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UNUSUAL CNS POLYNOMIALS

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Abstract: CNS polynomials which lose the CNS property under addition of 1 are studied.

1. Introduction

Canonical number systems (usually abbreviated by CNS) can be regarded as generalizations of the classical decimal or binary numeration systems. They have first been introduced by the Hungarian school some decades ago (see [23, 21, 22, 25]); special cases had already been studied in [19, 24, 18]. The works [8, 9] are recommended as profound surveys on this subject in a broader context.

The concept of CNS polynomials[1] (see Sec. 2 for the definition) was introduced by A. Pethő [29] and generalized in the sequel (see for example [2, 7, 32]). Some characterization results on these polynomials are known (see e.g., [21, 18] for quadratic polynomials, [4, 12, 6, 14] for some other classes of polynomials and [26, 20] for more general results). However, until now the complete description of these polynomials remains an open problem even for small degrees.

K. Scheicher and J. Thuswaldner [30, Sec. 7] published the first example of a CNS polynomial P such that $P+1$ is not a CNS polynomial.

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[1]CNS polynomials are named complete base polynomials in [17].

For convenience we call such polynomials unusual CNS polynomials here (see Def. 5). Three more examples of unusual CNS polynomials have subsequently become known (see Sec. 2 for details), and A. Pethő [28] put forward the following problem: Find an infinite sequence of unusual CNS polynomials and then prove that for any given positive integer k there exists a member P of this sequence such that $P - k$ is a CNS polynomial.

In this note we collect some results on addition of constants to CNS polynomials and state sufficient conditions for unusual CNS polynomials. Some examples of unusual CNS polynomials of higher degrees and an infinite sequence of unusual cubic CNS polynomials are exhibited. Thereby we provide a solution to the aforementioned problem of A. Pethő and settle a conjecture of S. Akiyama and A. Pethő [5] on addition of constants to CNS polynomials (see Conj. 11 below).

2. Addition of constants to CNS polynomials

Let us first recall the definition of a CNS polynomial. Let $P \in \mathbb{Z}[X]$ be a monic integer polynomial of positive degree with $P(0) \neq 0$. We call P a CNS polynomial if for every $A \in \mathbb{Z}[X]$ there exists a polynomial $B \in \{0, \ldots, |P(0)| - 1\}[X]$ such that $A \equiv B \pmod{P}$. Throughout we denote by $\mathbb{Z}$ ($\mathbb{N}$, respectively) the set of rational integers (the set of nonnegative rational integers, respectively).

Theorems 1 to 4 gather immediate consequences of well-known results and show that in numerous cases the CNS property is preserved if the constant term of a CNS polynomial is enlarged.

Theorem 1. *Let $P = X^d + p_{d-1}X^{d-1} + \cdots + p_1X + p_0 \in \mathbb{N}[X]$ be a non-constant polynomial. If*

$$p_1 + \cdots + p_{d-1} \leq p_0 \qquad \text{or} \qquad 1 \leq p_{d-1} \leq \cdots \leq p_1 \leq p_0,$$

then $P + k$ is CNS polynomial for every positive integer k.

Proof. By [4, Lemma 1] the first condition implies that $P + k$ is expanding, thus we can apply [6, Th. 3.2]. For the second condition a generalization of a theorem of B. Kovács and A. Pethő [27] can be used (see also [15, Cor. 6]). $\Diamond$

Theorem 2. *Let $P = X^d + p_{d-1}X^{d-1} + \cdots + p_1X + p_0 \in \mathbb{Z}[X]$ be a monic polynomial which fulfills the following properties:*

(i) $p_1 < 0$,
(ii) $p_2, \ldots, p_{d-1} \geq 0$,
(iii) $p_0 \geq \sum_{i=1}^{d} |p_i|$.

Then $P + k$ is a CNS polynomial for every positive integer k.

