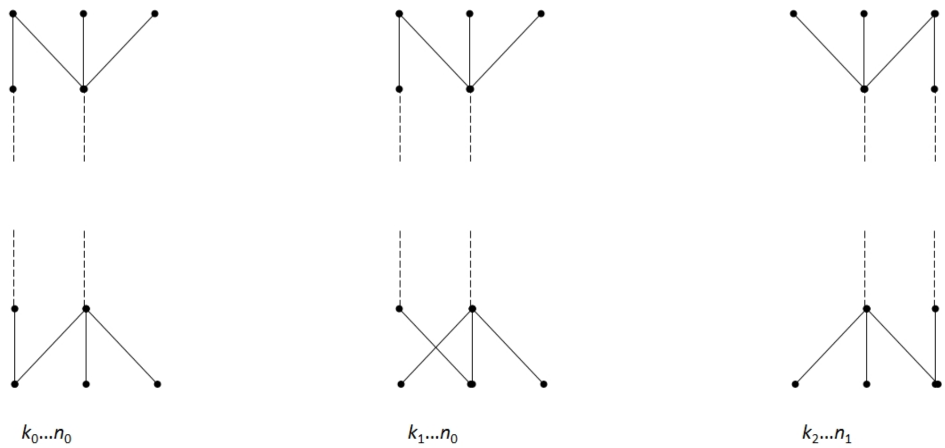

Figure 10

First we assume we start with seg_m. Put its minimals in chains C_1, C_2, and C_3 from left to right and put the elements in chains C_1 and C_3 as we go up. As we emerge into seg_l or seg_n, we can put the third maximal into C_2 since it lies above the minimal in C_2. If we start with seg_k and end with seg_l we can do something similar, putting the middle maximal in the unused chain.

So now assume we start with seg_k and end with seg_n. There are $3 \times 2 = 6$ options, and 3 are fine, because the remaining maximal lies above the minimal in the unused chain (Fig. 10).

Figure 11

In the remaining 3 possibilities, if we have two different orientations of seg_i, we are fine, since then we would have a substack like Fig. 11, where the point x is above the extra minimal and below the extra maximal. We are also fine if we have one of the situations in Fig. 12, for a similar reason.

Hence Q has width at most 3. $\diamondsuit$

Theorem 26. *Let L be a complete $2,3$-stack of width at most 3. Then $w = w(L)$ if and only if one of the following holds:*

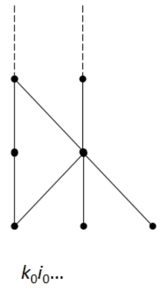

Figure 12

(1) *L uses only* seg_h, seg_i, *and* seg_j;
(2) *L uses* seg_b.

In case (1), $w = 2$. *In case* (2), $w = 3$.

Proof. If L only uses seg_h, seg_i, and seg_j, then L can be covered by 2 chains. Since L has 2 atoms, $P \cong \mathcal{J}(L)$ has width at least 2, hence exactly 2.

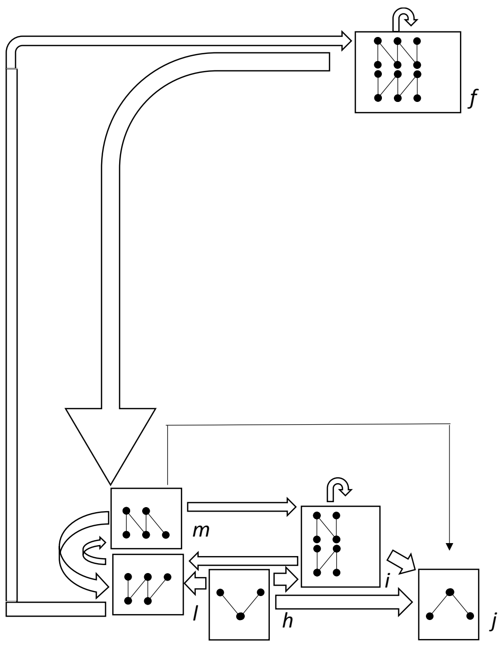

Figure 13

If L uses seg_b, then one of $\mathbf{2}^3$, $1 \oplus \mathbf{2}^3$, $\mathbf{2}^3 \oplus 1$, and $1 \oplus \mathbf{2}^3 \oplus 1$ is a $\{0,1\}$-sublattice of L, where n is the n-element antichain. Hence by

