Mathematica Pannonica
24/2 (2013), 197–210

THE CONCEPT OF ORTHOGONALITY IN CARTAN'S GEOMETRY BASED ON THE NOTION OF AREA

Imsatfia **Moheddine**

Institut Mathématiques de Jussieu, UMR 7586, Université Paris Diderot – Paris 7, Bâtiment Sophie Germain, Bureau 751, Case 7012 – 75205 Paris Cedex 13, France

Received: March 2013

MSC 2010: Primary 53 C 60; secondary 58 E 30

Keywords: Grassmannian bundle, areal space, calculus of variations.

Abstract: In 1933 Elie Cartan defined an infinitesimal metric ds starting from a variational problem on hypersurfaces in an n-dimensional manifold $\mathcal{M}$. This metric depends not only on the point $\mathrm{M} \in \mathcal{M}$ but also on the orientation of a hyperplane in the tangent space $T_{\mathrm{M}}\mathcal{M}$. His work is based in a natural definition of the orthogonal direction to such tangent hyperplane. In this paper we extend this orthogonality to an oriented vector subspace in $T_{\mathrm{M}}\mathcal{M}$ by using calculus of variation.

1. Introduction

Riemann considered the possibility to give to ds, the distance between two infinitesimally close points, a more general expression than $\sqrt{g_{ij}(dx^i, dx^j)}$ namely to choose any function of x and dx which is homogeneous of degree 1 in dx. P. Finsler defined this geometry in his thesis in 1917. It was later developed by E. Cartan [3], Chern [4] and Bryant [1]. . . .

In [2] Cartan proposed another generalisation of Riemannian geom-

E-mail address: imsatfia@math.jussieu.fr

etry where the distance between two infinitesimally closed points in $\mathcal{M}$ depends on the point M and on the choice of a hyperplane in the tangent space to the manifold, this geometry was latter studied by R. Debever in [8, 7], after some years Kawaguchi and Davies introduce the notion of areal spaces in [10, 5], more recently, in 2010 by Morales and Vilches in [11]. In the modern language, this amounts to define a metric on the vector bundle over the Grassmannian bundle of oriented hyperplanes, $Gr_{n-1}(\mathcal{M})$ whose fiber at $\mathrm{M} \in E$ is the set of oriented hyperplanes in $T_{\mathrm{M}}\mathcal{M}$ (where $\mathrm{M} \in \mathcal{M}$ and E called *"element"* by Cartan, denotes an oriented hyperplane in $T_{\mathrm{M}}\mathcal{M}$). Moreover Cartan found a way to canonically derive such a metric from a variational problem on hypersurfaces in $\mathcal{M}$. He simultaneously defined a connection on this bundle, more general explained in [6, 12]. The first step consists in choosing a natural definition for the orthogonal complement of an element E and the metric in the normal direction: The idea is to require that, for any extremal hypersurface $\mathcal{H}$ of $\mathcal{M}$ and any compact subset with a smooth boundary $\Sigma \in \mathcal{H}$ if we perform a deformation of $\partial\Sigma$ in the *normal direction* to Σ and with an arbitrary intensity and consider the family of extremal hypersurfaces whose boundaries are the images of $\partial\Sigma$ by this deformations, then the area of hypersurfaces is stationary. This uses a formula of De Donder (which is basically an extension to variational problems with several variables of a basic formula in the theory of integral invariants). Let us now present this idea for submanifolds of arbitrary codimension $n-p$. Such variational problem can be described as follows. Let β be a p-form which, in local coordinates $x^1, \ldots, x^n$, reads $\beta = dx^1 \wedge \cdots \wedge dx^p$. Any p-dimensional oriented submanifold $\mathcal{N}$ such that $\beta|_{\mathcal{N}} > 0$ can be locally represented as the graph of a function $f = (f^1, \ldots, f^{n-p})$ of the variables $(x^1, \ldots, x^p)$. We consider functional $\mathcal{L}$ of the form $\mathcal{L}(f) := \int_{\Omega \subset T_x\mathcal{N}} d\sigma$, when

$$d\sigma = L\left(x^1, \ldots, x^p, f^1, ..f^{n-p}, \nabla f\right)\beta.$$

Let $\mathcal{N}$ be the critical point of $\mathcal{L}$. To define the orthogonal subspaces to all tangent subspaces to $\mathcal{N}$ the idea is to consider a 1-parameter family $(\mathcal{N}_t)_t$ of submanifolds which forms locally a foliation of $(p+1)$-dimensional submanifold U of $\mathbb{R}^n$ and such that $\mathcal{N}_0 = \mathcal{N}$. Consider a vector field X on U witch induces the variation from $\mathcal{N}_t$ to $\mathcal{N}_{t+dt}$ and denote

$$\mathcal{A}(t) = \mathcal{L}(f_t).$$

According to Cartan [2] the condition for X to be orthogonal to $\mathcal{N} = \mathcal{N}_0$ is that the derivative of $\mathcal{A}(t)$ with respect to t at $t = 0$ is zero. The definition of the orthogonality actually dose not depend on the choice of $\mathcal{N}$ but uniquely on $E \in Gr_p^{\mathrm{M}}\mathcal{M}$.

