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ON ZARISKI TOPOLOGIES AND ALGEBRAIC COMBINATORICS

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Abstract: The "Zariski Topology" is an important tool within "Classical Algebraic Geometry" to study affine and projective varieties over fields. Andreas Dress and the author have extended this topology to socalled "Fuzzy Geometries", which unify "Algebraic Geometry" and the relatively new field of "Tropical Geometry". In this paper, an even more general concept of the "Zariski Topology" is studied, and an application to the theory of "Ordered Sets" is given.

1. Introduction

"Classical Algebraic Geometry" and "Tropical Geometry" have much in common, cf. for instance [5] as well as [6], [7]. Therefore, Andreas Dress and the author have begun to present a unified theory of "Fuzzy Geometries" that encompasses "Algebraic Geometry" and "Tropical Geometry"; see [3]. These general "Fuzzy Geometries" are erected on "fuzzy rings", which include commutative rings, whence the branch of "Classical Algebraic Geometry" is covered. These fuzzy rings were already introduced in [1] by developing a "Theory of Matroids with Coefficients" that

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includes particularly representable, oriented, and valuated matroids; see also [2].

Usually, "Tropical Geometry" is considered as the geometry over the semiring of real numbers with the "Tropical Addition", which means taking the minimum of two numbers, and the "Tropical Multiplication"; that is ordinary addition. By slightly modifying this semiring, we get a fuzzy ring that controls "Tropical Geometry". Particularly, "Affine Varieties" and "Tropical Varieties" serve to be studied in a general framework; this has now been done in [3]. To this end, we have extended the classical concept of the "Zariski Topology" to "Fuzzy Geometries" over "quasi fuzzy domains"; these constitute the appropriate generalization of integral domains. Such a quasi fuzzy domain K contains a distinguished proper and nonempty subset K_0 which has similar properties as a prime ideal in a ring. In particular, for $a, b \in K$ one has $a \cdot b \in K_0$ if and only if $a \in K_0$ or $b \in K_0$. The Zariski topology as considered in [3] is defined on an abstract nonempty set M as follows: Assume that $\mathcal{F} \subseteq K^M$ is a set of maps satisfying certain axioms. Then the closed subsets A of M are those sets for which there exists a subset $\mathcal{T}$ of $\mathcal{F}$ such that

$$A = Z(\mathcal{T}) := \{a \in M | f(a) \in K_0 \ \text{ for all } \ f \in \mathcal{T}\}.$$

Thus, the "zero set" of $\mathcal{T}$ as studied in "Algebraic Geometry" is replaced by the corresponding intersection of preimages of K_0. It is just the property of K_0 concerning the generalization of prime ideals described above which ensures that the union of two closed sets is closed again. In the present paper, it is analysed what is actually needed to define an even much more general concept of the "Zariski Topology". With regard to the idea concerning the union of two closed sets just mentioned, we can more generally consider arbitrary sets D with a nontrivial partition $D = D_0 \dot\cup D_1$ and a "multiplication" $\cdot : D \times D \to D$ that merely fulfills the following axiom:

For $a, b \in D$, one has $a \cdot b \in D_1$ if and only if $a \in D_1$ and $b \in D_1$.

Such an algebraic structure will be called a "bipartite domain". If, in addition, M is an arbitrary nonempty set, consider a set $\mathcal{F} \subseteq D^M$ of maps from M into D that is closed under "multiplication" and, additionally, has the property that for every $a \in M$ there exists some $f \in \mathcal{F}$ with $f(a) \in D_1$. Then it turns out that the system of all sets

$$Z(\mathcal{T}) := \{a \in M | f(a) \in D_0 \ \text{ for all } \ f \in \mathcal{T}\},$$

where $\mathcal{T}$ runs through $\mathcal{P}(\mathcal{F})$, is the system of closed sets of a "general Zariski Topology"; cf. Prop. and Def. 2.8. We study irreducible closed subsets of M and, thereby extend the well known correspondence between irreducible algebraic sets and prime ideals in polynomial rings over fields to this much more general framework.