Proof. Observe that $P + k$ is expanding, therefore the result follows by using [6, Th. 3.2]. $\Diamond$

Theorem 3. *Let $P = p_d X^d + p_{d-1} X^{d-1} + \cdots + p_1 X + p_0 \in \mathbb{Z}[X]$ be a monic expanding polynomial with all coefficients nonnegative except $p_j < 0$ for a single index $0 < j < d$. If*

$$p_0 \geq \sum_{i=1}^{d} |p_i| \quad \text{and} \quad \sum_{1 \leq ij \leq d} p_{ij} \geq 0,$$

then $P + k$ is a CNS polynomial for every $k \in \mathbb{N}$.

Proof. This statement immediately stems from [6, Th. 3.5]. $\Diamond$

Theorem 4. *Let $0 < m < d$ and $b \in \mathbb{Z}$. If one of the two conditions*

(i) *$m = 1$ and $b \geq -1$*

or

(ii) *m does not divide d and $b \geq 0$*

holds, then $X^d + bX^m + k$ is a CNS polynomial for every integer $k \geq b+2$.

Proof. This is clear by [12, Th. 3]. $\Diamond$

Having in mind these results and the definition of a CNS polynomial one might tend to expect that the addition of a positive constant to a CNS polynomial yields a CNS polynomial. However, K. Scheicher and J. Thuswaldner [30, Sec. 7] observed that this is not always true. For convenience, we therefore introduce the following terminology.

Definition 5. The CNS polynomial P is called unusual if $P + 1$ is not a CNS polynomial.

In view of the well known characterization of CNS polynomials of degree at most two, unusual CNS polynomials must have degree at least three as is shown in the following lemma.

Lemma 6. *Let P be CNS polynomial.*

(i) *If $\deg(P) \leq 2$, then $P + 1$ is a CNS polynomial.*
(ii) *If $\deg(P) = 3$, then $P + 1$ satisfies Gilbert's conditions (see [3, Th. 3.1]).*
(iii) *If $r \in \mathbb{R}$ is a real root of $P + 1$, then $r < -1$.*

Proof. (i) Recall that $P(0) \geq 2$ (see [18, 27]), thus for $\deg(P) = 1$ our statement follows trivially from Th. 1. For $\deg(P) = 2$ our assertion follows from the well-known characterization of quadratic CNS polynomials (see [21, 22, 18, 12, 34, 6]).

(ii) P satisfies Gilbert's conditions by [3, Th. 3.1], thus the assertion can easily be verified.

(iii) We have $P(r) = -1$, hence $r < -1$ because, by the so-called analytic conditions (e.g., see [1]), we have $P(t) > 0$ for all $t \geq -1$. $\Diamond$

Remark 7. It was conjectured in [13] that every monic cubic polynomial with integer coefficients and all roots real and less than -1 is a CNS polynomial. If this conjecture is true then by Lemma 6 (iii) every cubic unusual CNS polynomial must have non-real roots.

Since Gilbert's conditions do not characterize cubic CNS polynomials [3, Sec. 3], we may search for cubic unusual CNS polynomials. In Table 1 below the known examples of unusual CNS polynomials are listed; there Δ denotes the discriminant of the cubic CNS polynomial $X^3 + p_2X^2 + p_1X + p_0$. Furthermore, in all cases $(3, -1, -1) \in \mathbb{Z}^3$ generates a periodic element of period length 8 of the map[2]

$$(a_1, a_2, a_3) \mapsto \left(a_2, a_3, -\left\lfloor \frac{a_1 + p_2a_2 + p_1a_3}{p_0 + 1} \right\rfloor\right) \qquad ((a_1, a_2, a_3) \in \mathbb{Z}^3).$$

For the background and details the reader is referred to [2, Sec. 3].

p_0	p_1	p_2	Δ	roots (approximately)	reference
89	117	80	$-86,287,999$	$-78.524,\ -0.73777 \pm 0.76752\,\mathrm{i}$	[28]
109	143	97	$-190,323,500$	$-95.514,\ -0.74260 \pm 0.76793\,\mathrm{i}$	[28]
198	257	173	$-2,034,469,219$	$-171.50,\ -0.74586 \pm 0.77339\,\mathrm{i}$	[30]
473	611	410	$-66,428,866,967$	$-408.50,\ -0.74642 \pm 0.77506\,\mathrm{i}$	[28]