2. Cartan geometry based on the notion of area

Let $\mathcal{M}$ be a manifold of n-dimensional, then we define the *Grassmannian bundle* or *Grassmannian* by

$$Gr_p\mathcal{M} = \{(\mathrm{M}, E) | \mathrm{M} \in \mathcal{M}; E \text{ an oriented } p\text{-dimensional vector subspace in } T_{\mathrm{M}}\mathcal{M}\}.$$

If β is a p-form which in local coordinates $(x^1, \ldots, x^n)$, reads $\beta = dx^1 \wedge \cdots \wedge dx^p$ where $1 \leq p \leq n-1$, then

$$Gr_p^{\beta}\mathcal{M} = \{(\mathrm{M}, E) \in T_{\mathrm{M}}\mathcal{M} | \beta = dx^1 \wedge \cdots \wedge dx^p|_E > 0\}.$$

Let $(p^{\jmath})_{1 \leq \jmath \leq p(n-p)}$ be coordinate functions on $Gr_p^{\beta}\mathcal{M}$ such that $(x^{\imath}, p^{\jmath})$ are local coordinates on $Gr_p^{\beta}\mathcal{M}$. We denote the projection π by:

$$\pi : Gr_p^{\beta}\mathcal{M} \longrightarrow \mathcal{M},$$
$$(\mathrm{M}, E) \longmapsto \mathrm{M}.$$

We consider $\pi^* T\mathcal{M}$ the bundle over the Grassmannian whose fiber at (M, E) is $T_{\mathrm{M}}\mathcal{M}$, we denote a metric g on $\pi^* T\mathcal{M}$ by

$$g_{(\mathrm{M},E)} = g_{\imath\jmath}(x^k, p^k) dx^{\imath} dx^{\jmath}.$$

We see that the coefficients $g_{\imath\jmath}$ not only depend on coordinates of M, but they also depend on the orientation of the element at M.

Remark 2.1. If $p = n-1$, then

$$Gr_{n-1}(\mathcal{M}) \sim (T^*\mathcal{M} \setminus \{0\}) \diagup \mathbb{R}^*.$$

Definition 2.2. A *geometry based on the notion of area* $(\mathcal{M}, F)$ is a differential manifold $\mathcal{M}$ equipped with a function F defined over $T^*\mathcal{M}$ with values in $\mathbb{R}+$

$$F : T^*\mathcal{M} \to \mathbb{R}+,$$

which satisfies the following conditions:

1. F is $\mathcal{C}^{\infty}$ over $T^*\mathcal{M} \setminus \{0\} := \bigcup_{\mathrm{M} \in \mathcal{M}} T^*_{\mathrm{M}}\mathcal{M} \setminus \{0\}$.
2. F is homogeneous of degree one in p^k

$$F(x^k, \lambda p^k) = \lambda F(x^k, p^k).$$

3. The Hessian matrix defined by
$$(g_{\imath\jmath}) := \left[\frac{1}{2}(F^2)_{p^\imath p^\jmath}\right]$$
is positive definite at any point of $Gr_p(\mathcal{M})$.

In other words, $F\,|_{T^*_{\mathrm{M}}\mathcal{M}}$ is a Minkowski norm for all $\mathrm{M} \in \mathcal{M}$.

Remark 2.3. Catan's spaces are the dual of Finsler spaces under the Legendre transformation. Both are generalized by Kawaguchi by introducing the notion of AREAL SPACES [10].

3. The concept of orthogonality in Cartan's space

In the following, since we work locally we shall identify $\mathcal{M}$ with $\mathbb{R}^n$ to the coordinate system $(x^\imath)_\imath$.

3.1. Lagrangian formulation

Let $L : Gr_p^\beta(\mathbb{R}^p \times \mathbb{R}^{n-p}) := \{(x^1, \ldots, x^n, (p^\imath_\jmath)_{\substack{1\le\imath\le n-p\\1\le\jmath\le p}}) \longrightarrow \mathbb{R}\}$ be the Lagrangian function. For any function $f : \Omega \subset \mathbb{R}^p \to \mathbb{R}^{n-p}$ of class $\mathcal{C}^\infty$, we denote by Γ_f its graph. A point $x \in \Gamma_f$ is defined by $(x^{p+1}, \ldots, x^n) = (f^1(x^1, \ldots, x^p), \ldots, f^{n-p}(x^1, \ldots, x^p))$ and values of the coordinates $(p^\imath_\jmath)$ at the tangent space to Γ_f are given by $(\nabla f)(x)$. Let $\beta = dx^1 \wedge \cdots \wedge dx^p$ be a p-form, the action integral [9] is given by
$$\mathcal{L}(f) = \int_\Omega L(x^1, \ldots, x^p, f^1, \ldots, f^{n-p}, \nabla f)\beta = \int_\Omega L(x, f, \nabla f)\beta.$$
The bundle over the Grassmannian of Γ_f given by
$$Gr_p^\beta(\Gamma_f) := \{(x, E); x \in \Gamma_f, \quad E = T_x\Gamma_f\}.$$

Definition 3.1. Let Γ be an oriented p-dimensional submanifold of $\mathcal{M}$ with boundary Γ_0 which is a critical point of $\mathcal{L}$. A distribution of vector lines $\mathcal{D}$ in $T\mathcal{M}$ along Γ_0 is called *normal* if for any vector field N defined along Γ_0 such that $\forall \mathrm{M} \in \Gamma_0,\ N(\mathrm{M}) \in \mathcal{D}(\mathrm{M})$, and if
$$\partial\Gamma_t := \{e^{tN}(\mathrm{M}) | \mathrm{M} \in \partial\Gamma, t \in (-\varepsilon, \varepsilon)\}$$
and $\mathcal{A}(t) := \mathcal{L}(\Gamma_t)$, then $\frac{d}{dt}\left(\mathcal{A}(t)\right)|_{t=0} = 0$.

