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CRELLE–BROCARD POINTS OF THE TRIANGLE IN AN ISOTROPIC PLANE

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Abstract: In this paper the concept of Crelle–Brocard points of the triangle in an isotropic plane is defined. A number of statements about the relationship between Crelle–Brocard points and some other significant elements of a triangle in an isotropic plane are also proved. Some analogies with the Euclidean case are considered as well.

The isotropic (Galilean) plane (see [12] and [13]) is defined as a projective–metric plane with an absolute which consists of a line, absolute line ω and a point on that line, absolute point Ω. The lines through the

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point Ω are isotropic lines, and the points on the line ω are isotropic points (the points at infinity). Two lines through the same isotropic point are parallel, and two points on the same isotropic line are parallel points.

In an isotropic plane the *distance* between two non–parallel points $T_i = (x_i, y_i)$ $(i = 1, 2)$ is defined by $T_1T_2 = x_2 - x_1$. For two non-isotropic lines $y = k_1x + l_1$ and $y = k_2x + l_2$ the isotropic angle is defined by $k_2 - k_1$.

In [7] it is shown that each allowable triangle in an isotropic plane, if none of its sides is isotropic, can be set, by a suitable choice of the coordinates, in the so called *standard position*, where its vertices are of the form $A = (a, a^2)$, $B = (b, b^2)$, $C = (c, c^2)$ while $a + b + c = 0$.

With the labels $p = abc$, $q = bc + ca + ab$, $p_1 = \frac{1}{3}(bc^2 + ca^2 + ab^2)$, $p_2 = \frac{1}{3}(b^2c + c^2a + a^2b)$,

$$\omega = -\frac{1}{3q}(b-c)(c-a)(a-b) \tag{1}$$

it can be shown that the following equalities

$$\begin{aligned}
(b-c)^2 &= -(q+3bc), & (c-a)^2 &= -(q+3ca),\\
(a-b)^2 &= -(q+3ab), & (c-a)(a-b) &= 2q-3bc,\\
(a-b)(b-c) &= 2q-3ca, & (b-c)(c-a) &= 2q-3ab,\\
a^2+b^2+c^2 &= -2q, & q &= bc-a^2 = ca-b^2 = ab-c^2,\\
b^2+bc+c^2 &= -q, & a^2+ab+b^2 &= -q,\\
c^2+ca+a^2 &= -q, & p+p_1+p_2 &= 0,\\
p_1p_2 &= p^2+\frac{1}{9}q^3, & p_1^2+p_1p_2+p_2^2 &= p^2+pp_1+p_1^2 = -\frac{q^3}{9},
\end{aligned}$$

$$9q^2\omega^2 = (b-c)^2(c-a)^2(a-b)^2 = -(27p^2+4q^3)$$

are valid (see [7]).

To prove the geometric facts for each allowable triangle it is sufficient to give a proof for a standard triangle.

Theorem 1. *Let $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, respectively $\mathcal{A}'$, $\mathcal{B}'$, $\mathcal{C}'$, be the lines through the points A, B, C such that*

$$\angle(AB,\mathcal{A}) = \angle(BC,\mathcal{B}) = \angle(CA,\mathcal{C}) = \varphi,$$
$$\angle(\mathcal{A}',AC) = \angle(\mathcal{B}',BA) = \angle(\mathcal{C}',CB) = \psi.$$

If the triangle ABC is in a standard position, then the lines $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ pass through one point Ω_1, if and only if, $\varphi = \omega$, and the lines $\mathcal{A}'$, $\mathcal{B}'$, $\mathcal{C}'$ pass through one point Ω_2, if and only if, $\psi = \omega$ where ω is given by the formula (1).

The statement of this theorem, without the formula (1), has been proved by Tölke in [15].

Proof. The line BC has the equation $y = -ax - bc$, and then from $\angle(BC, \mathcal{B}) = \varphi$ and $\angle(BC, \mathcal{C}') = -\psi$ we get the equations of the lines $\mathcal{B}$ and $\mathcal{C}'$

$$y = (\varphi - a)(x - b) + b^2 \qquad \text{and} \qquad y = -(\psi + a)(x - c) + c^2,$$

i.e., owing to $a + b + c = 0$ we obtain the equations

$$y = (\varphi - a)x - bc - b\varphi, \qquad y = -(\psi + a)x - bc + c\psi.$$

Analogously the lines $\mathcal{C}$ and $\mathcal{A}'$, and lines $\mathcal{A}$ and $\mathcal{B}'$ have the equations