Table 1: Some unusual CNS polynomials

We now give a criterion for a CNS polynomial to be unusual. We fix a monic non-constant polynomial $P = \sum_{i=0}^{d} p_iX^i \in \mathbb{Z}[X]$ with $p_0 \neq 0$ and set

$$T_P(A) = \sum_{i=1}^{d-1}(a_i - \mathrm{sign}\,(p_0)cp_i)X^{i-1} - \mathrm{sign}\,(p_0)cX^{d-1}$$

[2] $\lfloor \dots \rfloor$ denotes the floor function.

for $A = \sum_{i=0}^{d-1} a_i X^i \in \mathbb{Z}[X]$ and $c = \lfloor a_0/|p_0| \rfloor$. Thus T_P is a mapping from the set of integer polynomials of degree less than d into itself (see [15, Sec. 3] for more details).

Theorem 8. *Let P be a CNS polynomial of degree at least 3. Then P is unusual if one of the following conditions is satisfied.*

(i) *There exists a polynomial $A \in \mathbb{Z}[X] \setminus \{0\}$ of degree less than $\deg(P)$ and a positive integer n such that*[3]
$$T_{P+1}^n(A) = A.$$

(ii) *There are polynomials $h \in \mathbb{Z}[X]$, $g \in \{0, 1, \ldots, P(0)\}[X] \setminus \{0\}$ and a positive integer n such that $\deg(g) < n$ and $h \cdot (P + 1) \equiv g \pmod{X^n - 1}$.*

Proof. (i) Clear by [2, Sec. 3].

(ii) Clear by (i) and the proof of [20, Th. 4]. $\Diamond$

Given an unusual CNS polynomial we can easily construct unusual CNS polynomials of higher degrees.

Proposition 9. *If P is an unusual CNS polynomial, then $P(X^n)$ is an unusual CNS polynomial for every $n \in \mathbb{N}_{>0}$.*

Proof. Clear by [12, Th. 1]. $\Diamond$

Now we extend the list of examples of unusual CNS polynomials and firstly present an infinite family of cubic unusual CNS polynomials.

Proposition 10. *Let $n \in \mathbb{N}$. Then*
$$P_n = X^3 + (15n + 50)X^2 + (22n + 73)X + 17n + 55$$
is an unusual CNS polynomial and $P_{100n} - n$ is a CNS polynomial.

Proof. First we show that $P_n = X^3 + p_2 X + p_1 X + p_0$ with $p_0 = 17n + 55$, $p_1 = 22n + 73$, $p_2 = 15n + 50$ is indeed a CNS polynomial. This is checked algorithmically for $n < 4$ (e.g., see [2, Sec. 5] or [33, 10, 17] for an algorithm). Let now $n \geq 4$. To $\mathbf{r} = (r_1, r_2, r_3) = (1/p_0, p_2/p_0, p_1/p_0)$ we associate the mapping $\tau_{\mathbf{r}} \colon \mathbb{Z}^3 \to \mathbb{Z}^3$ in the following way:
$$\tau_{\mathbf{r}}(a_1, a_2, a_3) = (a_2, a_3, -\lfloor r_1 a_1 + r_2 a_2 + r_3 a_3 \rfloor).$$
A rather tedious but straightforward calculation shows

$$\overline{F_i} \subset N_{\mathbf{r}} := \{x \in \mathbb{Z}^3 : \text{ there exists } k \in \mathbb{N} \text{ with } \tau_{\mathbf{r}}^k(x) = 0\},\ \tau_{\mathbf{r}}(\overline{F_i}) \subset G_i \cup \overline{F_{i+1}} \quad (i = 0, \ldots, 20), \tag{1}$$

[3]For a mapping f we write $f^0 = \mathrm{id}$ and $f^{k+1} = f \circ f^k$.