Theorem 3.2. *There exists a vector subbundle $\pi^*T^\perp\mathcal{M}$ of $\pi^*T\mathcal{M}$ of rank $n-p$ whose fiber at (x, E) is denoted by $(\pi^*T^\perp\mathcal{M})_{(x,E)}$ such that for* *any oriented p-dimensional critical point Γ of $\mathcal{L}$, a vector field N along $\partial\Gamma$ is normal if and only if $N_x \in (\pi^* T^\perp \mathcal{M})_{(x,T_x\Gamma)}$. In the following we write $(\pi^* T^\perp \mathcal{M})_{(x,T_x\Gamma)} = (T_x\Gamma)^\perp$. Moreover $(T_x\Gamma)^\perp$ is spanned by $(v^1, \dots v^{n-p})$, where*

$$v^1 = \begin{pmatrix} \frac{\partial L}{\partial p_1^1} \\ \vdots \\ \frac{\partial L}{\partial p_p^1} \\ -L + p_\jmath^1 \frac{\partial L}{\partial p_\jmath^1} \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad v^2 = \begin{pmatrix} \frac{\partial L}{\partial p_1^2} \\ \vdots \\ \frac{\partial L}{\partial p_p^2} \\ 0 \\ -L + p_\jmath^2 \frac{\partial L}{\partial p_\jmath^2} \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \dots,$$

$$v^{n-p} = \begin{pmatrix} \frac{\partial L}{\partial p_1^{n-p}} \\ \vdots \\ \frac{\partial L}{\partial p_p^{n-p}} \\ 0 \\ 0 \\ \vdots \\ 0 \\ -L + p_\jmath^{n-p} \frac{\partial L}{\partial p_\jmath^{n-p}} \end{pmatrix}.$$

Proof. *Consider first the case $p = 2$ and $n = 3$.*

The Grassmannian bundle is of dimension 5, the Lagrangian

$$(x, y, z, p, q) \mapsto L(x, y, z, p, q) := L(x, y, f(x, y), \nabla f(x, y)),$$

and the action integral is given by

$$\mathcal{L}(f) = \int_{\mathbb{R}^2} L(x, y, f(x, y), \nabla f(x, y))\beta.$$

Suppose that this integral is extended to a portion of extremal surface Σ limited by a contour $\mathcal{C}$, deform slightly Σ to a surface Σ' limited by a contour $\mathcal{C}'$. This amounts to change in the preceding integral f into $f + \varepsilon g$ where g has not necessarily a compact support. Then we consider a family $(\Sigma_t)_t$ of surfaces with boundary which forms locally a foliation of a domain $U \subset \mathbb{R}^3$ which coincides in $t = 0$ with Σ and in $t = 1$ with Σ', depending on a real parameter $t \in [0, 1]$. We suppose that for all t,

$(\Sigma_t)_t$ is a critical point of $\mathcal{L}$ that we will represent by the graph Σ_t of a function $f_t : \Omega_t \to \mathbb{R}$

$$\Sigma_t = \{(x, y, f_t(x, y)) \backslash (x, y) \in \Omega_t\}.$$

Let X be a vector field defined on U such that, if e^{sX} is the flow of X, then

$$e^{sX}(\Sigma_t) = \Sigma_{t+s}.$$

Note that

$$\begin{cases} f(t, x, y) = f_t(x, y), \\ \quad f(x, y) = f(0, x, y) = f_0(x, y), \\ \Phi(t, x, y) = e^{tX}(x, y, f(x, y)). \end{cases}$$

If $t = 0$ we have $\Phi = f = f_0$ and $\forall t \in]0, 1]$, the function $(x, y) \mapsto \Phi(t, x, y)$ is a parametrization of Σ_t, we denote by $\Phi = (\phi^1, \phi^2, \phi^3)$ and $\phi^3(t, x, y) = f(t, \phi^1(t, x, y), \phi^2(t, x, y))$ so, if we derive with respect to t, then

$$\frac{\partial \phi^3}{\partial t} = \frac{\partial f}{\partial t}(t, \phi^1, \phi^2) + \frac{\partial f}{\partial x}(t, \phi^1, \phi^2)\frac{\partial \phi^1}{\partial t} + \frac{\partial f}{\partial y}(t, \phi^1, \phi^2)\frac{\partial \phi^2}{\partial t}.$$

which gives for $t = 0$

$$X^3(x, y, f) = \frac{\partial f}{\partial t}(0, x, y) + \frac{\partial f}{\partial x}(0, x, y)X^1(x, y, f) + \frac{\partial f}{\partial y}(0, x, y)X^2(x, y, f).$$

Thus along $\Sigma = \Sigma_0$, we have:

$$\frac{\partial f}{\partial t} = X^3 - X^1\frac{\partial f}{\partial x} - X^2\frac{\partial f}{\partial y}. \tag{1}$$

Let the Lagrangian $(x, y, z, p, q) \mapsto L(x, y, z, p, q)$ and we consider

$$\mathcal{A}(t) = \int_{\Omega_t} L(x, y, f_t(x, y), \frac{\partial f_t}{\partial x}(x, y), \frac{\partial f_t}{\partial y}(x, y))dxdy.$$