As an application of these generalized “Zariski Topologies”, we study ordered sets $(M, \leq)$, cf. also [4], and let $\mathcal{F}$ denote the set of all increasing maps $f : M \to \{0, 1\}$. Based on the subject developed in this paper, it follows easily that the closed sets in the induced “Zariski Topology” are precisely the order ideals in $(M, \leq)$ and that such an order ideal $I \subseteq M$ is irreducible if and only if any two elements $a, b \in I$ have an upper bound $c \in I$.

2. Bipartite domains and Zariski topologies

In this section, we introduce the rather general concept of a bipartite domain D as well as an abstract “Zariski Topology” on a set M, which will be induced by a family of maps with values in D.

Definition 2.1. A *bipartite domain* $D = (D, \cdot, D_0, D_1)$ is a set D, together with an inner operation

$$\cdot : D \times D \to D : (a, b) \mapsto a \cdot b$$

and a specified partition $D = D_0 \dot{\cup} D_1$ into two nonempty subsets D_0, D_1 of D such that the following axiom holds:

(BD) For $a, b \in D$, one has $a \cdot b \in D_1$ if and only if $a \in D_1$ and $b \in D_1$.

Remark 2.2. Note that, by definition, a bipartite domain has at least two elements and that the operation “$\cdot$” need neither be commutative nor associative.

Example 2.3. Assume that D_0, D_1 are arbitrary disjoint and nonempty sets, put $D := D_0 \dot{\cup} D_1$, and define the inner operations: $\cdot : D \times D \to D$ and $\odot : D \times D \to D$ by

$$a \cdot b := \begin{cases} a \text{ if } a \in D_0 \text{ or } b \in D_1, \\ b \text{ if } a \in D_1 \text{ and } b \in D_0. \end{cases}$$

$$a \odot b := \begin{cases} a \text{ if } a \in D_0, \\ b \text{ if } a \in D_1. \end{cases}$$

Then both $(D, \cdot, D_0, D_1)$ and $(D, \odot, D_0, D_1)$ are bipartite domains.

Example 2.4. Suppose that $(D, \leq)$ is an ordered set such that any two elements $a, b \in D$ have an infimum $a \wedge b \in D$. Assume that $x_1 \in D$ is arbitrary such that

$$D_1 := \{x \in D | x_1 \leq x\} \neq D. \tag{2.1}$$

Thus, D_1 is a proper filter in $(D, \leq)$, while $D_0 := D \setminus D_1$ is a proper ideal in $(D, \leq)$.

Finally, put

$$a \cdot b := a \wedge b \text{ for } a, b \in D. \tag{2.2}$$

Then $(D, \cdot, D_0, D_1)$ is a bipartite domain.

Example 2.5. Assume that $(R, +, \cdot)$ is an *integral domain*; that means, R is a commutative unitary ring such that $x, y \in R$ satisfy $x \cdot y = 0$ if and only if $x = 0$ or $y = 0$. Then $(R, \cdot, \{0\}, R \setminus \{0\})$ is a bipartite domain.

Example 2.6. By generalizing Ex. 2.5, assume that $(K; +; \cdot; \varepsilon; K_0)$ is a *quasi fuzzy domain*, cf. [3]. – That means that K is a set with two inner operations $+, \cdot : K \times K \to K$, a specified element $\varepsilon \in K$ and a specified nonempty proper subset K_0 of K such that – among several other axioms – the following holds:

For $x, y \in K$ one has $x \cdot y \in K_0$ if and only if $x \in K_0$ or $y \in K_0$.

Then $(K, \cdot, K_0, K \setminus K_0)$ is a bipartite domain.

Such quasi fuzzy domains have been the foundation in [3] to study a unified approach to “Classical Algebraic Geometry” and to “Tropical Geometry”. It is this example that suggests to study abstract bipartite domains and more general “Zariski Topologies”.

Definition 2.7. Assume that $D = (D, \cdot, D_0, D_1)$ is a bipartite domain, that M is an arbitrary nonempty set, and that $\mathcal{F} \subseteq D^M$ is a set of maps from M into D. The triple $(M, D, \mathcal{F})$ is called a *Zariski system with coefficients in the bipartite domain* D (or shortly Zarisky system), if the following axioms hold:

(Z1) For $f, g \in \mathcal{F}$ one has also $f \cdot g \in \mathcal{F}$, where, of course, $f \cdot g : M \to D$ is defined by $(f \cdot g)(a) := f(a) \cdot g(a)$ for $a \in M$.