$$y = (\varphi - b)x - ca - c\varphi, \qquad y = -(\psi + b)x - ca + a\psi,$$

$$y = (\varphi - c)x - ab - a\varphi, \qquad y = -(\psi + c)x - ab + b\psi.$$

The lines $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ pass through one point provided that

$$0 = \begin{vmatrix} 1 & \varphi - a & b\varphi + bc \\ 1 & \varphi - b & c\varphi + ca \\ 1 & \varphi - c & a\varphi + ab \end{vmatrix} =$$

$$= \varphi(a^2 + b^2 + c^2 - bc - ca - ab) + a^2(b - c) + b^2(c - a) + c^2(a - b) =$$

$$= -3q\varphi - (b - c)(c - a)(a - b) = -3q\varphi + 3q\omega$$

and the lines $\mathcal{A}'$, $\mathcal{B}'$, $\mathcal{C}'$ pass through one point under condition

$$0 = \begin{vmatrix} 1 & \psi + a & c\psi - bc \\ 1 & \psi + b & a\psi - ca \\ 1 & \psi + c & b\psi - ab \end{vmatrix} =$$

$$= \psi(a^2 + b^2 + c^2 - bc - ca - ab) + a^2(b - c) + b^2(c - a) + c^2(a - b) =$$

$$= -3q\psi + 3q\omega,$$

i.e., if and only if, $\varphi = \omega$ respectively $\psi = \omega$. $\Diamond$

The angle ω from Th. 1 will be called, by the analogy with the Euclidean case, *Brocard angle* of the triangle ABC, and the points Ω_1 and Ω_2 *Crelle–Brocard points* of that triangle (shortly *CB–points*).

The equality (1) can also be written in the form $-3q\omega = 3p_1 - 3p_2$, i.e.

$$\omega = \frac{p_2 - p_1}{q}.$$

Corollary 1. *If Ω_1 and Ω_2 are CB–points of the standard triangle ABC, then the lines $A\Omega_1$, $B\Omega_1$, $C\Omega_1$ and the lines $A\Omega_2$, $B\Omega_2$, $C\Omega_2$ have successively the equations*

$$y = (\omega - c)x - a\omega - ab, y = (\omega - a)x - b\omega - bc, y = (\omega - b)x - c\omega - ca, \tag{2}$$

$$y = -(\omega + b)x + a\omega - ca, y = -(\omega + c)x + b\omega - ab, y = -(\omega + a)x + c\omega - bc, \tag{3}$$

where ω is the Brocard angle of that triangle.

Corollary 2. *If Ω_1 and Ω_2 are CB–points of the triangle ABC and ω the Brocard angle of that triangle then the following equalities are valid*

$$\angle(AB, A\Omega_1) = \angle(BC, B\Omega_1) = \angle(CA, C\Omega_1) = \omega, \tag{4}$$

$$\angle(A\Omega_2, AC) = \angle(B\Omega_2, BA) = \angle(C\Omega_2, CB) = \omega. \tag{5}$$

The points Ω_1 and Ω_2 are isogonal points with respect to the triangle ABC.

If the lines through the symmedian center K of the triangle ABC, which are parallel to the lines BC, CA, AB, intersect successively pairs of the lines CA, AB; AB, BC; BC, CA at the pairs of the points B_a, C_a, C_b, A_b; A_c, B_c, then the following equalities are valid (they are proved in [4, Th. 7])

$$\angle(BC, A_cB_a) = \angle(CA, B_aC_b) = \angle(AB, C_bA_c) = \omega, \tag{6}$$

$$\angle(C_aA_b, BC) = \angle(A_bB_c, CA) = \angle(B_cC_a, AB) = \omega. \tag{7}$$

According to (4) and (6) we get, for example the equality $\angle(AB, A\Omega_1) = \angle(AB, C_bA_c)$ which means that the lines $A\Omega_1$ and C_bA_c are parallel, and analogously $B\Omega_1 || A_cB_a$ and $C\Omega_1 || B_aC_b$. Besides that the lines B_aK, C_bK, A_cK are parallel to the lines BC, CA, AB, respectively. Thus, the triangles ABC and $B_aC_bA_c$ have the property that parallel lines through the vertices of the first triangle with the corresponding sides of the second triangle pass through one point and parallel lines through the vertices of the second triangle with corresponding sides of the first triangle pass also through one point. For these two triangles we shall say *parallelogonic*, and two mentioned points are the *centers of parallelogy* of these triangles. In the same way, by means of the equalities (5) and (7), it can be shown the triangles ABC and $C_aA_bB_c$ are parallelogonic, so we have the following theorem