Assuming that Ω_t is regular (i.e., $\partial\Omega_t$ is a curve $\mathcal{C}^1$ of plan $\mathbb{R}^2$), then we have

$$\frac{d\mathcal{A}(t)}{dt} = \int_{\Omega_t} \frac{\partial}{\partial t} L\left(x, y, f_t, \frac{\partial f_t}{\partial x}, \frac{\partial f_t}{\partial y}\right) dxdy + \\ + \int_{\partial\Omega_t} L\left(x, y, f_t, \frac{\partial f_t}{\partial x}, \frac{\partial f_t}{\partial y}\right) \langle (X^1, X^2), \nu\rangle d\ell,$$

where ν is a exterior normal of Ω_t in $\mathbb{R}^2$ and $\langle (X^1, X^2), \nu\rangle$ is the horizontal change in the area of Ω_t and $d\ell$ is a measure of one dimension $\partial\Omega$, hence

$$\begin{aligned}
\frac{d\mathcal{A}(t)}{dt}\ |_{t=0} &= \int_{\Omega_0} \frac{\partial}{\partial t} L\left(x,y,f_t,\frac{\partial f_t}{\partial x},\frac{\partial f_t}{\partial y}\right)dxdy+\\
&+\int_{\partial\Omega_0} L\left(x,y,f_t,\frac{\partial f_t}{\partial x},\frac{\partial f_t}{\partial y}\right)\langle (X^1,X^2),\nu\rangle d\ell =\\
&=\int_\Omega \frac{\partial L}{\partial z}\frac{\partial f}{\partial t}+\frac{\partial L}{\partial p}\frac{\partial^2 f}{\partial x\partial t}+\frac{\partial L}{\partial q}\frac{\partial^2 f}{\partial y\partial t}+\int_{\partial\Omega} L\langle (X^1,X^2),\nu\rangle d\ell =\\
&=\int_\Omega \frac{\partial L}{\partial z}\frac{\partial f}{\partial t}+\frac{\partial}{\partial x}\left(\frac{\partial L}{\partial p}\frac{\partial f}{\partial t}\right)+\frac{\partial}{\partial y}\left(\frac{\partial L}{\partial q}\frac{\partial f}{\partial t}\right)-\\
&-\frac{\partial f}{\partial t}\left[\frac{\partial}{\partial x}\left(\frac{\partial L}{\partial p}\right)+\frac{\partial}{\partial y}\left(\frac{\partial L}{\partial q}\right)\right]+\\
&+\int_{\partial\Omega} L\langle (X^1,X^2),\nu\rangle d\ell =\\
&=\int_\Omega \frac{\partial f}{\partial t}\left[\frac{\partial L}{\partial z}-\frac{\partial}{\partial x}\left(\frac{\partial L}{\partial p}\right)-\frac{\partial}{\partial y}\left(\frac{\partial L}{\partial q}\right)\right]+\\
&+\int_{\partial\Omega}\left\langle\left(\frac{\partial L}{\partial p}\frac{\partial f}{\partial t},\frac{\partial L}{\partial q}\frac{\partial f}{\partial t}\right),\nu\right\rangle d\ell+\int_{\partial\Omega} L\langle (X^1,X^2),\nu\rangle d\ell =\\
&=\int_\Omega \frac{\partial f}{\partial t}\left[\frac{\partial L}{\partial z}-\frac{\partial}{\partial x}\left(\frac{\partial L}{\partial p}\right)-\frac{\partial}{\partial y}\left(\frac{\partial L}{\partial q}\right)\right]+\\
&+\int_{\partial\Omega}\left\langle\left(\frac{\partial L}{\partial p}\frac{\partial f}{\partial t}+LX^1,\frac{\partial L}{\partial q}\frac{\partial f}{\partial t}+LX^2\right),\nu\right\rangle d\ell.
\end{aligned}$$

But we know that $\Sigma=\Sigma_0$ is a critical point of $\int_\Omega L$, then the Euler–Lagrange equations are satisfied, thus

$$\frac{d\mathcal{A}(t)}{dt}\ |_{t=0}=\int_{\partial\Omega}\left\langle\left(\frac{\partial L}{\partial p}\frac{\partial f}{\partial t}+LX^1,\frac{\partial L}{\partial q}\frac{\partial f}{\partial t}+LX^2\right),\nu\right\rangle dl.$$

We now assume that $X|_{\partial\Gamma}$ has the form $\psi N_0\in\mathcal{D}$ where $\psi\in\mathcal{C}^\infty(\partial\Gamma)$ with values in $\mathbb{R}$ and where N_0 is a fixed non-vanishing tangent defined along $\partial\Gamma$, to be determined we seek a condition to N_0 such that for any regular function ψ, $\frac{d\mathcal{A}(t)}{dt}\ |_{t=0}=0$. We can choose a function f_t depends on ψ such that $\frac{\partial f_{t,\psi}}{\partial t}\ |_{t=0}=\psi\frac{\partial f_t}{\partial t}\ |_{t=0}$, thus

$$\frac{d\mathcal{A}(t)}{dt}\ |_{t=0}=\int_{\partial\Gamma}\left\langle\left(\frac{\partial L}{\partial p}\psi\frac{\partial f}{\partial t}+\psi LN_0^1,\psi\frac{\partial L}{\partial q}\frac{\partial f}{\partial t}+\psi LN_0^2\right),\nu\right\rangle dl=$$

$$= \int_{\partial\Gamma} \psi \left\langle \left(\frac{\partial L}{\partial p}\frac{\partial f}{\partial t} + LN_0^1, \frac{\partial L}{\partial q}\frac{\partial f}{\partial t} + LN_0^2 \right), \nu \right\rangle dl.$$

The condition for $\frac{d\mathcal{A}(t)}{dt} \mid_{t=0}= 0$ for all ψ regular function on $\partial\Gamma$ and ν exterior normal of Γ is that $\left\langle \left(\frac{\partial L}{\partial p}\frac{\partial f}{\partial t} + LN_0^1, \frac{\partial L}{\partial q}\frac{\partial f}{\partial t} + LN_0^2 \right), \nu \right\rangle = 0$. If we denote by $\lambda = \frac{-\frac{\partial f}{\partial t}}{L}$, then

$$\begin{cases} N_0^1 = \lambda \frac{\partial L}{\partial p}, \\ N_0^2 = \lambda \frac{\partial L}{\partial q}. \end{cases}$$