(Z2) Every point $a \in M$ is *nondegenerate*; that means, there exists some $f = f_a \in \mathcal{F}$ with $f(a) \in D_1$.

If $(M, D, \mathcal{F})$ is a Zariski system and $\mathcal{T} \subseteq \mathcal{F}$, put

$$Z(\mathcal{T}) := \{a \in M | f(a) \in D_0 \text{ for all } f \in \mathcal{T}\}, \tag{2.3}$$

$$Z(f) := Z(\{f\}) \text{ for } f \in \mathcal{F}, \tag{2.3a}$$

$$\mathcal{V} := \{Z(\mathcal{T}) | \mathcal{T} \subseteq \mathcal{F}\}. \tag{2.4}$$

We have the following

Proposition and Definition 2.8. Suppose that $(M, D, \mathcal{F})$ is a Zariski system with coefficients in the bipartite domain D. Then the set system $\mathcal{V}$ satisfies the following conditions:

(A1) $M = Z(\phi) \in \mathcal{V}, \ \phi = Z(\mathcal{F}) \in \mathcal{V}$.

(A2) If $(\mathcal{T}_i)_{i \in I}$ is a family of subsets of $\mathcal{F}$, then we have

$$\bigcap_{i \in I} Z(\mathcal{T}_i) = Z(\bigcup_{i \in I} \mathcal{T}_i) \in \mathcal{V}.$$

(A3) If $\mathcal{T}_1, \mathcal{T}_2 \subseteq \mathcal{F}$ and

$$\mathcal{T} := \mathcal{T}_1 \cdot \mathcal{T}_2 := \{f_1 \cdot f_2 | f_1 \in \mathcal{T}_1, f_2 \in \mathcal{T}_2\},$$

then we have

$$Z(\mathcal{T}_1) \cup Z(\mathcal{T}_2) = Z(\mathcal{T}) \in \mathcal{V}.$$

In particular, $\mathcal{V}$ is the system of closed sets of a topology defined on M, called the *Zariski topology of the Zariski system* $(M, D, \mathcal{F})$.

Proof. The relation $M = Z(\phi)$ is trivial, while $\phi = Z(\mathcal{F})$ is nothing but a reformulation of (Z2). (A2) holds trivially.

Verification of (A3). By axiom (Z1), one has $\mathcal{T} = \mathcal{T}_1 \cdot \mathcal{T}_2 \subseteq \mathcal{F}$ and, hence, $Z(\mathcal{T}) \in \mathcal{V}$. It remains to prove: $Z(\mathcal{T}_1) \cup Z(\mathcal{T}_2) = Z(\mathcal{T})$. Assume that $a \in Z(\mathcal{T}_1)$. Then we have $f_1(a) \in D_0$ for all $f_1 \in \mathcal{T}_1$, and hence also $f_1(a) \cdot f_2(a) \in D_0$ for all $f_1 \in \mathcal{T}_1$ and all $f_2 \in \mathcal{T}_2$. This means $Z(\mathcal{T}_1) \subseteq Z(\mathcal{T})$. Similarly, we get $Z(\mathcal{T}_2) \subseteq Z(\mathcal{T})$.

To complete the proof, assume that $a \in Z(\mathcal{T}) \setminus Z(\mathcal{T}_1)$. Then there exists some $f_1 \in \mathcal{T}_1$ with $f_1(a) \in D_1$. But for every $f_2 \in \mathcal{T}_2$ we have $f_1(a) \cdot f_2(a) \in D_0$ and, hence, $f_2(a) \in D_0$ by axiom (BD). This means $a \in Z(\mathcal{T}_2)$ as claimed. $\Diamond$

3. Irreducibility and an application to ordered sets

Throughout this section, assume that $(M, D, \mathcal{F})$ is a Zariski system with coefficients in the bipartite domain $D = (D, \cdot, D_0, D_1)$. We want to extend the classical concept of an "irreducible algebraic set" from "Algebraic Geometry" to this more general framework.