Theorem 2. *If the lines through the symmedian center K of the triangle ABC, which are parallel to the lines BC, CA, AB, intersect successively pairs of the lines CA, AB; AB, BC; BC, CA at the pairs of the points B_a, C_a; C_b, A_b; A_c, B_c, then the triangle ABC is parallelogonic with the triangles $B_aC_bA_c$ and $C_aA_bB_c$. The centers of parallelogy are the points Ω_1, K and Ω_2, K where Ω_1 and Ω_2 are CB–points of the triangle ABC.*

Theorem 3. *CB–points of the standard triangle ABC are the points*

$$\Omega_1 = \left(\frac{p-p_1}{q}, \frac{3p_1^2}{q^2} - \frac{2}{9}q\right), \qquad \Omega_2 = \left(\frac{p-p_2}{q}, \frac{3p_2^2}{q^2} - \frac{2}{9}q\right). \tag{8}$$

Proof. From the last two equations (2) for abscissa x of the point Ω_1 we get the equation $(a-b)x = c(a-b)-(b-c)\omega$. Applying (1) we obtain

$$\begin{aligned} x &= c - \frac{b-c}{a-b}\omega = c + \frac{1}{3q}(b-c)^2(c-a) = \frac{1}{3q}[3cq-(q+3bc)(c-a)] = \\ &= \frac{1}{3q}(2cq+aq+3p-3bc^2) = \frac{1}{3q}[2c(bc-a^2)+a(ca-b^2)-3bc^2+3p] = \\ &= \frac{1}{3q}(3p-bc^2-ca^2-ab^2) = \frac{1}{3q}(3p-3p_1) = \frac{p-p_1}{q} = -\frac{2p_1+p_2}{q}. \end{aligned}$$

If the last two equations (2) are multiplied successively by $-b$ and a and if we add these two obtained equations we get for the ordinate y of the point Ω_1

$$\begin{aligned} y &= \omega x - \frac{q\omega}{a-b} - c(a+b) = \frac{1}{q^2}(2p_1+p_2)(p_1-p_2) + \frac{1}{3}(b-c)(c-a) + c^2 = \\ &= \frac{1}{q^2}(2p_1^2 - p_1p_2 - p_2^2) + \frac{1}{3}(2q-3ab) + ab - q = \\ &= \frac{3p_1^2}{q^2} - \frac{1}{q^2}(p_1^2 + p_1p_2 + p_2^2) - \frac{q}{3} = \frac{3p_1^2}{q^2} + \frac{q}{9} - \frac{q}{3} = \frac{3p_1^2}{q^2} - \frac{2q}{9}. \end{aligned}$$

The substitution $b \leftrightarrow c$ results in the substitutions $p_1 \leftrightarrow p_2$ and $\omega \leftrightarrow -\omega$, and the equations (2) become the equations (3) and vice versa. Because of that the same substitution leads to the substitution $\Omega_1 \leftrightarrow \Omega_2$ in the equations (8). $\lozenge$

Referring to [5], Th. 2 the second Lemoine circle of the standard triangle ABC is the pair of isotropic lines with the equations

$$x = \frac{p - p_1}{q}, \qquad x = \frac{p - p_2}{q},$$

and because of (8) these lines pass through the points Ω_1 and Ω_2. So we have the following statement.

Corollary 3. *CB-points of the standard triangle ABC lie on its second Lemoine circle, which consists of two isotropic lines where each of these lines passes through one CB-point.*

Theorem 4. *The joint line of CB–points Ω_1 and Ω_2 of the standard triangle ABC has the equation*

$$y = \frac{3p}{q}x - \frac{6p^2}{q^2} - \frac{5}{9}q \tag{9}$$

or

$$y = \frac{3p}{q}x + 2\omega^2 + \frac{1}{3}q.$$

Proof. For example for the point Ω_1, owing to (8), we get

$$\frac{3p}{q} \cdot \frac{p - p_1}{q} - \frac{6p^2}{q^2} - \frac{5}{9}q = -\frac{3}{q^2}(p^2 + pp_1 + p_1^2) + \frac{3p_1^2}{q^2} - \frac{5}{9}q =$$
$$= \frac{q}{3} + 3\frac{p_1^2}{q^2} - \frac{5}{9}q = \frac{3p_1^2}{q^2} - \frac{2}{9}q.$$

Besides that we have

$$-\frac{6p^2}{q^2} - \frac{5}{9}q - \frac{q}{3} = -\frac{6p^2}{q^2} - \frac{8}{9}q = -\frac{2}{9q^2}(27p^2 + 4q^3) = -2\omega^2. \qquad \lozenge$$