From (1), we have

$$N_0^3 = -\lambda L + \lambda \frac{\partial L}{\partial p}\frac{\partial f}{\partial x} + \lambda \frac{\partial L}{\partial q}\frac{\partial f}{\partial y}.$$

Hence,

$$X = \left(\frac{\partial L}{\partial p}, \frac{\partial L}{\partial q}, p\frac{\partial L}{\partial p} + q\frac{\partial L}{\partial q} - L \right).$$

Let now $n > 0$ ***and*** $p = n-1$. The Grassmannian bundle is of dimension $2n-1$. By same as previous, thus the orthogonal of $\pi^*T\mathcal{M}$ of rank $n-1$ is spanned by

$$X = \left(\frac{\partial L}{\partial p^1}, \ldots, \frac{\partial L}{\partial p^{n-1}}, \sum_{\imath=1}^{n-1} p^\imath \frac{\partial L}{\partial p^\imath} - L \right), \tag{2}$$

where $p^\imath = \frac{\partial f}{\partial x_\imath}$ for $\imath = 1, \ldots n-1$, and L be the Lagrangian on $Gr_{n-1}(\Sigma)$.

Now the case $n > 3$ ***and*** $p < n$. For $1 \le p \le n-1$ let Ω be a regular open set of $\mathbb{R}^p$ and $f = (f^1, \ldots f^{n-p}) : \Omega \longrightarrow \mathbb{R}^{n-p}$, we denote its graph by

$$\mathcal{N} := \{(x, f(x)) \mid x \in \Omega\}.$$

Let $\beta = dx^1 \wedge \cdots \wedge dx^p$ be a p-form, and L be the Lagrangian on $Gr_p^\beta(\mathcal{N})$, thus for an open set $\Omega \in T_x\mathcal{N}$ the action integral is given by

$$\mathcal{L}(f) := \int_\Omega L(x^1, \ldots, x^p, f^1, \ldots, f^{n-p}, \nabla f)\beta.$$

The family $(\mathcal{N}_t)_t$ of submanifolds with boundary forms locally a foliation in a $(p+1)$-dimensional submanifold U of $\mathbb{R}^n$, we suppose that for all t, $\mathcal{N}_t$ is a critical point of $\mathcal{L}$. Let X be a vector field defined on U such that, if e^{sX} is the flow of X, then $e^{sX}(\mathcal{N}_t) = \mathcal{N}_{t+s}$, denote

$$\begin{cases} f(t,x^1,\ldots,x^p)=f_t(x^1,\ldots x^p)\Leftrightarrow f^\imath(t,x^1,\ldots,x^p)=(f^\imath)_t(x^1,\ldots x^p), \\ \qquad\qquad \forall\imath=1,\ldots n-p, \\ \\ f(x^1,\ldots,x^p)=f(0,x^1,\ldots,x^p)=f_0(x^1,\ldots x^p)\Leftrightarrow\forall\imath=1,\ldots n-p \\ \qquad\qquad \text{we have} \\ f^\imath(x^1,\ldots,x^p)=f^\imath(0,x^1,\ldots,x^p)=(f^\imath)_0(x^1,\ldots x^p), \\ \\ \Phi(t,x^1,\ldots,x^p)=e^{tX}(x^1,\ldots,x^p,f^1,\ldots,f^{n-p}). \end{cases}$$

The function Φ is a parametrization of $\mathcal{N}_t$, we denote:

$$\begin{cases} \Phi=(\varphi^1,\ldots,\varphi^p,\varphi^{p+1},\ldots,\varphi^n), \\ \varphi^{p+\imath}(t,x^1,\ldots,x^p)=f_t^\imath(\varphi^1,\ldots,\varphi^p) \text{ for } \imath=1,\ldots n-p, \end{cases}$$

where f_t is defined on a domain $\Omega_t\subset\mathbb{R}^p$. Thus, $\forall\imath=1,\ldots n-p$:

$$\frac{\partial\varphi^{p+\imath}}{\partial t}=\frac{\partial f^\imath}{\partial t}(t,\varphi^1,\ldots,\varphi^p)+\sum_{\jmath=1}^{p}\frac{\partial f^\imath}{\partial x^\jmath}(t,\varphi^1,\ldots,\varphi^p)\frac{\partial\varphi^\jmath}{\partial t}.$$

For $t=0$, $\forall\imath=1,\ldots n-p$, thus

$$X^{p+\imath}(x,f)=\frac{\partial f^\imath}{\partial t}(0,x^1,\ldots,x^p)+\sum_{\jmath=1}^{p}\frac{\partial f^\imath}{\partial x^\jmath}(0,x^1,\ldots,x^p)X^\jmath(x,f),$$

which gives along $\mathcal{N}=\mathcal{N}_0$ and $\forall\imath=1,\ldots n-p$

$$\frac{\partial f^\imath}{\partial t}=X^{p+\imath}-\sum_{\jmath=1}^{p}\frac{\partial f^\imath}{\partial x^\jmath}X^\jmath. \tag{3}$$