For $N \subseteq M$ put

$$I(N) := \{f \in \mathcal{F} | f(a) \in D_0 \text{ for all } a \in N\}. \tag{3.1}$$

Trivially, the operators $I : \mathcal{P}(M) \to \mathcal{P}(F)$ and $Z : \mathcal{P}(F) \to \mathcal{P}(M)$ as defined in (2.3) are order reversing. Moreover, we have the following

Lemma 3.1. *For all $N \subseteq M$ one has:*
i) $N \subseteq Z(I(N))$,
ii) $I(N) = I(Z(I(N)))$.
Moreover, we have for all $\mathcal{T} \subseteq \mathcal{F}$:
iii) $\mathcal{T} \subseteq I(Z(\mathcal{T}))$,
iv) $Z(\mathcal{T}) = Z(I(Z(\mathcal{T})))$.

Proof. i) and iii) are trivial.

iii) yields, applied to $I(N)$ instead of $\mathcal{T}$:

$$I(N) \subseteq I(Z(I(N))).$$

But i) implies $I(Z(I(N))) \subseteq I(N)$, because the operator I is order reversing. This proves ii), and, similarly, iv) follows. $\Diamond$

Definition 3.2. A nonempty closed subset $A = Z(\mathcal{T})$ of M is called *irreducible* in $(M, \mathcal{V})$ (or in $(M, D, \mathcal{F})$ or simply in M, if no misunderstanding is possible), if A is not the union $A_1 \cup A_2$ of two proper closed subsets $A_1, A_2 \in \mathcal{V}$.

The next result generalizes the well known correspondence between irreducible algebraic sets and prime ideals in polynomial rings over fields, cf. for instance Section 1.1 in [5].

Proposition 3.3. *For $\mathcal{T} \subseteq \mathcal{F}$ with $Z(\mathcal{T}) \neq \phi$, the following two statements are equivalent:*

(i) *$Z(\mathcal{T})$ is irreducible in $(M, D, \mathcal{F})$.*

(ii) *For all $f, g \in \mathcal{F} \setminus I(Z(\mathcal{T}))$ one has $f \cdot g \in \mathcal{F} \setminus I(Z(\mathcal{T}))$; that means, the complement of $I(Z(\mathcal{T}))$ is multiplicatively closed.*

Proof. (i) $\Rightarrow$ (ii): Suppose that $f, g \in \mathcal{F}$ satisfy $f \cdot g \in I(Z(\mathcal{T}))$. Then Lemma 3.1 iv) yields

$$Z(\mathcal{T}) = Z(I(Z(\mathcal{T}))) \subseteq Z(f \cdot g) = Z(f) \cup Z(g),$$

where the last equation follows from axiom (BD) (or (A3)).

Consequently, we get

$$Z(\mathcal{T}) = (Z(\mathcal{T}) \cap Z(f)) \cup (Z(\mathcal{T}) \cap Z(g)).$$

Since $Z(\mathcal{T}) \cap Z(f)$ and $Z(\mathcal{T}) \cap Z(g)$ are closed, (i) yields without loss of generality:

$$Z(\mathcal{T}) = Z(\mathcal{T}) \cap Z(f).$$

This means $Z(\mathcal{T}) \subseteq Z(f)$, whence

$$f \in I(Z(f)) \subseteq I(Z(\mathcal{T})).$$

(ii) $\Rightarrow$ (i): We assume that there exist $\mathcal{T}_1, \mathcal{T}_2 \subseteq \mathcal{F}$ with $Z(\mathcal{T}) = Z(\mathcal{T}_1) \cup Z(\mathcal{T}_2)$, but $Z(\mathcal{T}_i) \neq Z(\mathcal{T})$ for $i \in \{1, 2\}$. If $I(Z(\mathcal{T}_1)) \subseteq I(Z(\mathcal{T}_2))$, Lemma 3.1 iv) would imply

$$Z(\mathcal{T}_2) = Z(I(Z(\mathcal{T}_2))) \subseteq Z(I(Z(\mathcal{T}_1))) = Z(\mathcal{T}_1),$$

a contradiction to $Z(\mathcal{T}_1) \neq Z(\mathcal{T})$.