Acoording to [16] the Lemoine line of the triangle ABC has the slope $\dfrac{3p}{q}$, so it follows

Corollary 4. *The joint line of CB–points of an allowable triangle is parallel to its Lemoine line.*

Theorem 5. *If R is the radius of the circumscribed circle of a triangle and ω its Brocard angle, then the distance of CB–points is equal to $2R\omega$.*

Proof. According to [7] $2R = 1$ for the standard triangle ABC. Therefore from (8) we get

$$\Omega_2\Omega_1 = \frac{p-p_1}{q} - \frac{p-p_2}{q} = \frac{p_2-p_1}{q} = \omega = 2R\omega. \qquad \Diamond$$

In [15] Tölke proved a theorem wherefrom we get

Theorem 6. *CB-points of an allowable triangle in an isotropic plane are conjugated with regard to its polar circle.*

Proof. In [1] it is proved that polar circle $\mathcal{K}_p$ of the standard triangle has the equation $x^2+2y+q=0$, so the points (x_1,y_1), (x_2,y_2) are conjugated with respect to this circle under condition $x_1x_2+y_1+y_2+q=0$. For the points Ω_1, Ω_2 owing to (8) we get

$$\begin{aligned}
q^2(x_1x_2+y_1+y_2+q) &= (p-p_1)(p-p_2)+3p_1^2+3p_2^2-\frac{4}{9}q^3+q^3 = \\
&= p^2-(p_1+p_2)p+3(p_1^2+p_1p_2+p_2^2)-2p_1p_2+\frac{5}{9}q^3 = \\
&= 2p^2+3(-\frac{1}{9}q^3)-2\left(p^2+\frac{1}{9}q^3\right)+\frac{5}{9}q^3 = 0. \qquad \Diamond
\end{aligned}$$

Theorem 7. *CB–points Ω_1 and Ω_2 of the triangle ABC satisfy the following equalities*

$$A\Omega_1 = -\omega\,\tfrac{CA}{BC}, \quad B\Omega_1 = -\omega\,\tfrac{AB}{CA}, \quad C\Omega_1 = -\omega\,\tfrac{BC}{AB},$$

$$A\Omega_2 = \omega\,\tfrac{AB}{BC}, \qquad B\Omega_2 = \omega\,\tfrac{BC}{CA}, \qquad C\Omega_2 = \omega\,\tfrac{CA}{AB},$$

where ω is the Brocard angle of the triangle ABC.

Proof. From (1) we obtain

$$\begin{aligned}
3p-3p_1-3aq &= 3p-bc^2-ca^2-ab^2-3p-3a^2(b+c) = \\
&= c^2(c+a)-ca^2-a(c+a)^2+3a^3 = 2a^3-3a^2c+c^3 = \\
&= (a-c)^2(2a+c) = (a-c)^2(a-b) = \\
&= \frac{a-c}{c-b}(b-c)(c-a)(a-b) = -\frac{CA}{BC}\,3q\omega, \\
3p-3p_2-3aq &= 3p-b^2c-c^2a-a^2b-3p-3a^2(b+c) = \\
&= b^2(a+b)-a(a+b)^2-a^2b+3a^3 = 2a^3-3a^2b+b^3 = \\
&= (a-b)^2(2a+b) = (a-b)^2(a-c) = \\
&= -\frac{b-a}{c-b}(b-c)(c-a)(a-b) = \frac{AB}{BC}\,3q\omega
\end{aligned}$$

wherefrom it follows

$$A\Omega_1 = \frac{p-p_1}{q} - a = \frac{3p-3p_1-3aq}{3q} = -\frac{CA}{BC}\,\omega,$$

$$A\Omega_2 = \frac{p-p_2}{q} - a = \frac{3p-3p_2-3aq}{3q} = \frac{AB}{BC}\,\omega. \qquad \Diamond$$

Corollary 5. *With the labels from Th. 6 the following equalities are valid*

$$A\Omega_1 \cdot B\Omega_1 \cdot C\Omega_1 = -\omega^3, \qquad A\Omega_2 \cdot B\Omega_2 \cdot C\Omega_2 = \omega^3,$$
$$B\Omega_1 \cdot C\Omega_2 = -\omega^2, \quad C\Omega_1 \cdot A\Omega_2 = -\omega^2, \quad A\Omega_1 \cdot B\Omega_2 = -\omega^2,$$
$$A\Omega_1 : A\Omega_2 = -CA : AB, \quad B\Omega_1 : B\Omega_2 = -AB : BC,$$
$$C\Omega_1 : C\Omega_2 = -BC : CA.$$

TÖLKE ([15]) has the equalities

$$\frac{A\Omega_1}{A\Omega_2} : \frac{B\Omega_1}{B\Omega_2} : \frac{C\Omega_1}{C\Omega_2} = \frac{CA}{AB} : \frac{AB}{BC} : \frac{BC}{CA}$$

and shows that these equalities characterize CB–points among the pairs of isogonal points Ω_1 and Ω_2 of the triangle ABC.