We have

$$\mathcal{A}(t)=\mathcal{L}(f_t)=\int_{\Omega_t}L(x^1,\ldots,x^p,(f^1)_t,\ldots,(f^{n-p})_t,\nabla f_t)\beta,$$

thus

$$\frac{d\mathcal{A}(t)}{\partial t}=\int_{\Omega_t}\frac{\partial}{\partial t}L(x^1,\ldots,x^p,(f^1)_t,\ldots,(f^{n-p})_t,\nabla f_t)\beta+$$
$$+\int_{\partial\Omega_t}L(x^1,\ldots,x^p,(f^1)_t,\ldots,(f^{n-p})_t,\nabla f_t)\langle(X^1,\ldots,X^p),\nu\rangle d\ell,$$

where ν is the exterior normal to Ω_t in $\mathbb{R}^p$, $\langle(X^1,\ldots,X^p),\nu\rangle$ represents the horizontal change in volume of Ω_t and $d\ell$ is a measure of $p-1$ dimension $\partial\Omega_t$. Thus for $t=0$ we have

$$\frac{d\mathcal{A}(t)}{\partial t}\ |_{t=0}\ = \int_{\Omega_t} \frac{\partial}{\partial t} L(x^1, \dots, x^p, (f^1)_t, \dots, (f^{n-p})_t, \nabla f_t)\beta +$$
$$+ \int_{\partial\Omega_t} L\langle (X^1, \dots, X^p), \nu \rangle d\ell.$$

We calculate $\frac{d\mathcal{A}(t)}{\partial t}\ |_{t=0}$

$$\int_\Omega \frac{\partial}{\partial t} L(x, f_t, \nabla f_t)\beta =$$
$$= \int_\Omega \left(\sum_{\imath=1}^{n-p} \frac{\partial L}{\partial x^\imath}\frac{\partial f^\imath}{\partial t} + \sum_{\substack{1\le\jmath\le p\\ 1<\imath\le n-p}} \frac{\partial F}{\partial p^\imath_\jmath}\frac{\partial^2 f^\imath}{\partial x^\jmath \partial t} \right)\beta =$$
$$= \sum_{\imath=1}^{n-p}\int_\Omega \left(\frac{\partial L}{\partial x^\imath}\frac{\partial f^\imath}{\partial t} + \sum_{\jmath=1}^{p} \frac{\partial}{\partial x^\jmath}\left(\frac{\partial L}{\partial p^\imath_\jmath}\frac{\partial f^\imath}{\partial t}\right) - \frac{\partial f^\imath}{\partial t}\sum_{\jmath=1}^{p}\frac{\partial}{\partial x^\jmath}\left(\frac{\partial L}{\partial p^\imath_\jmath}\right)\right)\beta =$$
$$= \sum_{\imath=1}^{n-p}\int_\Omega \frac{\partial f^\imath}{\partial t}\left[\frac{\partial L}{\partial x^\imath} - \sum_{\jmath=1}^{p}\frac{\partial}{\partial x^\jmath}\left(\frac{\partial L}{\partial p^\imath_\jmath}\right)\right]\beta +$$
$$+ \sum_{\imath=1}^{n-p}\int_{\partial\Omega}\left\langle \left(\frac{\partial L}{\partial p^\imath_1}\frac{\partial f^\imath}{\partial t}, \dots, \frac{\partial L}{\partial p^\imath_p}\frac{\partial f^\imath}{\partial t}\right), \nu\right\rangle d\ell.$$

We have that $\mathcal{N} = \mathcal{N}_0$ is a critical point of $\mathcal{L}$, thus the Euler–Lagrange equations are satisfied $\frac{\partial L}{\partial x^\imath} - \sum_{\jmath=1}^{p}\frac{\partial}{\partial x^\jmath}\left(\frac{\partial L}{\partial p^\imath_\jmath}\right) = 0$ which gives

$$\frac{d\mathcal{A}(t)}{\partial t}\ |_{t=0} = \sum_{\imath=1}^{n-p}\int_{\partial\Omega}\left\langle\left(\frac{\partial L}{\partial p^\imath_1}\frac{\partial f^\imath}{\partial t} + LX^1, \dots, \frac{\partial L}{\partial p^\imath_p}\frac{\partial f^\imath}{\partial t} + LX^p\right), \nu\right\rangle d\ell.$$

Using Def. 3.1 we can consider a regular function ψ changing X in ψX, where $\psi : \partial\mathcal{N} \to \mathbb{R}$ and hence $\frac{\partial f^\imath_{t,\psi}}{\partial t} = \psi \frac{\partial f^\imath_t}{\partial t}$. By the same as previous, so that $\frac{d\mathcal{A}(t)}{\partial t}\ |_{t=0} = 0$, it suffices that for all $\jmath = 1, \dots p$ we have

$$\sum_{\imath=1}^{n-p}\frac{\partial L}{\partial p^\imath_\jmath}\frac{\partial f^\imath}{\partial t} + LX^\jmath = 0$$

if we denote by $\lambda_\imath = \frac{-\frac{\partial f^\imath}{\partial t}}{L}$ and $\nabla f := \left(\frac{\partial f^\imath}{\partial x^\jmath}\right)_{\substack{1<\imath\le n-p\\ 1\le\jmath\le p}} = (p^\imath_\jmath)_{\substack{1<\imath\le n-p\\ 1\le\jmath\le p}}$, then

$$X^\jmath = \sum_{\imath=1}^{n-p}\lambda_\imath \frac{\partial L}{\partial p^\imath_\jmath} \qquad \text{for all } \jmath = 1, \dots p,$$

from (3), for $\imath = 1, \dots n-p$ thus

$$X^{p+\imath} = -\lambda_\imath L + \sum_{\jmath=1}^{p} \lambda_\imath p^\imath_\jmath \frac{\partial L}{\partial p^\imath_\jmath},$$