where we set
$$\overline{S} = S \cup (-S)$$
for subsets S of $\mathbb{Z}^3$,
$$G_k = \bigcup_{i=0}^{k} \overline{F_i} \qquad (k = 0, 1, \ldots, 21),$$
and the sets F_i are defined as follows (the choice of these sets is motivated by [2, Th. 5.1]):

$$\begin{aligned}
F_0 =& \{(1,0,0),(0,1,0),(0,0,1),(0,0,0)\},\\
F_1 =& \{(0,1,-2),(0,1,-1),(1,-2,1),(1,-1,0),(1,-1,1),(1,0,-1),\\
&(2,-1,0)\},\\
F_2 =& \{(1,-2,2),(1,1,-2),(2,-2,0),(2,-1,-1),(2,0,-1)\},\\
F_3 =& \{(0,2,-2),(1,-3,3),(1,-3,4),(1,1,-3),(2,-2,1),(2,0,-2),\\
&(2,1,-3),(3,-4,2),(3,-3,2),(3,-2,-1),(4,-2,0)\},\\
F_4 =& \{(0,2,-3),(1,-4,4),(2,-3,2),(2,1,-4),(3,-4,3),(3,-3,1),\\
&(3,-2,0),(3,-1,-1),(3,0,-2),(4,-4,2),(4,-3,0),(4,-2,-1)\},\\
F_5 =& \{(0,3,-3),(1,-4,5),(1,2,-4),(1,2,-3),(2,-4,3),(2,-3,3),\\
&(3,-1,-2),(3,0,-3),(4,-5,2),(4,-4,1),(4,-3,1),(4,-1,-2),\\
&(5,-2,-1)\},\\
F_6 =& \{(0,3,-4),(1,3,-4),(2,-4,4),(2,1,-3),(2,2,-4),(3,-4,2),\\
&(3,-3,3),(3,1,-3),(4,-5,3),(4,-1,-3),(5,-3,-1),(5,-2,-2)\},\\
F_7 =& \{(1,3,-5),(2,-5,4),(2,2,-5),(3,-5,3),(3,1,-4),(5,-4,1),\\
&(5,-3,0),(5,-3,1)\},\\
F_8 =& \{(2,-5,5),(3,-5,4),(4,0,-3),(5,-5,2),(5,-4,0)\},\\
F_9 =& \{(0,4,-5),(4,0,-4),(5,-5,3)\},\\
F_{10} =& \{(0,4,-6),(4,-6,4),(6,-4,0)\},\\
F_{11} =& \{(4,-6,5),(4,1,-4),(5,-1,-3),(6,-5,1),(6,-4,-1)\},\\
F_{12} =& \{(1,-5,5),(1,-4,4),(1,4,-6),(4,1,-5),(5,-1,-4),(6,-5,2)\},\\
F_{13} =& \{(1,-5,6),(1,4,-7),(3,1,-4),(4,-7,5),(5,-6,3),(6,-3,-1),\\
&(7,-5,1)\},\\
F_{14} =& \{(3,2,-5),(4,-7,6),(5,-6,4),(5,0,-4),(6,-3,-2),(6,-1,-4),\\
&(7,-6,1),(7,-5,0)\},
\end{aligned}$$

$$
\begin{aligned}
F_{15} =&\{(0,5,-6),(2,-6,6),(2,3,-5),(3,2,-6),(5,0,-5),(6,-6,3),\\
&(6,-2,-3),(7,-6,2)\},\\
F_{16} =&\{(0,5,-7),(2,-6,7),(2,3,-6),(3,-6,5),(5,-7,4),(6,-7,3),\\
&(6,-6,2),(6,-2,-2),(7,-4,0),(7,-3,-2)\},\\
F_{17} =&\{(3,-6,6),(3,3,-6),(5,-7,5),(6,-7,4),(7,-4,-1),(7,-3,-3)\},\\
F_{18} =&\{(3,-7,6),(3,3,-7)\},\\
F_{19} =&\{(3,-7,7),(7,-7,2),(7,-2,-3)\},\\
F_{20} =&\{(2,4,-6),(4,-6,5),(7,-7,3),(7,-2,-4)\},\\
F_{21} =&\{(2,4,-7)\}.
\end{aligned}
$$

Let us exemplify the calculation for $i = 0$. Here we simply write an arrow for the action of $\tau_{\mathbf{r}}$ and successively find the following relations:

(i) $(1,0,0) \mapsto (0,0,0)$, hence $(1,0,0) \in N_{\mathbf{r}}$ and $\tau_{\mathbf{r}}(1,0,0) \in F_0 \subset G_0$,

(ii) $(0,1,0) \mapsto (1,0,0)$, hence $(0,1,0) \in N_{\mathbf{r}}$ and $\tau_{\mathbf{r}}(0,1,0) \in G_0$,

(iii) $(0,0,1) \mapsto (0,1,-1)$, hence $\tau_{\mathbf{r}}(0,0,1) \in G_0 \cup F_1$, and $(0,0,1) \in N_{\mathbf{r}}$ because we have

$$(0,1,-1) \mapsto (1,-1,1) \mapsto (-1,1,0) \mapsto (1,0,0). \tag{2}$$

Now we turn to $-F_0$ and derive

$$(-1,0,0) \mapsto (0,0,1), \text{ hence } (-1,0,0) \in N_{\mathbf{r}} \text{ and } \tau_{\mathbf{r}}(-1,0,0) \in G_0,$$

then

$$(0,-1,0) \mapsto (-1,0,1) \mapsto (0,1,-1), \tag{3}$$

hence $(0,-1,0) \in N_{\mathbf{r}}$ by (2), and $\tau_{\mathbf{r}}(0,-1,0) \in (-F_1) \subset G_0 \cup \overline{F_1}$, and finally

$$(0,0,-1) \mapsto (0,-1,2) \mapsto (-1,2,-1) \mapsto (2,-1,0) \mapsto (-1,0,1),$$

hence $(0,0,-1) \in N_{\mathbf{r}}$ by (3), and $\tau_{\mathbf{r}}(0,0,-1) \in (-F_1) \subset G_0 \cup \overline{F_1}$. Since $(0,0,0) \in N_{\mathbf{r}}$ is fixed by $\tau_{\mathbf{r}}$, we thus have shown $\overline{F_0} \subset N_{\mathbf{r}}$ and $\tau_{\mathbf{r}}(\overline{F_0}) \subset$ $\subset G_0 \cup \overline{F_1}$.

The verification of (1) for $i = 1, \ldots, 20$ is performed analogously and left to the reader. Moreover, we find $G_{21} \subset N_{\mathbf{r}}$ and $\tau_{\mathbf{r}}(G_{21}) \subset G_{21}$, i.e., G_{21} is a set of witnesses, and therefore P_n is a CNS polynomial; for definitions and further details the reader is referred to [2, Sec. 5].

Setting

$$T := T_{P_n+1} \quad \text{and}$$
$$A := p_1 + 2p_2 - 3 + (p_2+2)X + X^2 = 52n + 170 + (15n+52)X + X^2$$

we find

$$\begin{aligned} T^8(A) &= T^7(-51n - 167 + (-45n-149)X - 3X^2) = \\ &= T^6(21n + 70 + (45n+147)X + 3X^2) = \\ &= T^5(23n + 74 + (-15n-47)X - X^2) = \\ &= T^4(-37n - 120 + (-15n-51)X - X^2) = \\ &= T^3(51n + 168 + (45n+149)X + 3X^2) = \\ &= T^2(-21n - 70 + (-45n-147)X - 3X^2) = \\ &= T(-n - 1 + (30n+97)X + 2X^2) = A, \end{aligned}$$

thus an application of Th. 8 (i) concludes the proof of our first assertion; alternatively, we can check that $(-3,2,1) \in \mathbb{Z}^3$ generates a periodic element under the mapping $\tau := \tau_{(1/(p_0+1),\, p_2/(p_0+1),\, p_1/(p_0+1))}$, namely

$$\begin{aligned} \tau^8(-3,2,1) &= \tau^7(2,1,-3) = \tau^6(1,-3,3) = \tau^5(-3,3,-1) = \tau^4(3,-1,-1) = \\ &= \tau^3(-1,-1,3) = \tau^2(-1,3,-3) = \tau(3,-3,2) = (-3,2,1). \end{aligned}$$