Theorem 8. *Let Ω_1, Ω_2 be CB-points of the triangle ABC and*

$$A_1 = B\Omega_1 \cap C\Omega_2, \quad B_1 = C\Omega_1 \cap A\Omega_2, \quad C_1 = A\Omega_1 \cap B\Omega_2,$$
$$A_2 = B\Omega_2 \cap C\Omega_1, \quad B_2 = C\Omega_2 \cap A\Omega_1, \quad C_2 = A\Omega_2 \cap B\Omega_1. \tag{10}$$

The lines AA_1, BB_1,CC_1 pass through the one point Ω_1', and the lines AA_2, BB_2, CC_2 pass through the one point Ω_2'.

This theorem generalizes the Euclidean result (see Emmerich [2]).

Proof. Let us multiply the equations $(2)_2$ and $(3)_3$, of the lines $B\Omega_1$ and $C\Omega_2$, by $(a-b)^2$ and $-(c-a)^2$ respectively, and then add the obtained equalities. Because of

$$(a-b)^2 - (c-a)^2 = q + 3ca - (q+3ab) = -3a(b-c)$$

$$\begin{aligned}
&(a-b)^2(\omega-a) + (c-a)^2(\omega+a) = \\
&= -(q+3ab)(\omega-a) - (q+3ca)(\omega+a) = \\
&= -\omega(2q+3ca+3ab) + 3a^2(b-c) = \\
&= \frac{1}{3q}(b-c)(c-a)(a-b)(5q-3bc) + 3a^2(b-c) = \\
&= \frac{b-c}{3q}[(2q-3bc)(5q-3bc) + 9q(bc-q)] = \frac{b-c}{3q}(q^2-12bcq+9b^2c^2),
\end{aligned}$$

$$
\begin{aligned}
&- b\omega(a-b)^2 - c\omega(c-a)^2 - bc[(a-b)^2 - (c-a)^2] = \\
&= \omega[b(q+3ab) + c(q+3ca)] + bc \cdot 3a(b-c) = \\
&= -\frac{1}{3q}(b-c)(c-a)(a-b)[-aq + 3a(b^2+c^2)] + 3p(b-c) = \\
&= -\frac{b-c}{3q}(2q-3bc)a[-q+3(-q-bc)] + 3p(b-c) = \\
&= \frac{a(b-c)}{3q}[(2q-3bc)(4q+3bc)+9bcq] = \frac{a(b-c)}{3q}(8q^2+3bcq-9b^2c^2)
\end{aligned}
$$

we get, after multiplication by $-3q$ and dividing by $b-c$, the equality

$$
(11) \quad 9aq\,y = -(q^2 - 12bcq + 9b^2c^2)x - a(8q^2 + 3bcq - 9b^2c^2)
$$

of the line, which passes through the point $B\Omega_1 \cap C\Omega_2 = A_1$. However, this line also passes through the point $A = (a, a^2)$ because of
$-a(q^2 - 12bcq + 9b^2c^2) - a(8q^2 + 3bcq - 9b^2c^2) = 9aq(bc-q) = 9aq \cdot a^2$.
Therefore (11) is the equation of the line AA_1. This line passes through the point

$$
(12) \qquad \Omega_1{}' = \left(-\frac{3p}{q},\ \frac{3p^2}{q^2} - \frac{8}{9}q\right)
$$

owing to

$$
\begin{aligned}
&\frac{3p}{q}(q^2 - 12bcq + 9b^2c^2) - a(8q^2 + 3bcq - 9b^2c^2) = \\
&= \frac{a}{q}(3bcq^2 - 36b^2c^2q + 27b^3c^3 - 8q^3 - 3bcq^2 + 9b^2c^2q) = \\
&= \frac{a}{q}(-8q^3 - 27b^2c^2q + 27b^3c^3) = \frac{a}{q}(-8q^3 + 27b^2c^2 \cdot a^2) = \\
&= 9aq\left(\frac{3p^2}{q^2} - \frac{8}{9}q\right),
\end{aligned}
$$