which gives

$$\left\{\begin{aligned} X^1 &= \lambda_1 \frac{\partial L}{\partial p^1_1} + \lambda_2 \frac{\partial L}{\partial p^2_1} + \cdots + \lambda_{n-p} \frac{\partial L}{\partial p^{n-p}_1} \\ &\vdots \\ X^p &= \lambda_1 \frac{\partial L}{\partial p^1_p} + \lambda_2 \frac{\partial L}{\partial p^2_p} + \cdots + \lambda_{n-p} \frac{\partial L}{\partial p^{n-p}_p} \\ X^{p+1} &= \lambda_1 \left(-L + p^1_1 \frac{\partial L}{\partial p^1_1} + \cdots + p^1_p \frac{\partial L}{\partial p^1_p}\right) \\ &\vdots \\ X^n &= \lambda_{n-p} \left(-L + p^{n-p}_1 \frac{\partial L}{\partial p^{n-p}_1 1} + \cdots + p^{n-p}_p \frac{\partial L}{\partial p^{n-p}_p}\right) \end{aligned}\right. =$$

$$= \lambda_1 \begin{pmatrix} \frac{\partial L}{\partial p^1_1} \\ \vdots \\ \frac{\partial L}{\partial p^1_p} \\ -L + p^1_\jmath \frac{\partial L}{\partial p^1_\jmath} \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} \frac{\partial L}{\partial p^2_1} \\ \vdots \\ \frac{\partial L}{\partial p^2_p} \\ 0 \\ -L + p^2_\jmath \frac{\partial L}{\partial p^2_\jmath} \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \cdots$$

$$\cdots + \lambda_{n-p} \begin{pmatrix} \frac{\partial L}{\partial p^{n-p}_1} \\ \vdots \\ \frac{\partial L}{\partial p^{n-p}_p} \\ 0 \\ 0 \\ \vdots \\ 0 \\ -L + p^{n-p}_\jmath \frac{\partial L}{\partial p^{n-p}_\jmath} \end{pmatrix}$$

$$= \lambda_1 v^1 + \lambda_2 v^2 + \cdots + \lambda_{n-p} v^{n-p},$$

so the theorem is proved. $\Diamond$

Example 3.3. We take $n = 4$ and $p = 2$, then $f : \mathbb{R}^2 \to \mathbb{R}^2$, Σ_t are a domains with boundary of dimension 2 in $\mathbb{R}^4$, we define the functional area by

$$L(x, y, f^1, f^2, p_1^1, p_2^1, p_1^2, p_2^2) :=$$
$$:= \sqrt{1 + (p_1^1)^2 + (p_2^1)^2 + (p_1^2)^2 + (p_2^2)^2 + (p_1^1 p_2^2 - p_2^1 p_1^2)^2},$$

hence the normal subspace to $\mathcal{N}_x$ is $\mathcal{V} = (v^1, v^2)$ with

$$\mathcal{V} = \frac{1}{L}\left(\begin{pmatrix} p_1^1 + p_2^2(p_1^1 p_2^2 - p_2^1 p_1^2) \\ p_2^1 - p_1^2(p_1^1 p_2^2 - p_2^1 p_1^2) \\ -1 - (p_2^2)^2 - (p_1^2)^2 \\ 0 \end{pmatrix}, \begin{pmatrix} p_1^2 - p_2^1(p_1^1 p_2^2 - p_2^1 p_1^2) \\ p_2^2 + p_1^1(p_1^1 p_2^2 - p_2^1 p_1^2) \\ 0 \\ -1 - (p_1^1)^2 - (p_2^1)^2 \end{pmatrix}\right).$$

Remark 3.4. In Euclidean case, the subspace orthogonal to the tangent space to Σ_{M} is spanned by

$$(T\Sigma_{\text{M}})^{\perp} = \left\langle \begin{pmatrix} p_1^1 \\ p_2^1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} p_1^2 \\ p_2^2 \\ 0 \\ -1 \end{pmatrix} \right\rangle,$$

which coincides with our result when

$$\begin{pmatrix} p_1^1 & p_2^1 \\ p_1^2 & p_2^2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

4. Determination of the normal unit vector to a hypersurface

Definition 4.1. We denote by $(e_1^*, \dots, e_n^*)$ the dual basis of a vector space E of n dimension. We consider that $1 \le \imath_1 < \cdots < \imath_p \le n$. Note $\xi_\jmath = (\xi_\jmath^{\imath_k})$, we have

$$e_{\imath_1}^* \wedge \cdots \wedge e_{\imath_p}^*(\xi_1, \xi_2, \dots, \xi_p) = \begin{vmatrix} \xi_1^{\imath_1} & \dots & \xi_p^{\imath_1} \\ \vdots & \ddots & \vdots \\ \xi_1^{\imath_p} & \dots & \xi_p^{\imath_p} \end{vmatrix}.$$

Theorem 4.2. *The length ℓ of the normal vector v to the hypersurface Σ is given by*

$$\sqrt{g}.$$

Proof. We recall that locally the tangent space of hypersurface Σ_{M} generated by $n-1$ vectors $p_\imath = (p_\imath^1, \dots, p_\imath^n)$ for $\imath = 1, \dots, n-1$. Note by $d\sigma$ the volume of the parallelepiped of n dimension spanned by $p_\imath = (p_\imath^1, \dots, p_\imath^n)$ and v. We have $\mathcal{V} = \ell d\sigma$, we introduce the variables $\xi_1, \dots, \xi_n$ such that $\xi_n = -\frac{\xi_1}{p_1} = \cdots = -\frac{\xi_{n-1}}{p_{n-1}}$. We define a function F by

$$F(x^1, \dots, x^n; \xi_1, \dots \xi_n) = \xi_n L\left(x^1, \dots, x^n; \frac{\xi_1}{\xi_n}, \dots, -\frac{\xi_{n-1}}{\xi_n}\right).$$