Thus, we can choose some $f \in I(Z(\mathcal{T}_1)) \setminus I(Z(\mathcal{T}_2))$ and, similarly, some $g \in I(Z(\mathcal{T}_2)) \setminus I(Z(\mathcal{T}_1))$. Then axiom (BD) yields:

$$f \cdot g \in I(Z(\mathcal{T}_1) \cup Z(\mathcal{T}_2)) = I(Z(\mathcal{T})).$$

On the other hand, we have $I(Z(\mathcal{T})) \subseteq I(Z(\mathcal{T}_1))$, whence $g \notin I(Z(\mathcal{T}))$ and, similarly, $f \notin I(Z(\mathcal{T}))$, what contradicts (ii). $\Diamond$

Warning. If $Z(\mathcal{T}) = \phi$, then we have $I(Z(\mathcal{T})) = \mathcal{F}$, whence (ii) is true, while (i) is wrong by definition.

Remark 3.4. Consider $\overline{D} := \{0, 1\}$ as a subset of $\mathbb{Z}$. Then, with the ordinary multiplication, $(\overline{D}, \cdot, \{0\}, \{1\})$ is a bipartite domain; it is the *final bipartite domain.* For every bipartite domain $D = (D, \cdot, D_0, D_1)$, we have a canonical map $\varphi : D \to \overline{D}$ given by

$$\varphi(a) := \begin{cases} 0 & \text{for} \quad a \in D_0, \\ 1 & \text{for} \quad a \in D_1, \end{cases} \tag{3.2}$$

and, by (BD), one has of course

$$\varphi(a \cdot b) = \varphi(a) \cdot \varphi(b) \ \text{ for all } \ a, b \in D. \tag{3.3}$$

If $(M, D, \mathcal{F})$ is a Zariski system with coefficients in $(D, \cdot, D_0, D_1)$, then we get a new Zariski system $(M, \overline{D}, \overline{\mathcal{F}})$ with

$$\overline{\mathcal{F}} := \{\varphi \circ f | f \in \mathcal{F}\}. \tag{3.4}$$

Since $f^{-1}(D_0) = (\varphi \circ f)^{-1}(\{0\})$ holds for all $f \in \mathcal{F}$, both of these Zariski systems exhibit one and the same Zariski topology and, hence, also the same system of irreducible sets.

In the rest of this paper, let $(M, \leq)$ denote an arbitrary – non-empty – ordered set. By definition, an *order ideal* in $(M, \leq)$ is a subset J of M satisfying the following condition:

For $a \in J$ and $b \in M$ with $b \leq a$ one has $b \in J$.

Note that we consider particularly the empty set as an order ideal.

We want to apply the general concept of the Zariski topology developed here to order ideals. To this end, assume that $\overline{D} = \{0, 1\}$ is as in Remark 3.4 – and that $\overline{D}$ is equipped with the usual order; that means $0 < 1$. Moreover, let $\mathcal{F}$ denote the set of all order morphisms from $(M, \leq)$ to $(\overline{D}, \leq)$; that means, for $a, b \in M$ with $a \leq b$ one has $f(a) \leq f(b)$.

We have the following

Proposition 3.5.

i) *The triple* $(M, \overline{D}, \mathcal{F})$ *is a Zariski system with coefficients in* $\overline{D} = (\overline{D}, \cdot, \{0\}, \{1\})$.

ii) *The closed sets of the induced Zariski topology are precisely the order ideals in* $(M, \leq)$.

Proof. i) Assume that $f, g \in \mathcal{F}$ and that $a, b \in M$ satisfy $a \leq b$. Then one has

$$(f \cdot g)(a) = f(a) \cdot g(a) \leq f(b) \cdot g(b) = (f \cdot g)(b),$$

whence (Z1) is verified.