and the same is true for the lines BB_1 and CC_1. Let us multiply the equalities $(2)_3$ and $(3)_2$ of the lines $C\Omega_1$ and $B\Omega_2$ with $a-b$ and $a-c$ respectively. If we add the obtained equalities, because of $2a-b-c=3a$ and

$$\begin{aligned}
&(a-b)(\omega-b)-(a-c)(\omega+c)=-(b-c)\omega+b^2+c^2-a(b+c)=\\
&=\frac{1}{3q}(b-c)^2(c-a)(a-b)+a^2+b^2+c^2=\\
&=\frac{1}{3q}[-(q+3bc)(2q-3bc)-2q\cdot 3q]=-\frac{1}{3q}(8q^2+3bcq-9b^2c^2),\\
&-c\omega(a-b)+b\omega(a-c)-ca(a-b)-ab(a-c)=\\
&=a(b-c)\omega-a(ca+ab-2bc)=\\
&=-\frac{a}{3q}(b-c)^2(c-a)(a-b)-a(q-3bc)=\\
&=\frac{a}{3q}[(q+3bc)(2q-3bc)-3q(q-3bc)]=-\frac{a}{3q}(q^2-12bcq+9b^2c^2),
\end{aligned}$$

we get, after multiplication by $3q$ and dividing by a, the equality

$$(13)\quad 9aq\,y=-(8q^2+3bcq-9b^2c^2)x-a(q^2-12bcq+9b^2c^2)$$

of the line, which passes through the point $B\Omega_2\cap C\Omega_1=A_2$. However, this line also passes through the point A because of
$-a(8q^2+3bcq-9b^2c^2)-a(q^2-12bcq+9b^2c^2)=9aq(bc-q)=9aq\cdot a^2$, and (13) is the equation of the line AA_2. This line passes through the point

$$(14)\qquad \Omega_2{}'=\left(\frac{3p}{2q},\ \frac{3p^2}{2q^2}-\frac{q}{9}\right)$$

owing to

$$\begin{aligned}
&-\frac{3p}{2q}(8q^2+3bcq-9b^2c^2)-a(q^2-12bcq+9b^2c^2)=\\
&=-\frac{a}{2q}(24bcq^2+9b^2c^2q-27b^3c^3+2q^3-24bcq^2+18b^2c^2q)=\\
&=-\frac{a}{2q}(2q^3+27b^2c^2q-27b^3c^3)=-\frac{a}{2q}(2q^3-27b^2c^2\cdot a^2)=\\
&=9aq\left(\frac{3p^2}{2q^2}-\frac{q}{9}\right). \qquad \diamondsuit
\end{aligned}$$

Theorem 9. *With the labels from Th. 8 the lines A_1A_2, B_1B_2, C_1C_2 pass through one point Ω_0, which lies on the line $\Omega_1'\,\Omega_2'$ as well as on the Euler line of the triangle ABC.*

This theorem generalizes the Euclidean result from Emmerich [2] and Nehring [8].

Proof. The point $\Omega_0 = (0, d)$, where

$$d = \frac{54p^2 - 10q^3}{27q^2}, \tag{15}$$

lies on the Euler line of the triangle ABC which, according to [7], has the equation $x = 0$. As

$$\frac{27p^2 - 8q^3}{9q^2} + 2 \cdot \frac{27p^2 - 2q^3}{18q^2} = \frac{1}{9q^2}(54q^2 - 10q^3) = 3d,$$

the equality ${\Omega_1}' + 2{\Omega_2}' = 3\Omega_0$ is true, and the point Ω_0 lies also on the line ${\Omega_1}'{\Omega_2}'$. Let us multiply the equations $(2)_2$ and $(3)_3$ with $d + bc - c\omega$ and $-(d + bc + b\omega)$, respectively. The addition of the obtained results leads to the equation

$$\begin{aligned} -(b + c)\omega\, y = {}& [(d + bc - c\omega)(\omega - a) + (d + bc + b\omega)(\omega + a)]\, x - \\ & - \omega[b(d + bc - c\omega) + c(d + bc + b\omega)] + bc(b + c)\omega \end{aligned}$$

of a line through the point A_1. The terms in brackets are equal to

$$(b - c)\omega^2 + (2d + 2bc + ca + ab)\omega = [(b - c)\omega + 2d + 2bc - a^2]\omega$$

and $-(b + c)(d + bc)\omega$, and the final form of this equation, without the factor ω, is

$$ay = [(b - c)\omega + 2d + 2bc - a^2]x + ad \tag{16}$$

owing to $a(d + bc) - abc = ad$. Therefore, this line passes through the point $\Omega_0 = (0, d)$. Multiplying equations $(2)_3$ and $(3)_2$ with $d + ab - b\omega$ and $-(d + ca + c\omega)$ respectively, and adding the obtained equations, we get the equation