F is homogeneous of degree 1 in $\xi_\imath$, then

$$v = \left(\frac{\partial F}{\partial \xi_1}, \dots, \frac{\partial F}{\partial \xi_n}\right). \tag{4}$$

But we know that

$$\mathcal{V} = \sqrt{g}\begin{vmatrix} \frac{\partial F}{\partial \xi_1} & \cdots & \frac{\partial F}{\partial \xi_n} \\ \xi_1^1 & \cdots & \xi_1^n \\ \vdots & \ddots & \vdots \\ \xi_{n-1}^1 & \cdots & \xi_{n-1}^n \end{vmatrix}$$

and $d\sigma = \sum_{\imath=1}^n (-1)^{\imath-1} \frac{\partial F}{\partial \xi_\imath} e_1^* \wedge \cdots \wedge e_{\imath-1}^* \wedge e_{\imath+1}^* \cdots \wedge e_n^*$. Now it remains to calculate

$$d\sigma(\xi_1, ..., \xi_{n-1}) = \sum_{\imath=1}^n \frac{\partial F}{\partial \xi_\imath} e_1^* \wedge \cdots \wedge e_{\imath-1}^* \wedge e_{\imath+1}^* \cdots \wedge e_n^*(\xi_1, \dots, \xi_{n-1}) =$$

$$= \sum_{\imath=1}^n (-1)^{\imath-1} \frac{\partial F}{\partial \xi_\imath} \begin{vmatrix} \xi_1^1 & \cdots & \xi_{n-1}^1 \\ \vdots & \ddots & \vdots \\ \xi_1^{\imath-1} & \cdots & \xi_{n-1}^{\imath-1} \\ \xi_1^{\imath+1} & \cdots & \xi_{n-1}^{\imath+1} \\ \vdots & \ddots & \vdots \\ \xi_1^n & \cdots & \xi_{n-1}^n \end{vmatrix} = \begin{vmatrix} \frac{\partial F}{\partial \xi_1} & \cdots & \frac{\partial F}{\partial \xi_n} \\ \xi_1^1 & \cdots & \xi_1^n \\ \vdots & \ddots & \vdots \\ \xi_{n-1}^1 & \cdots & \xi_{n-1}^n \end{vmatrix}$$

which gives $\ell = \sqrt{g}$. $\Diamond$

Consequence 4.3. *The components of* ν *on the dual basis are:*

$$\sqrt{g}\left(\frac{\xi_1}{F}, \dots, \frac{\xi_n}{F}\right).$$

Proof. Denote respectively $\ell^\imath$ and $\ell_\imath$ the components of ν in the basis and in the dual basis, then by using (4) we have $\ell^\imath = \frac{1}{\sqrt{g}} \frac{\partial F}{\partial \xi_\imath}$. We recall that ν is normal hence $\ell^\imath \ell_\imath = 1$. Since F is homogeneous of degree one in $\xi_\imath$, then

$$\frac{1}{\sqrt{g}} \frac{\partial F}{\partial \xi_\imath} \xi_\imath = \frac{1}{\sqrt{g}} F \Rightarrow \frac{1}{\sqrt{g}} \frac{\partial F}{\partial \xi_\imath} \sqrt{g} \frac{\xi_\imath}{F} = 1$$

which gives the normal component of unit vector in the dual basis. ♢

Acknowledgement. I am grateful to Frédéric Hélein who suggested to me this problem as a part of my Ph.D.

References

[1] BRYANT, R. L.: Some remarks on Finsler manifolds with constant flag curvature, *Houston J. Math.* **28** (2) (2002), 221–262. Special issue for S. S. Chern.

[2] CARTAN, E.: *Les espaces métriques fondés sur la notion d'aire*, Hermann, 6, rue de la Sorbonne, Paris, 1933.

[3] CARTAN, E.: *Les espaces de Finsler*, Hermann, 6, rue de la Sorbonne, 1934.

[4] CHERN, SHIING SHEN: On Finsler geometry, *C. R. Acad. Sci. Paris Sér. I Math.* **314** (10) (1992), 757–761.

[5] DAVIES, E. T.: Areal spaces, *Ann. Mat. Pura Appl. (4)* **55** (1961), 63–76.

[6] DAVIES, E. T.: On the notion of Euclidean connection in areal spaces, *Tensor (N.S.)* **24** (1972), 53–59. Commemoration volumes for Prof. Dr. Akitsugu Kawaguchi's seventieth birthday, Vol. I.

[7] DEBEVER, ROBERT: *Une classe d'espaces à connexion euclidienne*, PhD thesis, La faculté des sciences de l'Université libre de Bruxelles.

[8] DEBEVER, ROBERT: Sur une classe de formes quadratiques extérieures et la géométrie fondée sur la notion d'aire, *C. R. Acad. Sci. Paris* **224** (1947), 1269–1271.

[9] HÉLEIN, FRÉDÉRIC: Hamiltonian formalisms for multidimensional calculus of variations and perturbation theory, in: *Non-compact problems at the intersection of geometry, analysis, and topology*, Contemp. Math. 350, pp. 127–147, Amer. Math. Soc., Providence, RI, 2004.

[10] KAWAGUCHI, A.: On areal spaces, I. Metric tensors in n-dimentional spaces based on the notion of two-dimentional area, *Tensor N.S.*, **1** (1950), 14–45.

[11] MORALES, C. and VILCHES, M.: On 2-Riemannian manifolds, *SUT J. Math.* **46** (1) (2010), 119–153.

[12] TAMÁSSY, LAJOS and KOZMA, LÁSZLÓ: Connections in k-vector bundles, *Math. Pannon.* **6** (1) (1995), 105–114.