Analogously we settle our second statement, i.e., we show that $P_{100n} - n$ is a CNS polynomial. This is clear for $n = 0$ by the above and is checked algorithmically for $1 \le n \le 3$. Now we fix $n > 3$ and set

$$H_k = \bigcup_{i=0}^{k} \overline{E_i} \qquad (k = 0, 1, \ldots, 21)$$

with

$$E_i = F_i \qquad (i \in \{0, 1, \ldots, 6, 8, \ldots, 14\})$$

and

$$\begin{aligned} E_7 &= F_7 \setminus \{(5,-3,1)\}, \\ E_{15} &= F_{15} \cup \{(1,5,-7), (5,-7,5), (6,-1,-5)\}, \\ E_{16} &= \{(0,5,-7), (1,5,-8), (2,-6,7), (2,3,-6), (3,-6,5), (5,-8,5), \\ &\quad (5,-7,4), (6,-7,3), (6,-6,2), (6,-2,-2), (7,-4,-1), (7,-3,-2), \\ &\quad (8,-5,0)\}, \end{aligned}$$

$$
\begin{aligned}
E_{17} = & \{(2,2,-4),(3,-6,6),(3,3,-6),(4,2,-6),(5,-8,6),(5,1,-5),\\
& (6,-7,4),(6,0,-5),(7,-4,-2),(7,-3,-3),(8,-6,0),(8,-5,-1)\},\\
E_{18} = & \{(0,6,-7),(1,-6,6),(2,-7,7),(3,-7,6),(3,3,-7),(4,2,-7),\\
& (5,1,-6),(6,0,-6),(7,-7,3),(8,-6,1)\},\\
E_{19} = & \{(0,6,-8),(1,-6,7),(2,-7,8),(3,-7,7),(6,-8,5),(6,-7,3),\\
& (7,-8,4),(7,-7,2),(7,-2,-3),(8,-4,-1)\},\\
E_{20} = & \{(2,4,-6),(4,-6,5),(6,-8,6),(7,-8,5),(7,-2,-4),(8,-4,-2)\},\\
E_{21} = & \{(2,4,-7),(8,-6,1)\}.
\end{aligned}
$$

Replacing $\mathbf{r}$ by

$$\mathbf{s} = \frac{1}{1699n+55} \cdot (1,\ 1500n+50,\ 2200n+73)$$

we find $H_{21} \subset N_{\mathbf{s}}$ and $\tau_{\mathbf{s}}(H_{21}) \subset H_{21}$ which completes the proof. $\diamondsuit$

Obviously, the sequence $(P_n)_{n\in\mathbb{N}}$ given in Prop. 10 solves the problem of A. Pethő mentioned in the introduction. Furthermore, it settles the following conjecture of S. Akiyama and A. Pethő [5]:

Conjecture 11. *For any positive integer k there exists a CNS polynomial P of degree at least 3 such that $P+k$ is not a CNS polynomial.*

Indeed, if $k > 0$ then $P = P_{100(k-1)}$ is an unusual CNS polynomial of degree 3, $Q = P - (k-1)$ is a CNS polynomial and

$$Q + k = P - (k-1) + k = P + 1$$

is not a CNS polynomial.

Remark 12. (i) Observe that the polynomial P_2 of Prop. 10 coincides with the first polynomial of Table 1.

(ii) The polynomial P_0 seems to be an unusual CNS polynomial with smallest known constant term.

(iii) The factor 100 in the second part of Prop. 10 was chosen to downsize the quotient of the linear and the constant terms of the polynomial. Numerical experiments suggest that this factor may be replaced by an arbitrary positive integer.