$$\begin{aligned} (ab - ca - b\omega - c\omega)y = {}& [(d + ab - b\omega)(\omega - b) + (d + ca + c\omega)(\omega + c)]x - \\ - \omega[c(d+ab-b\omega)+b(d + ca + c\omega)] & - a[c(d + ab - b\omega) - b(d + ca + c\omega)] \end{aligned}$$

of a line through the point A_2. The terms in square brackets are equal to

$$\begin{aligned} & - (b - c)\omega^2 + (2d + ab + b^2 + ca + c^2)\omega - (b - c)d - a(b^2 - c^2) = \\ & = -(b - c)\omega^2 + (2d - bc - bc)\omega - (b - c)d + a^2(b - c) = \\ & = 2(d - bc)\omega + (a^2 - \omega^2 - d)(b - c) \end{aligned}$$

and $(b+c)d+2p$, $-(b-c)d-2bc\omega$, so this equation is of the form

$$[a(b-c)+a\omega]y=[2(d-bc)\omega+(a^2-\omega^2-d)(b-c)]x-$$
$$-(2p-ad)\omega+a[(b-c)d+2bc\omega]$$

and a free coefficient is $ad(b-c+\omega)$, i.e., it is of the form

$$a(b-c+\omega)y=[2(d-bc)\omega+(a^2-\omega^2-d)(b-c)]x+ad(b-c+\omega). \tag{17}$$

Therefore this line passes through the point $\Omega_0=(0,d)$. For the proof of the equivalency of the equations (16) and (17) it is necessary to prove the equality

$$(b-c+\omega)[(b-c)\omega+2d+2bc-a^2]=2(d-bc)\omega+(a^2-\omega^2-d)(b-c),$$

i.e.,

$$(b-c)^2\omega+2(b-c)\omega^2+2(d+bc)(b-c)+4bc\omega-2a^2(b-c)-a^2\omega+(b-c)d=0.$$

As

$$(b-c)^2+4bc-a^2=(b+c)^2-a^2=0,$$

without the factor $b-c$, it is necessary to prove the equality $2\omega^2+3d+$ $+2bc-2a^2=0$, i.e. the equality $2\omega^2=-3d-2q$. However we get according to (15) and (1)

$$-3d-2q=\frac{1}{9q^2}(10q^3-54p^2)-2q=-2\frac{27p^2+4q^3}{9q^2}=$$
$$=\frac{2}{9q^2}(b-c)^2(c-a)^2(a-b)^2=2\omega^2. \qquad \Diamond$$

In the Euclidean geometry Emmerich ([2]) and Nehring ([8]) have the equality $\Omega_1{}'\Omega_0:\Omega_2{}'\Omega_0=-2\cos 2\omega$, and in the isotropic geometry the equality $\Omega_1{}'+2\Omega_2{}'=3\Omega_0$ implies the following

Corollary 6. *The point from Th. 9 satisfies the equality*

$$\Omega_1'\,\Omega_0:\Omega_2'\,\Omega_0=-2.$$

Theorem 10. *The points Ω_1' and Ω_2' from Th. 8 satisfy the equality*

$$\frac{A\Omega_1'}{A\Omega_2'}:\frac{B\Omega_1'}{B\Omega_2'}:\frac{C\Omega_1'}{C\Omega_2'}=BC^3:CA^3:AB^3.$$

In the Euclidean geometry Nehring [8] has this statement.

Proof. From (12) and (14) we get

$$A\Omega_1'=-\frac{3p}{q}-a=-\frac{a}{q}(q+3bc)=\frac{a}{q}(b-c)^2,$$
$$A\Omega_2'=\frac{3p}{2q}-a=\frac{a}{2q}(3bc-2q)=\frac{a}{2q}(c-a)(a-b)$$

and therefore

$$\frac{A\Omega_1'}{A\Omega_2'} = -\frac{2(b-c)^2}{(c-a)(a-b)} = \frac{2(b-c)^3}{3q\omega} = -\frac{2}{3q\omega}BC^3. \qquad \Diamond$$

Theorem 11. *With the labels from Th. 8 the equalities*

$$(18) \qquad \frac{BA_2}{CA_2} = -\frac{AB^3}{CA^3}, \quad \frac{CB_2}{AB_2} = -\frac{BC^3}{AB^3}, \quad \frac{AC_2}{BC_2} = -\frac{CA^3}{BC^3}$$

are valid.