Axiom (Z2) holds trivially, because the constant function $f_1 : M \to \overline{D}$ given by $f_1(a) := 1$ for all $a \in M$ lies in $\mathcal{F}$.

ii) Assume that $\mathcal{T} \subseteq \mathcal{F}$ and that $a \in Z(\mathcal{T})$. Then we have $f(a) = 0$ for all $f \in \mathcal{T}$. This means $0 \leq f(b) \leq f(a) = 0$ for all $b \in M$ with $b \leq a$ and all $f \in \mathcal{T}$. Thus we get $b \in Z(\mathcal{T})$ whenever $b \leq a$. Hence, $Z(\mathcal{T})$ is an order ideal.

If, vice versa, $J \subseteq M$ is an order ideal, define $f : M \to \overline{D}$ by

$$f(a) := \begin{cases} 0 & \text{for} \quad a \in J, \\ 1 & \text{for} \quad a \notin J. \end{cases}$$

One has $f \in \mathcal{F}$ – just because J is an order ideal. Moreover, we have $Z(f) = J$ as claimed. $\Diamond$

Now we are ready to give a rather conceptual and transparent proof of the following result.

Proposition 3.6. *For a nonempty order ideal J in the ordered set $(M, \leq)$, the following statements are equivalent*:

(i) *Any two elements $a, b \in J$ have an upper bound $c \in J$.*

(ii) *J is not the union $J_1 \cup J_2$ of two order ideals J_1, J_2 that are properly contained in J.*

(iii) *J is irreducible in the Zariski system $(M, \overline{D}, \mathcal{F})$.*

(iv) *For $f, g \in \mathcal{F}$ with $f \cdot g \in I(J)$ one has $f \in I(J)$ or $g \in I(J)$.*

Proof. (ii) $\Leftrightarrow$ (iii) holds by Def. 3.2 and Prop. 3.5 ii).

(iii) $\Leftrightarrow$ (iv) holds by Prop. 3.3, because J is closed and nonempty.

(i) $\Rightarrow$ (iv): We assume that there exist $f, g \in \mathcal{F}$ with $f \cdot g \in I(J)$ but $f \notin I(J)$ as well as $g \notin I(J)$. Choose $a, b \in J$ with $f(a) = g(b) = 1$. By (i) there exists some $c \in J$ with $a \leq c$ and $b \leq c$. Since $f, g \in \mathcal{F}$, we get

$$(f \cdot g)(c) = f(c) \cdot g(c) = 1 \cdot 1 = 1,$$

a contradiction to $f \cdot g \in I(J)$.

(iv) $\Rightarrow$ (i): Suppose that $a, b \in J$, and define $f, g \in \mathcal{F}$ by

$$f(x) := \begin{cases} 1 & \text{for } a \leq x, \\ 0 & \text{otherwise,} \end{cases}$$

$$g(x) := \begin{cases} 1 & \text{for } b \leq x, \\ 0 & \text{otherwise.} \end{cases}$$

Since $f(a) = g(b) = 1$, one has $f, g \notin I(J)$. Hence, by (iv), we have also $f \cdot g \notin I(J)$. Thus we get for an appropriate element $c \in J$:

$$f(c) \cdot g(c) = (f \cdot g)(c) = 1.$$

This means $a \leq c$ and $b \leq c$ as claimed. $\Diamond$

Remark 3.7. If the ordered set $(M, \leq)$ has finite height, then statements (i) and (ii) are easily seen to be equivalent to the following condition:

(v) There exists some $x_1 \in J$ with $J = \{x \in M | x \leq x_1\}$.

However, in general, (v) is stronger than (i) and (ii) – even if $(M, \leq)$ is a totally ordered set. Consider, for instance, the ordered set $(\mathbb{Q}, \leq)$ of rational numbers and Dedekind's cuts.

Note that, for any totally ordered set $(M, \leq)$, the statements (i)–(iv) are always true, because (i) is trivial.

Finally, we present the following result, which follows immediately by dualizing Prop. 3.6; that means, one has merely to invert the given ordering.

Proposition 3.8. *Assume that F is a nonempty filter in $(M, \leq)$; that means, for $a \in F$ and $b \in M$ with $a \leq b$ one has $b \in F$. Then the following two statements are equivalent*:

(i) *Any two elements $a, b \in F$ have a lower bound $c \in F$.*

(ii) *F is not the union of two filters that are properly contained in F.*

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