(iv) Using the notation of [2, Sec. 5] we give a hint on the construction of the example given in Prop. 10. Let $P = \sum_{i=1}^{d} p_i X^i \in \mathbb{N}[X]$ be an unusual CNS polynomial and assume that for a set of witnesses $W = \{k_1, \dots, k_m\} \subset \mathbb{Z}^d$ for P and certain $b_0, \dots, b_m, B_0, \dots, B_m \in \mathbb{N}_{>0}$ we have

$$-b_j < \sum_{i=1}^{d} k_{ji} r_i < -b_j + 1 \quad \text{or} \quad B_j - 1 < \sum_{i=1}^{d} k_{ji} r_i < B_j \quad (j = 1, \dots, m),$$

where $r_1 = 1/p_0, \dots, r_d = p_1/p_0$ are the components of the coefficient vector $\mathbf{r}$ associated to P. Certainly W cannot be a set of witnesses for $P+1$ whose coefficient vector is

$$(r'_1 \dots, r'_d) := \frac{p_0}{p_0+1}\mathbf{r}\,.$$

Since

$$-b_j < -\frac{p_0}{p_0+1} b_j < \frac{p_0}{p_0+1} \sum_{i=1}^{d} k_{ji} r_i = \sum_{i=1}^{d} k_{ji} r'_i < \frac{p_0}{p_0+1} B_j < B_j,$$

there must be some j such that

$$\sum_{i=1}^{d} k_{ji} r'_i \geq -b_j + 1 \qquad \text{or} \qquad \sum_{i=1}^{d} k_{ji} r'_i \leq B_j - 1,$$

which means

$$\sum_{i=1}^{d} k_{ji} r_i \geq (-b_j + 1)\frac{p_0+1}{p_0} = \left(1 + \frac{1}{p_0}\right)(1 - b_j)$$

or

$$\sum_{i=1}^{d} k_{ji} r_i \leq (B_j - 1)\frac{p_0+1}{p_0} = \left(1 + \frac{1}{p_0}\right)(B_j - 1).$$

(v) It is immediate that our example described in Prop. 10 satisfies the prerequisites of (iv). Furthermore, we observe

$$r_1 + r_2 - 3r_3 = \frac{1}{p_0}(-3p_0 - 3) = \frac{p_0+1}{p_0}(1-4) < -4 + 1.$$

Some easy numerical calculations (or [13, Prop. 2.2]) show that the discriminants of the polynomials P_n are negative, that their real roots belong to the interval $(-(15c+4), -(15c+3))$ and that the moduli of the complex roots are less than 1.0705.

Let us present examples of unusual CNS polynomials of degrees 4 and 5.

Example 13. (i) $P = X^4 + X^3 + 410X^2 + 611X + 474$ is an unusual quartic CNS polynomial: One can check algorithmically that P is in fact a CNS polynomial and $(-3, 2, 1, -3) \in \mathbb{Z}^4$ yields a periodic element (of period length 8) for $P+1$ (compare the analogous statement in the proof of Prop. 10). P has only non-real roots, approximately given by $-0.747 \pm 0.775\,\mathrm{i}$ and $0.247 \pm 20.237\,\mathrm{i}$.

(ii) Similarly, one can check algorithmically that $P = X^5 + X^4 + 2X^3 + 97X^2 + 143X + 109$ is an unusual quintic CNS polynomial; here $(1, -3, 3, -1, -1) \in \mathbb{Z}^5$ defines a periodic element (of period length 8) for $P + 1$. The roots of P are approximately -4.301, $-0.741 \pm 0.771\,\mathrm{i}$, and $2.392 \pm 4.052\,\mathrm{i}$.

We conclude by speculating about the effect of adding constants to CNS polynomials.

Conjecture 14. *For every $d \geq 3$ there exists an unusual CNS polynomial of degree d.*

Our results show that Conj. 14 holds provided that d is divisible by 3, 4 or 5.

Remark 15. (i) We observe that all examples of unusual CNS polynomials given in Table 1, Prop. 10 and Ex. 13 have only positive coefficients and are unimodal. Moreover, they admit a pair of complex conjugate roots with modulus close to 1.

(ii) The concept of CNS polynomials was extended to semi-CNS polynomials by P. Burcsi and A. Kovács [16, Def. 3.2]. In contrast to CNS polynomials semi-CNS have completely been described (see [31, 11]). This characterization shows that addition of constants to semi-CNS polynomials is not a reasonable problem: If P is a semi-CNS polynomial then $P - 1$ is a semi-CNS polynomial, and if $P(0) < -2$ then also $P + 1$ is a semi-CNS polynomial.

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