This theorem generalizes the Euclidean result (see Piggott [9]).

Proof. Let us find the abscissa of the point $A_2 = C\Omega_1 \cap B\Omega_2$. By the subtracting equations $(2)_3$ and $(3)_2$ of the lines $C\Omega_1$ and $B\Omega_2$, we get the equation $(2\omega - b + c)x - (b+c)\omega + a(b-c) = 0$ with the solution $x = a_2$ given by the formula

$$a_2 = \frac{a(c-b-\omega)}{2\omega - b + c}.$$

Since

$$\begin{aligned} c - b - \omega &= -(b-c) + \frac{1}{3q}(b-c)(c-a)(a-b) = \\ &= \frac{b-c}{3q}(-3q + 2q - 3bc) = -\frac{b-c}{3q}(q+3bc), \\ 2\omega - b + c &= -\frac{2}{3q}(b-c)(c-a)(a-b) - (b-c) = \\ &= -\frac{b-c}{3q}[2(2q-3bc)+3q] = -\frac{b-c}{3q}(7q-6bc), \end{aligned}$$

it follows

$$a_2 = \frac{a(q+3bc)}{7q-6bc}.$$

Because of that we get

$$\begin{aligned} (7q-6bc)\cdot BA_2 &= (7q-6bc)(a_2-b) = a(q+3bc) - b(7q-6bc) = \\ &= (a-7b)q + 3abc + 6b^2c = \\ &= -(a-7b)(a^2+ab+b^2) - 3b(a+2b)(a+b) = \\ &= -a^3 + 3a^2b - 3ab^2 + b^3 = (b-a)^3 = AB^3, \end{aligned}$$

and analogously

$$(7q-6bc)\cdot CA_2 = AC^3 = -CA^3,$$

so the first of the equalities (18) is valid. $\diamond$

In the Euclidean geometry the statement of Th. 13 can be found at [10], Pranesachar. The same statement can also be found at Piggott [9]. He states that in the considered case the line through the points A, Ω_1, Ω_2 is the angle bisector of the angle A. It is a consequence of the isogonality of the points Ω_1 and Ω_2.

It could be useful to know the midpoint of the points Ω_1 and Ω_2. So, it can be proved.

Theorem 12. *The midpoint of CB–points of the standard triangle ABC is the point* $S = \left(\frac{3p}{2q}, \frac{1}{2}\omega^2 - \frac{1}{3}q\right)$.

Proof. According to (8) we get

$$\begin{aligned} x &= \frac{2p - p_1 - p_2}{2q} = \frac{3p}{2q}, \\ y &= \frac{3}{2q^2}(p_1^2 + p_2^2) - \frac{2}{9}q = \frac{3}{2q^2}\left[(p_1 + p_2)^2 - 2p_1p_2\right] - \frac{2}{9}q = \\ &= \frac{3}{2q^2}\left[p^2 - 2\left(p^2 + \frac{1}{9}q^3\right)\right] - \frac{2}{9}q = -\frac{3}{2q^2}\left(p^2 + \frac{2}{9}q^3\right) + \frac{1}{9}q - \frac{1}{3}q = \\ &= -\frac{3p^2}{2q^2} - \frac{2}{9}q - \frac{1}{3}q = -\frac{1}{18q^2}(27p^2 + 4q^3) - \frac{1}{3}q = \frac{1}{18q^2} \cdot 9q^2\omega^2 - \frac{1}{3}q = \\ &= \frac{1}{2}\omega^2 - \frac{1}{3}q. \qquad \diamond \end{aligned}$$

Owing to [7] the Brocard diameter of the triangle ABC is the isotropic line with the equation $x = \frac{3p}{2q}$, so it follows

Corollary 7. *The midpoint of CB–points of an allowable triangle lies on its Brocard diameter.*

The following interesting statement can be also proved.

Theorem 13. *CB-points of the triangle ABC are collinear with its vertex A if and only if the equality* $BC^2 + CA \cdot AB = 0$ *is valid.*

Proof. The lines $A\Omega_1$ and $A\Omega_2$ with the first equations (2) and (3) are coincident under the condition $2\omega = -(b - c)$, i.e. provided that $2(c - a)(a - b) = 3q$ or $2(2q - 3bc) = 3q$ which gives the final condition $q = 6bc$.

On the other hand, we get
$BC^2+CA\cdot AB = (b-c)^2+(c-a)(a-b) = -(q+3bc)+2q-3bc = q-6bc$, so the obtained condition is equivalent to the desired equality BC^2+ $+CA\cdot AB = 0$. $\Diamond$

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