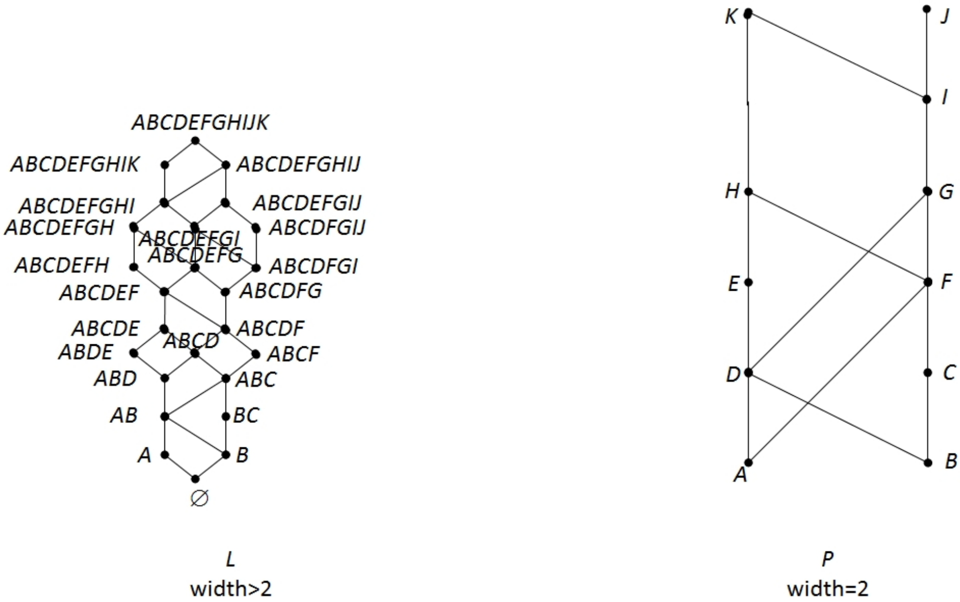

Figure 14

Priestley duality one of 3, $1 \oplus 3$, $3 \oplus 1$, and $1 \oplus 3 \oplus 1$ is a surjective image of P under an order-preserving map, so P has a 3-element antichain. Thus P has width at least 3, so exactly 3.

Conversely, suppose L does not use seg_b, but does use a segment other than seg_h, seg_i, and seg_j. Then L must use one of seg_f, seg_l, and seg_m, and cannot use seg_a, seg_b, seg_c, seg_d, seg_e, seg_g, seg_k, or seg_n. We have the finite-state diagram of Fig. 13. In this case, a join-irreducible will always be a right or a left node, and the left nodes form a chain, as do the right nodes. Thus $\mathcal{J}(L)$ has width at most 2, but since L uses seg_f, seg_l, or seg_m, L has width at least 3. $\Diamond$

Fig. 14 shows $hi_0i_1lmi_0lf_0mi_1j$.

Theorem 27. *The width of* **2** *equals* $w(1)$.

Corollary 28. *Let* P *be a finite poset. Let* $L = \mathcal{O}(P)$. *Then* $\mathrm{w}(P) = \mathrm{w}(L)$ *only if* $1 \le \mathrm{w}(P) \le 3$. *Also:*

(1) $\mathrm{w}(P) = 1 = \mathrm{w}(L)$ *if and only if* P *is a non-empty chain;*

(2) $\mathrm{w}(P) = 2 = \mathrm{w}(L)$ *if and only if* L *is a coalesced ordinal sum of copies of* **2** *and complete* $2,3$*-stacks that only use* seg_h, seg_i, *and* seg_j, *with at least one complete* $2,3$*-stack;*

(3) $\mathrm{w}(P) = 3 = \mathrm{w}(L)$ *if and only if* L *is a coalesced ordinal sum of copies of* **2** *and complete* $2,3$*-stacks such that* seg_e *and substacks of the form* $k_0 i_1{}^* n_1$*,* $k_1 i_1{}^* n_1$*, and* $k_2 i_0{}^* n_0$ *are never used, with at least one complete* $2,3$*-stack that uses* seg_b.

Proof. If $P = \emptyset$ then $\mathrm{w}(P) = 0 < \mathrm{w}(L)$, since $L \neq \emptyset$. The other inequality was proven earlier.

Part (1) is clear.

If $\mathrm{w}(P) = \mathrm{w}(L)$, then L is a coalesced ordinal sum of copies of **2** and complete $2,3$-stacks. If $w = 2$, we use Th. 26. If $w = 3$, we use Prop. 25 and Th. 26.

For the other direction of (2), use Prop. 25 and Th. 26. For the other direction of (3), use Prop. 25 and Th. 26. $\Diamond$

Thus we have answered the question of Rosenberg from the 1981 Banff Conference on Ordered Sets.

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